Micro Prelim June 2018 – ANSWER KEYS

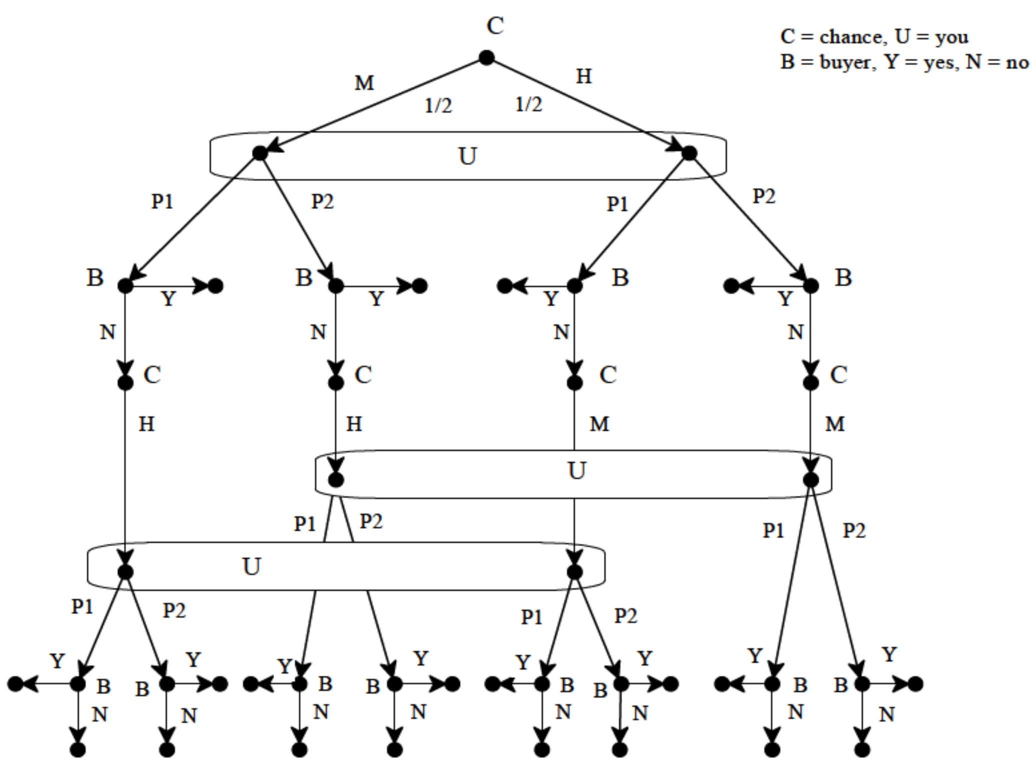
QUESTION 1

(a) **(a.1)** For every price $P \notin \{L, M, R\}$ there is a price in $\{L, M, R\}$ that strictly dominates P (for example, offering to sell at a price P such that $L < P < M$ is strictly dominated by offering to sell at price M (with $\frac{2}{3}$ probability P would be accepted, but with the same probability the higher price M would also be accepted). Thus we only need to consider prices in $\{L, M, R\}$. Offering to sell at price H gives rise to the wealth lottery $\begin{pmatrix} W_0 + H & W_0 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, whose expected utility is $U_H = \frac{1}{3}\sqrt{W_0 + H} + \frac{2}{3}\sqrt{W_0}$; offering to sell at price M gives rise to the wealth lottery $\begin{pmatrix} W_0 + M & W_0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$, whose expected utility is $U_M = \frac{2}{3}\sqrt{W_0 + M} + \frac{1}{3}\sqrt{W_0}$; offering to sell at price L guarantees a wealth of $U_L = W_0 + L$ with corresponding utility $\sqrt{W_0 + L}$. Thus

- You will offer to sell for \$*H* if $U_H > \max\{U_M, U_L\}$
- You will offer to sell for \$*M* if $U_M > \max\{U_H, U_L\}$
- You will offer to sell for \$*L* if $U_L > \max\{U_H, U_M\}$.

(a.2) When $W_0 = 16$, $H = 33$, $M = 20$, $L = 9$, $\frac{1}{3}\sqrt{W_0 + H} + \frac{2}{3}\sqrt{W_0} = 5$, $\frac{2}{3}\sqrt{W_0 + M} + \frac{1}{3}\sqrt{W_0} = 5.33$ and $\sqrt{W_0 + L} = 5$. Thus you will offer to sell for \$20.

(b) The game is as follows:



(c) By sequential rationality, at any weak sequential equilibrium, each buyer's strategy is as follows: if offered a price $P > r$ say No and if offered a price $P \leq r$ say Yes (recall that if a buyer says No then she has to leave and ceases to be a player and thus can no longer purchase the painting). Your beliefs at the first information set must assign probability $\frac{1}{3}$ to each of L, M and R.

