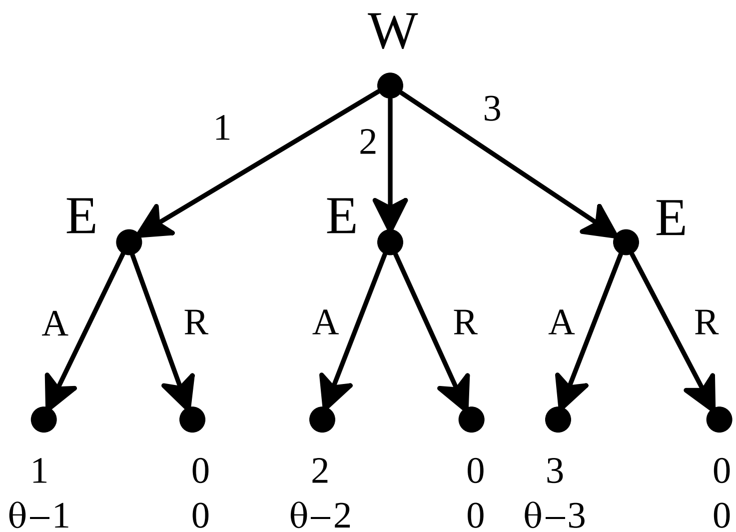
ANSWER KEYS Micro Prelim June 21, 2021

QUESTION 1 ANSWER KEYS

(a) (a.1) The extensive game is as follows:



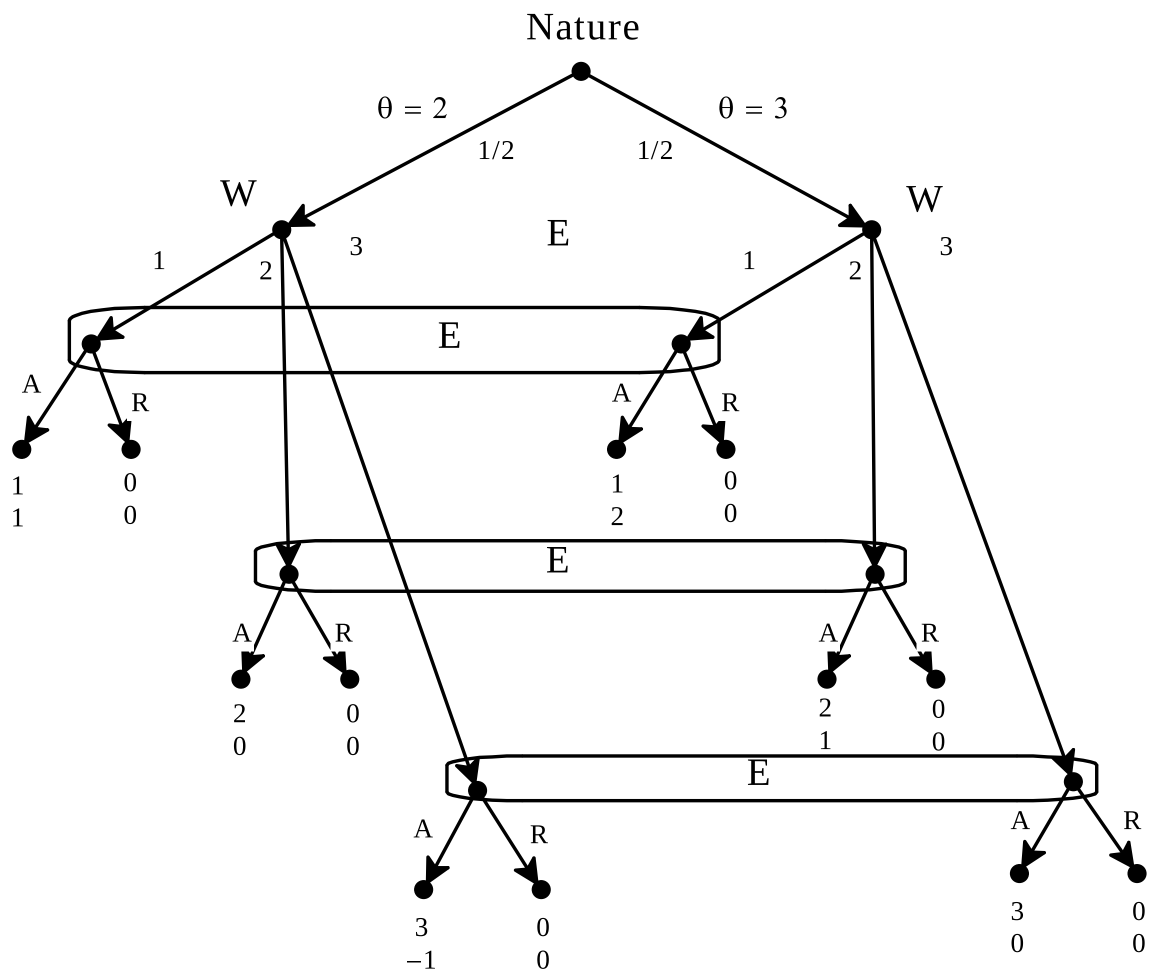
(a.2) The subgame-perfect equilibria are as follows (E's strategy is to be interpreted from left to right)

