Econometrics Preliminary Exam

Agricultural and Resource Economics, UC Davis

July, 2020

There are **THREE** questions. Choose and answer two of the three questions. Within each question, each part will receive equal weight in grading. You have 15 minutes to read the exam and then three hours to complete the exam.

I. Probability and Statistics

(a) Consider (X, Y) with joint p.d.f. $f_{X,Y}(x, y) = \begin{cases} (x+y) & 0 < x < 1,\ 0 < y < 1. \\ 0 & \text{otherwise.} \end{cases}$

(i) Obtain $f_X(X)$, the marginal density of X.

(ii) Find the mean of X.

(iii) Obtain the conditional density $f_{Y|X}(y|x)$.

(iv) Hence find the conditional mean $E[Y|X]$.

(b) Various unrelated questions

(i) Suppose random variable X has density moment generating function $M_X(t) = (1-2t)^{-\theta}$, $t < 0.5$. Find $E[X]$.

(ii) Obtain the density of $Y = X^2$ when X has density $f(x;\theta) = \exp(-x)$, $x \geqslant 0$.

(iii) Suppose a random variable X has mean μ. Provide an example of an estimator of μ that is biased for μ but is consistent for μ.

(iv) Suppose a random variable X has mean μ. Provide an example of an estimator of μ that is unbiased for μ but is inconsistent for μ.

(c) Suppose we have a random sample $x_1, ..., x_n$ of size n from a distribution with density $f(x;\theta) = \frac{\exp(\theta - x)}{[1+\exp(\theta-x)]^2}$ where $-\infty < x < \infty$, $-\infty < \theta < \infty$ and it can be shown that $\mathrm{E}[X] = \theta$ and $\mathrm{Var}[X] = \pi^2/3$ where $\pi = 3.14159....$

(i) Obtain the first-order conditions for the MLE of θ. Note that for this example the first-order conditions cannot be solved for an explicit solution for $\widehat{\theta}$.

(ii) Using standard results for the MLE, give the limit distribution of $\sqrt{n}(\widehat{\theta} - \theta)$. Note: there is no need to try to simplify your answer to this part.

(iii) A researcher claims that the variance in the limit distribution given in part (ii) is at most $\pi^2/3$. Is this a reasonable claim? Explain.

(d) In answering the following use the knowledge that for $Z \sim N(0,1)$, $\Pr[Z > 1.645] = 0.05$ and $\Pr[Z > 1.960] = 0.025$. Suppose we have an estimator $\widehat{\theta}$ which is exactly normally distributed with mean θ and known variance 4.

(i) Obtain a 95 percent confidence interval for θ if $\widehat{\theta} = 3$.

(ii) Consider a one-sided test of $H_0 : \theta = 0$ against $H_a : \theta > 0$ at significance level 0.05. Obtain the p-value for this test if $\widehat{\theta} = 3$. Your answer will involve the standard normal cumulative distribution function $\Phi(\cdot)$.

(iii) Obtain the critical region in terms of $\widehat{\theta}$ for the test in part (ii).

(iv) Obtain the power of the test in part (ii) when $\theta = 2$. Your answer will involve the standard normal cumulative distribution function $\Phi(\cdot)$.

II. Linear Regression

Consider the model $y_i = x_i'\beta + e_i$, where $x_i = [1 \quad w_i]'$ is a 2×1 vector, $E(x_i e_i) = 0$, and $E(e_i|x_i) = \theta(1 - w_i^2)$. You have a sample of size n. The OLS estimator is $\hat{\beta} = \left(\sum_{i=1}^n x_i x_i'\right)^{-1} \left(\sum_{i=1}^n x_i y_i\right)$. Credit will be given for answers that avoid imposing unnecessarily strong assumptions.

(a) Is $\hat{\beta}$ unbiased for β? If so, prove it. If not, state additional conditions you need for unbiasedness and prove unbiasedness under those conditions.

(b) Is $\hat{\beta}$ consistent for β? If so, prove it. If not, state additional conditions you need for consistency and prove consistency under those conditions.

(c) Following on from (b), find the asymptotic distribution of $\sqrt{n}(\hat{\beta} - \beta)$ as $n \to \infty$. State any additional assumptions you need.