- Offering to sell at price L leads to a payoff of $\boxed{L}$.
- **What if, initially, you offer to sell at price *M*?** Given the buyers' strategies, only type L will say No (types H and M would say Yes and give you a payoff of M); thus if the first buyer says No then you find yourself at an information set containing the following nodes:
  - two nodes which have zero prior probability (following a mistaken No by an H-type after which Chance sent you an M-type with probability 1/2 or an L-type with probability ½)
  - two nodes which have zero prior probability (following a mistaken No by an M-type after which Chance sent you an H-type with probability 1/2 or an L-type with probability ½)
  - two nodes which have positive prior probability: one with prior probability 1/6 where the first buyer was an L-type and the second is an H-type and the other, also with prior probability 1/6, where the first buyer was an L-type and the second is an M-type.

  Hence in period 2 Bayesian updating requires assigning probability $\frac{1}{2}$ to facing an H type and probability $\frac{1}{2}$ to facing an M-type. Thus if, in period 2, you offer to sell at price M your offer will be accepted and your payoff will be δM; if you offer to sell at price H then only type M will say No (type H will say Yes and your payoff will be δH) and at the next stage you will know that you are facing type H (the only one left) and thus you can charge H, obtaining a payoff of $\delta^2 H$. Thus at the information set under consideration you will offer M if $\delta M > \frac{1}{2}\delta H + \frac{1}{2}\delta^2 H$ (that is, if $M > \frac{H+\delta H}{2}$) and you will offer H otherwise.
  Thus offering to sell at price M at the very beginning leads to the following expected payoff: $\boxed{\begin{cases} \frac{2}{3}M + \frac{1}{3}\delta M & \text{if } \delta M > \frac{1}{2}\delta H + \frac{1}{2}\delta^2 H \\ \frac{2}{3}M + \frac{1}{3}\left(\frac{1}{2}\delta H + \frac{1}{2}\delta^2 H\right) & \text{if } \delta M < \frac{1}{2}\delta H + \frac{1}{2}\delta^2 H \end{cases}}$
- **What if, initially, you offer to sell at price *H*?** Given the buyers' strategies, type M and L will say No (type H would say Yes, giving you a payoff of H); hence if the first buyer says No then you find yourself at an information set containing the following nodes:
  - two nodes which have zero prior probability (following a mistaken No by an H-type after which Chance sent you an M-type with probability 1/2 or an L-type with probability ½)
  - two nodes, call them x_1 (where the first person was of type M and the current person is of type H; prior probability 1/6) and x_2 (where the first person was of type M and the current person is of type L; prior probability 1/6), and two nodes, call them x_3 (where the first person was of type L and the current person is of type H; prior probability 1/6) and x_4 (where the first person was of type L and the current person is of type M; prior probability 1/6). Thus Bayesian updating requires you to assign probability 1/2 to the current person being type H, 1/4 to the current person being M and 1/4 to the current person being L.

  Hence in period 2

  - if you offer L, the offer will be accepted and your payoff will be $\boxed{\delta L}$,

- if you offer M, then your offer will be accepted with probability ¾ (giving you a payoff of δM) and rejected with probability ¼, in which case in period 3 Bayesian updating requires you to assign probability 1 to type H and thus you will offer H, with a payoff of $\delta^2 H$. Hence offering M gives you an expected payoff of $\boxed{\tfrac{3}{4}\delta M + \tfrac{1}{4}\delta^2 H}$,

- if you offer H, then the offer will be accepted with probability ½ (with payoff δH) and rejected with probability ½ in which case in period 3 Bayesian updating requires you to assign probability 1 to type H and thus you will offer H, with a payoff of $\delta^2 H$. Hence offering H gives you an expected payoff of $\boxed{\tfrac{1}{2}\delta H + \tfrac{1}{2}\delta^2 H}$.

Thus at the information set under consideration your will offer

$$\begin{cases} L & \text{if } L > Max\left\{\tfrac{3}{4}M + \tfrac{1}{4}\delta H,\ \tfrac{1}{2}H + \tfrac{1}{2}\delta H\right\} \\ M & \text{if } \tfrac{3}{4}M + \tfrac{1}{4}\delta H > Max\left\{L,\ \tfrac{1}{2}H + \tfrac{1}{2}\delta H\right\} \\ H & \text{if } \tfrac{1}{2}H + \tfrac{1}{2}\delta H > Max\left\{L,\ \tfrac{3}{4}M + \tfrac{1}{4}\delta H\right\} \end{cases}$$

Thus offering to sell at price H at the very beginning leads to the following expected

payoff: $$\boxed{\boxed{\begin{cases} \tfrac{1}{3}H + \tfrac{2}{3}\delta L & \text{if } L > Max\left\{\tfrac{3}{4}M + \tfrac{1}{4}\delta H,\ \tfrac{1}{2}H + \tfrac{1}{2}\delta H\right\} \\ \tfrac{1}{3}H + \tfrac{1}{2}\delta M + \tfrac{1}{6}\delta^2 H & \text{if } \tfrac{3}{4}M + \tfrac{1}{4}\delta H > Max\left\{L,\ \tfrac{1}{2}H + \tfrac{1}{2}\delta H\right\} \\ \tfrac{1}{3}H + \tfrac{1}{3}\delta H + \tfrac{1}{3}\delta^2 H & \text{if } \tfrac{1}{2}H + \tfrac{1}{2}\delta H > Max\left\{L,\ \tfrac{3}{4}M + \tfrac{1}{4}\delta H\right\} \end{cases}}}$$

Hence the optimal choice at the first information set is the one that is associated with the largest of the above three double-boxed payoffs.