- If $\theta = 1$: (1,RRR), (2,RRR), (3,RRR), (1,ARR)
- If $1 < \theta < 2$: (1,ARR)
- If $\theta = 2$: (1,ARR), (2,AAR)
- If $2 < \theta < 3$: (2,AAR)
- If $\theta = 3$: (2,AAR), (3,AAA)
- If $\theta > 3$: (3,AAA)

(a.3) The Nash equilibria that are not subgame-perfect are the following:

- If $1 < \theta < 2$: (2,RRR), (3,RRR)
- If $\theta = 2$: (2,RRR), (3,RRR), (2,RAR)
- If $2 < \theta < 3$: (1,ARR), (2,RAR), (3,RRR)
- If $\theta = 3$: (1,ARR), (2,RAR), (3,RRR), (3,ARA), (3,RAA), (3,RRA)
- If $\theta > 3$: (1,ARR), (2,AAR), (2,RAR), (3,ARA), (3,RAA), (3,RRA)

(b) **(b.1)** The extensive game is as follows:



(b.2) First of all, for the employer A strictly dominates R at her top information set (where she is offered 1). Thus the choice there must be A.

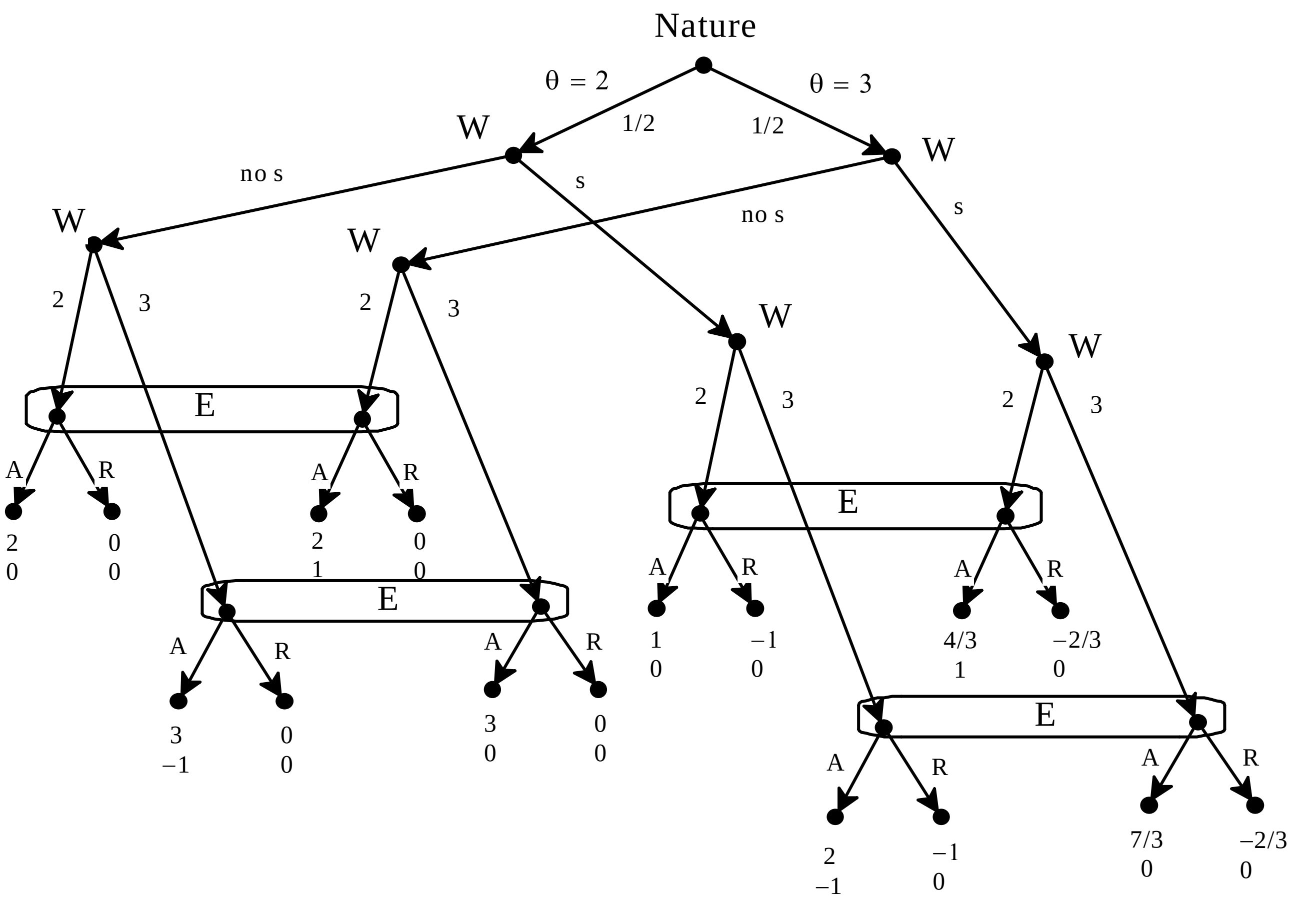
Pooling equilibria:

- ((1,1),(A,R,R)) with beliefs (½, ½) at *E*'s top information set, (1,0) at her middle information set and any beliefs at the bottom information set.

- ((2,2),(A,A,R)) with any beliefs at *E*'s top information set, beliefs (½, ½) at her middle information set and any beliefs at the bottom information set.

(b.3) There are no pure-strategy separating equilibria. First of all, note that at E's top information set A must be chosen because it strictly dominates R. If *E*'s choice at her bottom information set is *A*, then requesting 3 is a strictly dominant choice at both of *W*'s nodes, thus making different choice not rational. Suppose, therefore, that *E*'s choice at her bottom information set is *R*; then if her choice at her middle information set is *A* then requesting 2 is a strictly dominant choice at both of *W*'s nodes, thus making different choices not rational. Thus it must be that *E*'s strategy is (*A*,*R*,*R*) but then requesting 1 is a strictly dominant choice at both of *W*'s nodes, thus making different choices not rational.

(c) (c.1) The extensive game is as follows:



(c.2) A separating equilibrium is as follows:

Worker's strategy: (1) No signal if $\theta = 2$ and acquire signal (at cost 1.5) if $\theta = 3$, (2) if did not acquire signal request 2 (two nodes where this applies) and if acquired signal request 3 (two nodes where this applies).

Employer's strategy: (1) if no signal and request was 2, accept, (2) if no signal and request was 3, reject, (3) if signal and request was 2, accept, (4) if signal and request was 3, accept.