(d) Propose a test statistic of the null hypothesis $H_0 : \beta = \beta_0$. Derive the asymptotic null distribution of your test statistic. State any additional assumptions you need.

(e) Define a 2×1 vector $z_i = [1 \quad w_i^2]'$. Consider the estimator $\tilde{\beta} = \left(\sum_{i=1}^n z_i x_i'\right)^{-1} \left(\sum_{i=1}^n z_i y_i\right)$. Is $\tilde{\beta}$ consistent for β? If so, prove it. If not, state additional conditions you need for consistency and prove consistency under those conditions.

(f) Following on from part (e), propose a test statistic of the null hypothesis $H_0 : plim(\tilde{\beta}) = plim(\hat{\beta})$. State the asymptotic null distribution of the test statistic you propose and any assumptions required for this asymptotic distribution to be valid.

III. **Nonlinear Estimation and Panel Data Methods**

(a) Suppose that for $i = 1, \ldots, n$, $y_i^* = x_i'\beta_0 + u_i$. The researcher can *only* observe a binary version of y_i^*, specifically $y_i = 1\{y_i^* \geqslant 0\}$, and x_i, where $1\{A\}$ equals 1 when the event A holds and zero otherwise. Assume $dim(\beta_0) = k$ and maintain the i.i.d. assumption across i.

Suppose that $u_i | x_i \overset{i.i.d.}{\sim} N(0, 1)$.

Note: In your answer, you can use the following notation for the standard normal cdf and pdf, $\Phi(z)$ and $\phi(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, respectively.

(i) Propose a maximum likelihood estimator of β_0, $\hat{\beta}$. Provide sufficient conditions for its consistency.

(ii) Derive an expression for $\sqrt{n}(\hat{\beta} - \beta_0)$ and provide sufficient conditions for its asymptotic normality as $n \to \infty$. Discuss concisely how the conditions imply the result. Make sure to state the asymptotic distribution.

(iii) Propose two different estimators of the asymptotic variance you provide in (ii). Briefly discuss the conditions required for their consistency. (A discussion of high-level conditions is sufficient, no need to provide primitive conditions.)

(iv) Propose the Wald, score and likelihood ratio statistics to test $H_0 : \beta_0 = c$, where c is non-random. For each test statistic, state its asymptotic distribution under the null hypothesis and make sure to define all quantities that your statistic consists of.

(b) Let $y_{it} = x_{it}'\beta_0 + \alpha_i + \epsilon_{it}$, where x_{it} is a vector of time-varying variables, $dim(x_{it}) = k$. Let $i = 1, \ldots, n$, $t = 1, \ldots, T$, all asymptotic arguments pertain to $n \to \infty$ while holding T fixed. You can maintain the i.i.d. assumption across i, but make no additional assumptions along the time-series dimension. Let $X_i \equiv (x_{i1}, \ldots, x_{iT})$.

Note: For each of the following questions, make sure to solve for the closed-form solution of each estimator if it exists.

(i) Propose a consistent and asymptotically efficient estimator of β_0 assuming $E[\alpha_i | X_i] = 0$, $E[u_{it} | X_i, \alpha_i] = 0$, $E[\alpha_i^2 | X_i] = \sigma_\alpha^2$, $E[\epsilon_{it}^2 | X_i, \alpha_i] = \sigma_\epsilon^2$, $E[\epsilon_{it}\epsilon_{i,t-1} | X_i, \alpha_i] = \rho$, $E[\epsilon_{it}\epsilon_{i,t-\tau} | X_i, \alpha_i] = 0$ for $|\tau| > 1$. Make sure to define all objects that the estimator consists of clearly.

Note: No need to provide conditions for its consistency or asymptotic efficiency.

(ii) Now propose a method-of-moments estimator that is based on moments implied by the strict exogeneity of x_{it} ($E[u_{it} | X_i, \alpha_i] = 0$) only. Provide sufficient conditions for its consistency and briefly describe why they are sufficient.

Note: The set of moment conditions you propose can over- or just-identify the parameter.

(iii) Now suppose that x_{it} is only sequentially (not strictly) exogeneous, propose a method-of-moments estimator that would be consistent under this weaker exogeneity condition. Provide sufficient conditions for its consistency and briefly describe why they are sufficient.

Note: The set of moment conditions you propose can over- or just-identify the parameter.