(d) When H =140, M = 90, L = 70 and δ= 0.75, $\delta M = 67.5 < \tfrac{1}{2}\delta H + \tfrac{1}{2}\delta^2 H = 91.875$ so that the expected payoff from offering M at the beginning is $\tfrac{2}{3}M + \tfrac{1}{3}\left(\tfrac{1}{2}\delta H + \tfrac{1}{2}\delta^2 H\right) = 90.625$ and $Max\left\{L = 70,\ \tfrac{3}{4}M + \tfrac{1}{4}\delta H = 93.75,\ \tfrac{1}{2}H + \tfrac{1}{2}\delta H = 122.5\right\} = \tfrac{1}{2}H + \tfrac{1}{2}\delta H$ so that the expected payoff from offering H at the beginning is $\tfrac{1}{3}H + \tfrac{1}{3}\delta H + \tfrac{1}{3}\delta^2 H = 107.92$. Hence at the sequential equilibrium of Part (c) you will start by requesting H and, if rejected, you will request H again and, if rejected a second time, you will request H once more.

General Equilibrium Theory
Sketched answers to prelim exam, June 2018
Question 2

Consider a finite society $\mathcal{I} \quad \{1, \ldots, I\}$, where the preferences of the individuals are represented by the utility functions $(u^i : \mathbb{R}^L_+ \to \mathbb{R})_{i \in \mathcal{I}}$. Assume that all of these functions are continuous, strictly quasiconcave and strictly monotone. Unlike in class, we are going to treat individual endowments as variables. For individual i, her endowment is denoted by $w^i \in \mathbb{R}^L_+$.

Define the functions $x^i : \mathbb{R}^L_{++} \times \mathbb{R}^L_+ \to \mathbb{R}^L_+$ by

$$x^i(p, w^i) \quad \operatorname{argmax}_{x \in \mathbb{R}^L_+} \left\{ u^i(x) \mid p \cdot x \leqslant p \cdot w^i \right\}, \tag{1}$$

for each i. Define also the function $Z : \mathbb{R}^L_{++} \times \mathbb{R}^{LI}_+ \to \mathbb{R}^L$ by

$$Z(p, w) \quad \sum_i [x^i(p, w^i) - w^i] \tag{2}$$

and the set

$$\mathcal{M} \quad \left\{ (p, w) \in \mathbb{R}^L_{++} \times \mathbb{R}^{LI}_+ \mid Z(p, w) \quad 0 \right\}. \tag{3}$$

Here, x^i is individual i's demand function and Z is the aggregate excess demand function. Obviously, p is a vector of competitive equilibrium prices of exchange economy $\{\mathcal{I}, (u^i, w^i)_{i \in \mathcal{I}}\}$ if, and only if, $(p, w) \in \mathcal{M}$. Set $\mathcal{M}$ is hence called the *equilibrium set*, or *equilibrium manifold*, of preference profile $(u^i)_{i \in \mathcal{I}}$.

In this exercise you are going to prove that an observer loses no information when working with the equilibrium set, in comparison to the profile of individual demands. In technical language, you will argue that the equilibrium manifold *identifies* both the excess demand function and all the individual demand functions: given set $\mathcal{M}$, there is one and only one function Z that generates $\mathcal{M}$ by Eq. (3); and there is one and only one profile $(x^i)_{i \in \mathcal{I}}$ that generates that Z by Eq. (2).[1]

1. Suppose that p, w and $\hat{w}$ are such that: (a) (p, w) and $(p, \hat{w})$ are both in $\mathcal{M}$; and (b) for all i, $p \cdot w^i \quad p \cdot \hat{w}^i$. Argue that $\sum_i w^i \quad \sum_i \hat{w}^i$.

   Answer: By (b) $x^i(p, w^i) \quad x^i(p, \hat{w}^i)$ for all i, given Eq. (1). This obviously implies that

   $$\sum_i x^i(p, w^i) \quad \sum_i x^i(p, \hat{w}^i).$$

   Using Eq. (2), (a) now implies that

   $$\sum_i w^i \quad \sum_i x^i(p, w^i) \quad \sum_i x^i(p, \hat{w}^i) \quad \sum_i \hat{w}^i,$$

   where the first and third equalities come from Eq. (3)

[1] One can go further and show that, in fact, the set identifies all the individual preferences, in the sense that there is one and only one profile $(u^i)_{i \in \mathcal{I}}$ that generates $(x^i)_{i \in \mathcal{I}}$ by Eq. (1). Asking this further step in a prelim exam would probably qualify as a human rights violation, though.

2. Fix p and w, and suppose that $\hat{w}$ is such that: (a) $(p, \hat{w})$ is in $\mathcal{M}$; and (b) for all i, $p \cdot \hat{w}^i \quad p \cdot w^i$. Argue that $Z(p, w) \quad \sum_i (\hat{w}^i - w^i)$.