Employer's beliefs: (1) at top left information set (1,0) (from left to right), (2) at bottom left information set any beliefs, (3) at top right information set any beliefs, (4) at bottom right information set (0,1).

Question 2: Risk Apportionment

Considering only lotteries that pay in non-negative units of money, suppose that W, X, Y and Z are independent lotteries with bounded support. Assume that $W >_1 X$ and $Y >_1 Z$.[1] Consider the following compound lotteries:

- $\mathcal{K}$ pays $W + Y$ or $X + Z$ with equal probabilities, 1/2;
- $\mathcal{L}$ pays $W + Z$ or $X + Y$ with equal probabilities, 1/2.

In what follows, you will compare these two compound lotteries in terms of stochastic dominance.

(a) Argue that if a Bernoulli index $u : \mathbb{R}_+ \to \mathbb{R} \in \mathbb{C}^2$ is strictly increasing and strictly concave, then the function

$$v(r) = \mathrm{E}[u(X + r)] - \mathrm{E}[u(W + r)]$$

is strictly increasing.

Answer: It suffices to show that

$$v'(r) = \mathrm{E}[u'(X + r)] - \mathrm{E}[u'(W + r)] > 0.$$

For this, just note that, by strict concavity of u, function u' is decreasing. Since $W >_1 X$, it is immediate that $W + r >_1 X + r$, for all r. This implies that

$$\mathrm{E}[-u'(W + r)] > \mathrm{E}[-u'(X + r)],$$

which gives the desired result. □

(b) Argue now that, under the same assumptions about u,

$$\mathrm{E}[v(Y)] > \mathrm{E}[v(Z)].$$

Answer: Since v is increasing, the result is immediate because $Y >_1 Z$. □

(c) Argue that, still under the assumptions on u,

$$\tfrac{1}{2}\left\{\mathrm{E}[u(W + Y)] + \mathrm{E}[u(X + Z)]\right\} < \tfrac{1}{2}\left\{\mathrm{E}[u(W + Z)] + \mathrm{E}[u(X + Y)]\right\}.$$

Answer: Substituting directly,

$$\begin{aligned}
\mathrm{E}[u(X + Y)] - \mathrm{E}[u(W + Y)] &= \mathrm{E}[v(Y)] \\
&> \mathrm{E}[v(Z)] \\
&= \mathrm{E}[u(X + Z)] - \mathrm{E}[u(W + Z)].
\end{aligned}$$

Thus, reorganizing the terms yields the result. □

(d) Interpret the previous result in terms of an expected utility maximizer's preferences and attitudes towards wealth and risk. How does she rank the compound lotteries $\mathcal{K}$ and $\mathcal{L}$? Provide intuition for this result.

[1] Recall the notation used in class: $W >_1 X$ means that random variable W first-order stochastically dominates random variable X.

Answer: Let u be the agent's Bernoulli index. The expected utility under $\mathcal{K}$ is

$$\tfrac{1}{2}\left\{\mathrm{E}[u(W+Y)] + \mathrm{E}[u(X+Z)]\right\}$$

while under $\mathcal{L}$ it is

$$\tfrac{1}{2}\left\{\mathrm{E}[u(W+Z)] + \mathrm{E}[u(X+Y)]\right\}.$$

The previous results say that if the agent enjoys wealth and is strictly risk-averse, she strictly prefers $\mathcal{L}$.

The result is pretty intuitive: Since $W >_1 X$, $Y >_1 Z$ and the agent enjoys wealth, she strictly prefers W to X and Y to Z. In lottery $\mathcal{K}$, she gets the sum of the two best distributions with probability 1/2, but if she doesn't, she gets the sum of the two worst distributions. This implies a risk that is not present in lottery $\mathcal{L}$: by giving the agent the amounts $W + Z$ or $X + Y$, this lottery is less risky simply because it combines a good and a bad distribution in each of the states. Since the agent is strictly risk-averse, she strictly prefers $\mathcal{L}$ to $\mathcal{K}$. □

(e) How do lotteries $\mathcal{K}$ and $\mathcal{L}$ rank in terms of stochastic dominance?