   Answer: By (b) $x^i(p, \hat{w}^i) \quad x^i(p, w^i)$, again by Eq. (1). Then,

   $$\sum_i x^i(p, \hat{w}^i) \quad \sum_i x^i(p, w^i),$$

   as before. Now, by definition (Eq. 2),

   $$Z(p, w) \quad \sum_i x^i(p, w^i) - \sum_i w^i \quad \sum_i x^i(p, \hat{w}^i) - \sum_i w^i \quad \sum_i \hat{w}^i - \sum_i w^i,$$

   where the last equality comes from (b).

3. Argue that for any p and any w, there exists $\hat{w} \in \mathbb{R}_+^{LI}$ such that (a) $(p, \hat{w})$ is in $\mathcal{M}$; and (b) for all i, $p \cdot \hat{w}^i \quad p \cdot w^i$.

   Answer: Let $\hat{w}^i \quad x^i(p, w^i)$. By construction, pair $(p, \hat{w})$ is a competitive equilibrium for economy $\{\mathcal{I}, (u^i, \hat{w}^i)_{i \in \mathcal{I}}\}$. So $(p, \tilde{w}) \in \mathcal{M}$, by definition of $\mathcal{M}$. By local non-satiation of preferences, $p \cdot \bar{x}^i \quad p \cdot w^i$.

4. Use the previous steps to explain how an analyst who *only* observes $\mathcal{M}$ can construct function Z in a unique manner.

   Answer: Well, what else is there to say! Question 3 guarantees the existence of the point that the analyst uses in Question 2 to find a value for $Z(p, w)$. Question 1 then says that if there are multiple such points, they all give the same value for $Z(p, w)$. Thus, the analyst who observes $\mathcal{M}$ can uniquely recover Z. (In practice, she may need to use some search method to find the point of Question 2.)

5. Argue that there exists a sub-profile of individual endowments for all agents other than $i \quad 1$, say $(\hat{w}^2, \ldots, \hat{w}^I)$, such that for all p and all w^1,

   $$x^1(p, w^1) \quad Z(p, w^1, \hat{w}^2, \ldots, \hat{w}^I) + w^1.$$

   Answer: For all $i \geqslant 2$, let $\hat{w}^i \quad 0$. As long as $p \in \mathbb{R}_{++}^L$, we have that

   $$Z(p, w^1, \hat{w}^2, \ldots, \hat{w}^I) \quad x^1(p, w^1) + \sum_{i \geqslant 2} x^i(p, \hat{w}^i) - w^1 - \sum_{i \geqslant 2} \hat{w}^i \quad x^1(p, w^1) - w^1,$$

   since $x^i(p, \hat{w}^i) \quad 0$ for all individuals except $i \quad 1$.

6. Explain how, once the analyst of part 4 has constructed function Z, she can construct function x^1 in a unique manner.

   Answer: Question 5 immediately yields $x^1(p, w^1) \quad Z(p, w^1, 0, \ldots, 0) + w^1$.

7. Conclude.

   Answer: Upon observation of $\mathcal{M}$, Question 4 shows that there is only one Z that can generate it by Eq. (3). Once Z is identified, there is only one profile $(x^i)_{i \in \mathcal{I}}$ that generates it by Eq. (2), by applying the idea of Question 6 to each i.

Question 3

Emma is a first-year Ph.D. student of economics. She is very lucky because she got a place to stay in the Aggie Village, the nicest part of Davis. She lives in a beautiful cottage with a little garden. Her modest wealth from being a TA is w. She spends it on coffee and gardening. Let x_1 denote the amount of coffee and x_2 the amount of gardening and let p_1 and p_2 denote the corresponding unit prices. Her budget constraint is given by

$$p_1 x_1 + p_2 x_2 \leq w. \tag{1}$$

Coffee is really a private good in the sense that she is the sole beneficiary of caffeine in her coffee (unless she calls up in panic her fellow student in the middle of the night because she cannot solve her ECN200A homework problem). In contrast, gardening creates a positive externality on others. But so does the gardening of others create a positive externality on her. There is plenty of gardening in the Aggie Village. Denote by e the total externality or public good created from gardening in the community. Her utility function $u(x_1, x_2, e)$ is concave and continuously differentiable with a strictly positive gradient on the interior of its domain. The total externality depends in part on Emma's gardening x_2 and on the externality created by the gardening of others, denoted by e_{-i}. It is assumed to satisfy

$$e \leq e_{-i} + \alpha x_2 \tag{2}$$

for some parameter α satisfying $\alpha > 0$. Emma does not think that she can affect the level of externalities provided by others. For instance, Professor Schipper, who also lives in the Aggie Village, is so busy writing prelim exam questions that talking to him about keeping up his gardening is no use. Thus, we can safely assume that Emma takes e_{-i} as well as p_1, p_2, and w as given.

Since Emma diligently studies microeconomic theory for the prelims, she is eager to maximize her utility function subject to constraints (1) and (2). This yields demand functions $x_1(p_1, p_2, w, e_{-i})$ and $x_2(p_1, p_2, w, e_{-i})$ as well as her optimal desired amount of public good $e(p_1, p_2, w, e_{-i})$.

a.) Write down her Kuhn-Tucker-Lagrangian (ignore non-negativity constraints).

$$L(x_1, x_2, e, \lambda_1, \lambda_2) = u(x_1, x_2, e) - \lambda_1(p_1 x_1 + p_2 x_2 - w) - \lambda_2(e - e_{-i} - \alpha x_2)$$

b.) Derive the Kuhn-Tucker first-order conditions (ignore non-negativity constraints).

$$\begin{aligned}
\frac{\partial u(x_1, x_2, e)}{\partial x_1} - \lambda_1 p_1 &= 0 \\
\frac{\partial u(x_1, x_2, e)}{\partial x_2} - \lambda_1 p_2 + \lambda_2 \alpha &= 0 \\
\frac{\partial u(x_1, x_2, e)}{\partial e} - \lambda_2 &= 0 \\
\lambda_1 (p_1 x_1 + p_2 x_2 - w) &= 0 \\
\lambda_2 (e - e_{-i} - \alpha x_2) &= 0
\end{aligned}$$

c.) Use the Kuhn-Tucker conditions and the assumptions that the solution is interior, that it is unique, and that constraints (1) and (2) are satisfied with equality to derive a system of three equations and three unknowns that does not involve multipliers and whose solution defines $x_1(p_1, p_2, w, e_{-i})$, $x_2(p_1, p_2, w, e_{-i})$ and $e(p_1, p_2, w, e_{-i})$. (No need to solve it.)

`Solve the first and third conditions for multipliers` λ_1 `and` λ_2`, respectively. Plug them into the second condition. Assume constraints are satisfied with equality. Then we obtain the following system of three equations with three unknowns:`

$$\begin{aligned}
\frac{\partial u(x_1, x_2, e)}{\partial x_2} - \frac{\partial u(x_1, x_2, e)}{\partial x_1} \frac{p_2}{p_1} + \frac{\partial u(x_1, x_2, e)}{\partial e} \alpha &= 0 \\
p_1 x_1 + p_2 x_2 - w &= 0 \\
e - e_{-i} - \alpha x_2 &= 0
\end{aligned}$$

`Since the objective function is concave and constraints are linear, the Kuhn-Tucker conditions are also sufficient.`

d.) If you think that Professor Schipper acts in a strange way sometimes, this is not just due to being an economic theorist. The secret is he comes from another solar system. One feature of these aliens is that they can read immediately the utility function of others. (Although this sounds quite useful, it is rather a curse.)[1] Anyway, as a proof of this claim we print here Emma's utility function:

$$u(x_1, x_2, e) = x_1 + (a_2, a_e) \begin{pmatrix} x_2 \\ e \end{pmatrix} - \frac{1}{2}(x_2, e) B \begin{pmatrix} x_2 \\ e \end{pmatrix}, \tag{3}$$

where $a_2, a_e > 0$ and $B = \begin{pmatrix} b_{22} & b_{2e} \\ b_{e2} & b_{ee} \end{pmatrix}$ is symmetric positive definite. I know, you surely must think "Wow" but let's focus again on the prelim exam. Assume that solutions are interior and that constraints are satisfied with equality. Write out the system of equations from problem c.) for Emma's utility function.

[1] Any similarity with the novel "The Three-Body Problem" by Chinese writer Liu Cixin is just coincidental.

First observe

$$\frac{\partial u(x_1, x_2, e)}{\partial x_1} = 1.$$

Further

$$\begin{pmatrix} \frac{\partial u(x_1,x_2,e)}{\partial x_2} \\ \frac{\partial u(x_1,x_2,e)}{\partial e} \end{pmatrix} = \begin{pmatrix} a_2 \\ a_e \end{pmatrix} - \begin{pmatrix} b_{22} & b_{2e} \\ b_{e2} & b_{ee} \end{pmatrix} \begin{pmatrix} x_2 \\ e \end{pmatrix}$$

Thus, the system of equations becomes

$$\begin{aligned} (a_2 - b_{22}x_2 - b_{2e}e) - \frac{p_2}{p_1} + (a_e - b_{e2}x_2 - b_{ee}e)\alpha &= 0 \\ p_1x_1 + p_2x_2 - w &= 0 \\ e - e_{-i} - \alpha x_2 &= 0 \end{aligned}$$

e.) Provide an interpretation of the partial derivatives $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ and $\frac{\partial e(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ and their signs.

$\frac{\partial x_2(p_1,p_2,w,\lambda)}{\partial e_{-i}}$: This is Emma's rate of change of demand in good 2 when the externality by others increases.
$\frac{\partial e(p_1,p_2,w,\lambda)}{\partial e_{-i}}$: This is Emma's rate of change of desired total externality when the externality by others increases.

f.) Compute $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ and $\frac{\partial e(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$.