Answer: Since part (d) covers all strictly risk-averse expected utility maximizers who enjoy wealth, it follows from the results seen in class that $\mathcal{L} >_2 \mathcal{K}$. □

(f) Using a virtually identical argument, one can prove the following: As before, suppose that W, X, Y and Z are independent lotteries, but assume that $W >_1 X$ and $Y >_2 Z$ now.[2] If $u : \mathbb{R}_+ \to \mathbb{R} \in \mathbb{C}^3$ is strictly increasing, strictly concave, and has strictly positive third derivative everywhere, then

$$\mathrm{E}[u(W+Y)] + \mathrm{E}[u(X+Z)] < \mathrm{E}[u(W+Z)] + \mathrm{E}[u(X+Y)]. \tag{$*$}$$

Interpret this result in terms of the preferences of an expected utility maximizer and her attitudes towards wealth, risk and prudence: how does she rank the following compound lotteries:

- $\mathcal{M}$ pays $W + Y$ or $X + Z$ with equal probabilities, 1/2;
- $\mathcal{N}$ pays $W + Z$ or $X + Y$ with equal probabilities, 1/2?

Answer: To be sure, let's start by proving the result (which is not required by the question). As in part (a), we consider the function

$$v(r) = \mathrm{E}[u(X+r)] - \mathrm{E}[u(W+r)]$$

By construction,

$$v'(r) = \mathrm{E}[u'(X+r)] - \mathrm{E}[u'(W+r)] \quad \text{and} \quad v''(r) = \mathrm{E}[u''(X+r)] - \mathrm{E}[u''(W+r)]$$

Since $u'' < 0$ and $W >_1 X$, $\mathrm{E}[u'(X+r)] > \mathrm{E}[u'(W+r)]$ and $v' > 0$, as in (a). In addition, since $u''' > 0$ and $W >_1 X$, $\mathrm{E}[u''(W+r)] > \mathrm{E}[u''(X+r)]$ and $v'' < 0$.

It follows that, under the new assumptions, v is strictly increasing and strictly concave, so

$$\mathrm{E}[v(Y)] > \mathrm{E}[v(Z)],$$

[2] Recall again the notation from class: $>_2$ denotes second-order stochastic dominance.

since $Y >_2 Z$. By direct substitution, then,

$$\mathrm{E}[u(X+Y)] - \mathrm{E}[u(W+Y)] > \mathrm{E}[u(X+Z)] - \mathrm{E}[u(W+Z)].$$

Reorganizing, one gets Eq. $(*)$

Now, the interpretation is that any expected utility individual who enjoys wealth and is strictly risk-averse and strictly prudent prefers lottery $\mathcal{N}$ to lottery $\mathcal{M}$, strictly: multiply both sides of $(*)$ by 1/2.

The intuition of this result is more nuanced than part (d). Lottery W pays more generously than X, while lottery Y is less risky than Z. Since the agent enjoys wealth and is risk-averse, W and Y are better than X and Z, respectively. The question is how she would prefer to combine the higher risks and the less generous payoffs. Since the agent is prudent, she is more willing to bear high risks when she is wealthier than poorer. Thus, combining Z with W in one state and X with Y in the other, as lottery $\mathcal{N}$ does, is the more attractive of the two options. $\square$

Question **3:** Dynamic General Equilibrium

Consider an exchange economy with society $\mathcal{I} = \{1, \ldots, I\}$. There are $L+K$ commodities and trade takes place in two periods:

1. In the morning, L commodities are traded. Individual i is endowed with ω^i_ℓ units of commodity $\ell = 1, \ldots, L$, and her consumption is x^i_ℓ. The price per unit of commodity ℓ is p_ℓ.

2. In the afternoon, the other K commodities are traded. The endowment and consumption of commodity $k = 1, \ldots, K$ by individual i are ψ^i_k and y^i_k, respectively. The price per unit of commodity k is q_k.