We first compute the solutions and use them to compute their derivatives. (Alternatively, we could have used implicit differentiation.) Consider the system of three equations. Since the utility function is linear in x_1, only the equation for the budget line involves x_1. The remaining two equations can be solved for x_2 and e. Write

$$\begin{aligned} (a_2 - b_{22}x_2 - b_{2e}e) - \frac{p_2}{p_1} + (a_e - b_{e2}x_2 - b_{ee}e)\alpha &= 0 \\ e - e_{-i} - \alpha x_2 &= 0 \end{aligned}$$

in matrix-form

$$\begin{pmatrix} b_{22} + b_{e2}\alpha & b_{2e} + b_{ee}\alpha \\ \alpha & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ e \end{pmatrix} = \begin{pmatrix} a_2 - \frac{p_2}{p_1} + a_e\alpha \\ -e_{-i} \end{pmatrix}.$$

Solve for x_2 and e,

$$\begin{aligned} \begin{pmatrix} x_2 \\ e \end{pmatrix} &= \begin{pmatrix} b_{22} + b_{e2}\alpha & b_{2e} + b_{ee}\alpha \\ \alpha & -1 \end{pmatrix}^{-1} \begin{pmatrix} a_2 - \frac{p_2}{p_1} + a_e\alpha \\ -e_{-i} \end{pmatrix} \\ &= \frac{1}{-\Delta} \begin{pmatrix} -1 & -b_{2e} - b_{ee}\alpha \\ -\alpha & b_{22} + b_{e2}\alpha \end{pmatrix} \begin{pmatrix} a_2 - \frac{p_2}{p_1} + a_e\alpha \\ -e_{-i} \end{pmatrix} \end{aligned}$$

with $-\Delta = -(b_{22} + b_{e2}\alpha) - (b_{2e} + b_{ee}\alpha)\alpha = -b_{22} - b_{e2}\alpha - b_{2e}\alpha - b_{ee}\alpha^2$. Taking partial derivatives yields

$$\begin{pmatrix} \frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}} \\ \frac{\partial e(p_1,p_2,w,e_{-i})}{\partial e_{-i}} \end{pmatrix} = \begin{pmatrix} \frac{-b_{2e} - b_{ee}\alpha}{\Delta} \\ \frac{b_{22} + b_{e2}\alpha}{\Delta} \end{pmatrix}.$$

g.) Assume $b_{2e} \geq 0$. Derive the signs of $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ and $\frac{\partial e(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$.

Since B is positive definite, $b_{22}, b_{ee} > 0$. Note that we can rewrite the determinant

$$\Delta = b_{22} + b_{e2}\alpha + b_{2e}\alpha + b_{ee}\alpha^2 = (1, \alpha) \begin{pmatrix} b_{22} & b_{2e} \\ b_{e2} & b_{ee} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix}.$$

Thus, positive definitness of B also implies $\Delta > 0$. Together with $b_{2e} \geq 0$ and symmetry of B we have

$$\begin{aligned} \frac{\partial x_2(p_1, p_2, w, e_{-i})}{\partial e_{-i}} &= \frac{-b_{2e} - b_{ee}\alpha}{\Delta} < 0 \\ \frac{\partial e(p_1, p_2, w, e_{-i})}{\partial e_{-i}} &= \frac{b_{22} + b_{e2}\alpha}{\Delta} > 0 \end{aligned}$$

h.) Assume now $b_{2e} < 0$. Show that without additional assumptions, the signs of $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ and $\frac{\partial e(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ remain ambiguous in this case. Find additional assumptions on matrix B and α that allow you to determine the signs of $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ and $\frac{\partial e(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$.

If $b_{2e} < 0$ then the two terms in the right-hand side of $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}} = \frac{-b_{2e}-b_{ee}\alpha}{\Delta}$ have opposite signs and hence the sign of the partial derivative is ambiguious. The same hold for the other partial derivative. As for additional assumptions, we have for instance,

$$\frac{\partial x_2(p_1, p_2, w, e_{-i})}{\partial e_{-i}} < 0 \text{ iff } -b_{2e} - b_{ee}\alpha < 0.$$

Not sure how interesting this is though.

i.) We want to get a better understanding of whether good 2 and the externality are complements or substitutes. In our particular context, do we need to distinguish between *gross* complements/substitutes and (net) complements/substitutes?

No, since Emma has quasilinear preference that are linear in x_1, the demand for good x_2 and the externality do not depend on wealth.

j.) To figure out whether good 2 and the externality are complements or substitutes, we can use the Slutsky substitution matrix. To this end, compute first Walrasian demand functions as if the externality has a market price p_e. That is, compute Walrasian demand functions $x_2(p_1, p_2, p_e, w)$ and $e(p_1, p_2, p_e, w)$.

We maximize u subject to the budget constraint $p_1x_2 + p_2x_2 + p_e e \leq w$. The Lagrangian is

$$L(x_1, x_2, e, \lambda) = x_1 + (a_2, a_e) \begin{pmatrix} x_2 \\ e \end{pmatrix} - \frac{1}{2}(x_2, e)B \begin{pmatrix} x_2 \\ e \end{pmatrix} - \lambda(p_1x_1 + p_2x_2 + p_e e - w).$$

First-order conditions are

$$\begin{aligned} 1-\lambda p_1 &= 0 \\ \begin{pmatrix} a_2 \\ a_e \end{pmatrix} - B\begin{pmatrix} x_2 \\ e \end{pmatrix} - \lambda\begin{pmatrix} p_2 \\ p_e \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \lambda(p_1x_1+p_2x_2+p_ee-w) &= 0 \end{aligned}$$

Solve first condition for $\lambda = \frac{1}{p_1}$. Plug into second condition to solve for (x_2, e) by

$$\begin{pmatrix} a_2 \\ a_e \end{pmatrix} - B\begin{pmatrix} x_2 \\ e \end{pmatrix} = \begin{pmatrix} \frac{p_2}{p_1} \\ \frac{p_e}{p_1} \end{pmatrix}$$

which is equivalent to

$$B\begin{pmatrix} x_2 \\ e \end{pmatrix} = \begin{pmatrix} a_2 - \frac{p_2}{p_1} \\ a_e - \frac{p_e}{p_1} \end{pmatrix}$$

Multiply both sides by the inverse B^{-1}, we get

$$\begin{pmatrix} x_2 \\ e \end{pmatrix} = \frac{1}{\bar{\Delta}}\begin{pmatrix} b_{ee} & -b_{2e} \\ -b_{e2} & b_{22} \end{pmatrix}\begin{pmatrix} a_2 - \frac{p_2}{p_1} \\ a_e - \frac{p_e}{p_1} \end{pmatrix}$$

By quasilinearity, the Walrasian demands of neither good 2 nor the externality depend on wealth. Therefore, the required submatrix of Slutsky matrix is

$$\begin{pmatrix} s_{22} & s_{2e} \\ s_{e2} & s_{ee} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_2}{\partial p_2} & \frac{\partial x_2}{\partial p_e} \\ \frac{\partial e}{\partial p_2} & \frac{\partial e}{\partial p_e} \end{pmatrix} = \frac{1}{\bar{\Delta}}\begin{pmatrix} -\frac{b_{ee}}{p_1} & \frac{b_{2e}}{p_1} \\ \frac{b_{e2}}{p_1} & -\frac{b_{22}}{p_1} \end{pmatrix} = \frac{1}{p_1\bar{\Delta}}\begin{pmatrix} -b_{ee} & b_{2e} \\ b_{e2} & -b_{22} \end{pmatrix}.$$

Recall $\bar{\Delta} = b_{22}b_{ee} - b_{2e}b_{e2}$. Note that since B is positive definite, we have $\bar{\Delta} > 0$.

k.) How does the fact that good 2 and the externality are substitutes or complements depend on the sign of b_{2e}?

Good 2 and the externality are complements if $s_{2e} < 0$. From the answer to problem j.) it follows that this is the case when $b_{2e} < 0$. Good 2 and the externality are substitutes if $s_{2e} > 0$. From the answer to problem j.) it follows that this is the case when $b_{2e} > 0$.

ℓ.) How is the fact that good 2 and the externality are substitutes or complements related to the sign of the derivatives $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ and $\frac{\partial e(p_1,p_2,w,e_{-i})}{\partial e_{-i}}$ discussed earlier?

Using formulas for the partial derivatives from the answers to f.) and g.), we realize that if good 2 and the externality are substitutes then $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}} < 0$ and $\frac{\partial e(p_1,p_2,w,e_{-i})}{\partial e_{-i}} > 0$. When good 2 and the externalities are complements, then the signs are ambiguous. If the complementarity is sufficiently strong, i.e., negative b_{2e} with large absolute value, then we have $\frac{\partial x_2(p_1,p_2,w,e_{-i})}{\partial e_{-i}} > 0$.

Question 4

In ECN200A we assumed that consumers have preferences over consumption bundles. In many contexts though it is more natural to think that consumers have preferences over characteristics of consumption bundles like "lots of vitamins", "gluten free", "lots of horse power", "no tail pipe emissions", "many mega pixels" etc. In the following I outline an alternative model of consumer theory, in which preferences are defined over characteristics rather than consumption bundles. You are right, we have never discussed it in class. But there is no need to freak out since we know all the tools that are required to think about such a model.

Goods are indexed by $\ell = 1, ..., L$ and characteristics are indexed by $i = 1, ..., I$. We denote by $a_{i,\ell} > 0$ the quantity of characteristics i possessed by one unit of good ℓ. x_ℓ denotes the quantity of good ℓ. z_i is the quantity of characteristic i. We let $z = (z_1, ..., z_I)$ and assume that $z \in Z \subseteq \mathbb{R}^I_+$. Moreover, as usual we let $x = (x_1, ..., x_L) \in X \subseteq \mathbb{R}^L_+$.

We assume $z_i = \sum_{\ell=1}^L a_{i,\ell} x_\ell$ for $i = 1, ..., I$. This is the amount of characteristic z_i derived from a bundle of goods $x = (x_1, ..., x_L)$. We arrange $A = (a_{i,\ell})_{i=1,...,I,\ell=1,...,L}$ into a matrix, in which rows refer to characteristics and columns to goods.

a.) Consider a binary relation $\succsim$ on the space of characteristics, Z. State conditions on $\succsim$ that are sufficient for the existence of a utility function over characteristics $u : Z \longrightarrow \mathbb{R}$ that represents $\succsim$.

`Same as for a utility function in goods space: completeness, transitivity, and continuity.`

b.) Given prices of goods $p = (p_1, ..., p_L) >> 0$ and wealth $w \geq 0$, define the budget set on the characteristics space by

$$K_{p,w,A} := \{z \in Z : \text{ there exist } x \in X \text{ s.t. } z = Ax, p \cdot x \leq w\}.$$

Show that for any $p >> 0$ and $w \geq 0$, the budget set $K_{p,w,A}$ is convex.