3. In the evening, each agent consumes. If agent i has purchased the bundle $x = (x_1, \ldots, x_L)$ in the morning and the bundle $y = (y_1, \ldots, y_K)$ in the afternoon, her utility in the evening is $u^i(x) + v^i(y)$.

In the morning, besides trading the corresponding commodities, individual i chooses an amount m^i of nominal savings. If positive, m^i becomes nominal wealth in the afternoon; if negative, it is nominal debt that the agent must honor.

In the afternoon, given the prices $q = (q_1, \ldots, q_K)$ and her savings m^i, agent i solves the problem

$$\max_{y \in \mathbb{R}^K_+} \left\{ v^i(y) \, : \, q \cdot y \leq q \cdot \psi^i + m^i \right\}, \tag{1}$$

where $\psi^i = (\psi^i_1, \ldots, \psi^i_K)$. In the morning, given $p = (p_1, \ldots, p_L)$ and anticipating q, she solves

$$\max_{(x,m) \in \mathbb{R}^L_+ \times \mathbb{R}} \left\{ u^i(x) + V^i(m, q) \, : \, p \cdot x + m \leq p \cdot \omega^i \right\}, \tag{2}$$

where $\omega^i = (\omega^i_1, \ldots, \omega^i_L)$ and

$$V^i(m, q) = \max_{y \in \mathbb{R}^K_+} \left\{ v^i(y) \, : \, q \cdot y \leq q \cdot \psi^i + m \right\}.$$

The minister of finance of this economy is worried that the agents may be acting silly. She would prefer it if, instead of solving the two problems (2) and (1), agent i solves the intertemporal problem

$$\max_{(x,y,m) \in \mathbb{R}^L_+ \times \mathbb{R}^K_+ \times \mathbb{R}} \left\{ u^i(x) + v^i(y) \, : \, p \cdot x + m \leq p \cdot \omega^i \text{ and } q \cdot y \leq q \cdot \psi^i + m \right\}. \tag{3}$$

Not having taken the second-year GE course in his Ph.D., the dean of the most prominent economics department in the economy is worried that in the model he learned in the first-year course, people were assumed to solve the static problem

$$\max_{(x,y) \in \mathbb{R}^L_+ \times \mathbb{R}^K_+} \left\{ u^i(x) + v^i(y) \, : \, p \cdot x + q \cdot y \leq p \cdot \omega^i + q \cdot \psi^i \right\}, \tag{4}$$

as if all trade took place at the same time.

There are three definitions of equilibrium for this economy.

1. The definition that the dean understands is:[1] a tuple $(p, q, \mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (x^1, \ldots, x^I)$ and $\mathbf{y} = (y^1, \ldots, y^I)$, is a *static equilibrium* if

   - for each individual, the pair (x^i, y^i) solves the static problem (4);

[1] This is the definition that we learned in the course.

- $\sum_i x^i = \sum_i \omega^i$ and $\sum_i y^i = \sum_i \psi^i$.

2. The minister would wish that the following was the definition of equilibrium: a tuple $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$, where $\mathbf{m} = (m^1, \ldots, m^I)$, is an *intertemporal equilibrium* if
   - for each individual the triple (x^i, y^i, m^i) solves the intertemporal problem (3);
   - $\sum_i x^i = \sum_i \omega^i$, $\sum_i y^i = \sum_i \psi^i$, and $\sum_i m^i = 0$.
3. The actual definition of equilibrium is: a tuple $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is a *dynamic equilibrium* if
   - for each individual the pair (x^i, m^i) solves the morning problem (2), and the bundle y^i solves the afternoon problem (1);
   - $\sum_i x^i = \sum_i \omega^i$, $\sum_i y^i = \sum_i \psi^i$, and $\sum_i m^i = 0$.

The definition of efficiency, on the other hand, does not change: allocation $(\mathbf{x}, \mathbf{y})$ is efficient if there does not exist another allocation $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ such that $u^i(\tilde{x}^i) + v^i(\tilde{y}^i) \geq u^i(x^i) + v^i(y^i)$ for all i, with strict inequality for some.

The point of this question is to argue that the three definitions of equilibrium are not that different.

(a) Argue that if pair (x^i, m^i) solves problem (2) and bundle y^i solves problem (1), then triple (x^i, y^i, m^i) is feasible in problem (3).

Answer: That (x^i, m^i) solves (2) implies that $p \cdot x^i + m^i \leq p \cdot \omega^i$. That y^i solves (1) implies that $q \cdot y^i \leq q \cdot \psi^i + m^i$. These are the two constraints of (3). □

(b) Argue that if $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is a dynamic equilibrium, then it is also an intertemporal equilibrium.

Answer: Since the market clearing conditions of the two definitions of equilibrium are the same, it suffices to show that if (x^i, m^i) solves (2) and y^i solves (1), then we further have that (x^i, y^i, m^i) solves the intertemporal problem (3).

The simplest way to argue this is to observe that by replacing the definition of function V^i, we can re-write the problem (1) as

$$\max_{(x,m)\in\mathbb{R}^L_+\times\mathbb{R}} \left\{ u^i(x) + \max_{y\in\mathbb{R}^K_+} \left\{ v^i(y) \,:\, q \cdot y \leq q \cdot \psi^i + m \right\} \,:\, p \cdot x + m \leq p \cdot \omega^i \right\}.$$

This is equivalent to

$$\max_{(x,m)\in\mathbb{R}^L_+\times\mathbb{R}} \left\{ \max_{y\in\mathbb{R}^K_+} \left\{ u^i(x) + v^i(y) \,:\, q \cdot y \leq q \cdot \psi^i + m \right\} \,:\, p \cdot x + m \leq p \cdot \omega^i \right\},$$

which is equivalent to problem (3).

Alternatively, we can write a direct proof. Suppose that it is not the case that (x^i, y^i, m^i) solves the intertemporal problem (3). By part (a), we know that (x^i, y^i, m^i) is feasible in (3), so it must be that there exists an alternative triple (x, y, m) such that

$$u^i(x) + v^i(y) > u^i(x^i) + v^i(y^i), \tag{$*$}$$

while

$$p \cdot x + m \leq p \cdot \omega^i \tag{$**$}$$

and

$$q \cdot y \leq q \cdot \psi^i + m. \tag{***}$$

Eq. $(**)$ and the fact that (x^i, m^i) solves (2) together imply that

$$u^i(x^i) + V^i(m^i, q) \geq u^i(x) + V^i(m, q).$$

By definition, since y^i solves (1), we have that $V^i(m^i, q) = u^i(y^i)$. By substitution, then,

$$u^i(x^i) + v^i(y^i) \geq u^i(x) + V^i(m, q).$$

Eq. $(*)$ now implies that

$$v^i(y^i) > V^i(m, q).$$

The definition of V^i then implies that

$$q \cdot y > q \cdot \psi^i + m,$$

contradicting $(***)$. $\square$

(c) Argue that if $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is an intertemporal equilibrium, then $(p, q, \mathbf{x}, \mathbf{y})$ is a static equilibrium.

Answer: The argument is logically similar to the previous answer.

First, note that the market clearing conditions of an intertemporal equilibrium imply the market clearing conditions of static equilibrium. Thus, all we need to show is that (x^i, y^i, m^i) solving (3) implies that (x^i, y^i) solves (4).

As in part (a), it is useful to prove feasibility first. Suppose that (x^i, y^i, m^i) solves (3). This implies that

$$p \cdot x^i + m^i \leq p \cdot \omega^i \text{ and } q \cdot y^i \leq q \cdot \psi^i + m^i.$$

Adding these two inequalities yields

$$p \cdot x^i + m^i + q \cdot y^i \leq p \cdot \omega^i + q \cdot \psi^i + m^i.$$

Canceling m^i we get that (x^i, y^i) is feasible in (4).