Let $z', z'' \in K_{p,w,A}$. We need to show that for all $\alpha \in [0, 1]$, $\alpha z' + (1 - \alpha)z'' \in K_{p,w,A}$. Since $z', z'' \in K_{p,w,A}$, there exist $x', x'' \in B_{p,w}$ such that $z' = Ax'$ and $z'' = Ax''$, where $B_{p,w}$ is the budget set at p and w in space X. $\alpha z' + (1 - \alpha)z'' = \alpha Ax' + (1 - \alpha)Ax'' = A(\alpha x') + A(1 - \alpha)x'' = A(\alpha x' + (1 - \alpha)x'')$. Observe that $\alpha x' + (1 - \alpha)x'' \in B_{p,w}$ since $x', x'' \in B_{p,w}$ and $B_{p,w}$ being convex. It follows that $\alpha z' + (1 - \alpha)z'' \in K_{p,w,A}$.

c.) Assume that u is monotone and continuously differentiable. Consider the consumer problem

$$\max_{z \in Z} u(z) \text{ s.t. } z \in K_{p,w,A}.$$

Show that necessary conditions for a utility maximum are

$$p_\ell \geq \frac{1}{\lambda} \left(\sum_{i=1}^{I} a_{i,\ell} \frac{\partial u(z)}{\partial z_i} \right) \text{ for } \ell = 1, ..., L.$$

where λ is the Lagrange multiplier w.r.t. to the budget constraint $p \cdot x \leq w$.

Consider the equivalent problem in the goods space X given by

$$\max_{x \in X} u(Ax) \text{ s.t. } \quad x \in B_{p,w}.$$

Since u is monotone, the budget constraint binds.
Form the Kuhn-Tucker-Lagrangian

$$L(x, \lambda) = u(Ax) - \lambda(px - w).$$

Differentiate w.r.t. to x_ℓ, $\ell = 1, ..., L$, to obtain

$$\frac{\partial L(x, \lambda)}{\partial x_\ell} = \frac{\partial u(Ax)}{\partial x_\ell} - \lambda p_\ell \leq 0 \text{ for } \ell = 1, ..., L.$$

Solve for p_ℓ,

$$p_\ell \geq \frac{1}{\lambda} \frac{\partial u(Ax)}{\partial x}.$$

Use the chain rule to obtain

$$p_\ell \geq \frac{1}{\lambda} \left(\sum_{i=1}^{I} \frac{\partial u(z)}{\partial z_i(x)} \frac{\partial z_i}{\partial x_\ell} \right).$$

Since $z(x) = Ax$, $\frac{\partial z_i(x)}{\partial x_\ell} = a_{i,\ell}$ for $i = 1, ..., I$ and $\ell = 1, ..., L$. Thus

$$p_\ell \geq \frac{1}{\lambda} \left(\sum_{i=1}^{I} \frac{\partial u(z)}{\partial z_i} a_{i\ell} \right) \text{ for } \ell = 1, ..., L.$$

d.) Provide an economic interpretation of the term

$$\frac{1}{\lambda} \frac{\partial u(z)}{\partial z_i}.$$

The Lagrange multiplier is the marginal utility of money. So the term is a ratio of marginal utilities that is quite similar to what we know as marginal rate of substitution (MRS). In interior optimum, the MRS equals to the price ratio. The price of money is 1. Thus, we can interpret this term as the ''shadow price'' of a unit of characteristic i.

e.) Suppose you are a firm that introduces a new good into the market, let's call it good $L + 1$. Suppose the characteristics of your novel good are described by the vector $(a_{1,L+1}, ..., a_{I,L+1})$, for which each component is strictly positive. What price do you want to charge the consumer per unit of the new good?

The first-order conditions show that in interior optimum the price of a good consumed must be equal to the weighted average of the shadow prices of its characteristics, whereby the weights are given by matrix A. Hence, the price of the new good that makes the consumer indifferent between buying another unit of it or not is given by

$$p_{L+1} = \frac{1}{\lambda}\left(\sum_{i=1}^{I} a_{i,L+1}\frac{\partial u(z)}{\partial z_i}\right).$$

I hope you find this model interesting. Further details on this model can be found in:

Blow, L., Browning, M., and I. Crawford (2008). Revealed preferences analysis of the characteristics model, Review of Economic Studies 75, 371--389.
Gorman, W.M. (1956). A possible procedure for analysing quality differentials in the egg market, Review of Economic Studies 47 (1980), 843--856.
Lancaster, K.J. (1971). Consumer demand. A new approach, New York, London: Columbia University Press.
Lancaster, K.J. (1966). A new approach to consumer theory, Journal of Political Economy 74, 132--157.
Muellbauer, J. (1974). Household production theory, quality, and the ``Hedonic Technique'', American Economic Review 64, 977--994.
Rosen, S. (1974). Hedonic prices and implicit markets: product differentiation in pure competition, Journal of Political Economy 82, 34--55.