To argue optimality, suppose otherwise. By the previous observation, it must be that there exists (x, y) such that

$$u^i(x) + v^i(y) > u^i(x^i) + v^i(y^i), \tag{*}$$

while

$$p \cdot x + q \cdot y \leq p \cdot \omega^i + q \cdot \psi^i. \tag{**}$$

Defining $m = p \cdot (\omega^i - x)$, we get that $p \cdot x + m = p \cdot \omega^i$. On the other hand, by $(**)$,

$$q \cdot \psi^i + m = p \cdot \omega^i + q \cdot \psi^i - p \cdot x \geq q \cdot y.$$

It follows that (x, y, m) is feasible in (4), so, since (x^i, y^i, m^i) solves this problem,

$$u^i(x^i) + v^i(y^i) \geq u^i(x) + v^i(y).$$

Obviously, this contradicts $(*)$. $\square$

(d) State minimal conditions under which if tuple $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is a dynamic equilibrium, then allocation $(\mathbf{x}, \mathbf{y})$ is efficient.

Answer: All one needs to assume is that $u^i(x) + v^i(y)$ is locally non-satiated for all i, for which it suffices that *either* u^i *or* v^i be locally non-satiated. To see why this implies efficiency, note that if $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is a dynamic equilibrium, then it is also an intertemporal equilibrium by part (b), and, furthermore, $(p, q, \mathbf{x}, \mathbf{y})$ is a static equilibrium by part (c). The FFTWE then guarantees that $(\mathbf{x}, \mathbf{y})$ will be efficient, so long as all functions $u^i(x) + v^i(y)$ are locally non-satiated. □

(e) Suppose that function $v^i(y)$ is locally non-satiated for all i. Argue that if $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is a dynamic equilibrium, then there do not exist a coalition $\mathcal{H} \subseteq \mathcal{I}$ and a sub-allocation of afternoon bundles $(\tilde{y}^i)_{i \in \mathcal{H}}$ such that

(i) $\sum_{i \in \mathcal{H}} m^i \geq 0$,

(ii) $\sum_{i \in \mathcal{H}} \tilde{y}^i = \sum_{i \in \mathcal{H}} \psi^i$,

(iii) $v^i(\tilde{y}^i) \geq v^i(y^i)$ for all $i \in \mathcal{H}$, and

(iv) $v^i(\tilde{y}^i) > v^i(y^i)$ for some $i \in \mathcal{H}$.

Interpret this result, in particular for the case when $\mathcal{H} = \mathcal{I}$.

Answer: Suppose that one such $\mathcal{H} \subseteq \mathcal{I}$ and $(\tilde{y}^i)_{i \in \mathcal{H}}$ exist, even though $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is a dynamic equilibrium. Since each v^i is locally non-satiated and y^i solves (1), by (iii) it must be true that

$$q \cdot \tilde{y}^i \leq q \cdot \psi^i + m^i$$

for all $i \in \mathcal{H}$. The latter inequality is strict for the $i \in \mathcal{H}$ for whom (iv) holds true, so, aggregating,

$$q \cdot \textstyle\sum_{i \in \mathcal{H}} \tilde{y}^i > q \cdot \sum_{i \in \mathcal{H}} \psi^i + \sum_{i \in \mathcal{H}} m^i.$$

By (ii), it follows that $\sum_{i \in \mathcal{H}} m^i < 0$, contradicting (i).

The interpretation of this result is similar to the interpretation of the FFTWE:[2] no coalition that is *not* in debt with the rest of the economy will object to the equilibrium allocation of the afternoon commodities. Since the grand coalition $\mathcal{I}$ can never be in debt, the latter means that the whole society cannot find a Pareto improvement by re-allocating only the afternoon commodities. If the intertemporal allocation is inefficient for whatever reason, then any policy that induces a Pareto improvement must also intervene in the morning markets. □

[2] Indeed, the proof is identical to the proof of the FFTWE for the afternoon market, with the individual afternoon endowments adjusted by the morning savings.