The Economics of Perennial Orchards with Endogenous Age-Classes

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Abstract

Perennial crops exhibit boom and bust cycles. It has been conjectured that better price forecasts could reduce or eliminate production cycles in perennial crops, but an alternative hypothesis is that production cycles are optimal for a profit maximizing grower even under perfect information. We build a dynamic Lagrangian model of an orchard, and show in two- and three-age-class infinite horizon models that cycles are generally optimal, unless the grower starts from a balanced orchard (i.e. even share of land allocated to each age-class). Our results suggest that completely eliminating production cycles is suboptimal for orchard growers, and therefore better price forecasts should not be expected to eliminate production cycles. In addition, previous analyses of optimal orchard management assumed that yields were monotonically increasing over the life of the tree. In our the three-age-class model we are able to identify optimal stationary trajectories for age-yield relationships where yield declines at the end of the tree's life.

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1 Introduction

Perennial crops are plants that can be harvested multiple times before replanting. Fruit-bearing trees and vines, such as almonds, olives, apples, and grapes, are prominent examples. A notable feature of these crops is that they exhibit boom and bust cycles, increasing the risk to farmers of investing in these crops.

It has been conjectured that better price forecasts could reduce or eliminate acreage cycles in perennial crops. For example, in the case of lemons in California, French and Bressler (1962, p. 1036) stated: "If 'better' knowledge, including the realization that there is a cycle, leads to more realistic forecasts, then the cycle would be moderated or eliminated." Knapp (1987) has advanced an alternative hypothesis: acreage cycles are an inherent feature of perennial crop production systems, arising from the biology of the crop causing lags in production, rather than from imperfect information and foresight on the part of the growers.

Although perennial cycles have been identified in empirical and numerical work, there has been little work done to identify whether such cycles are consistent with growers optimizing the net present value of their production. We are only aware of two papers that address this question (Mitra et al., 1991; Wan, 1993). In this paper we develop a model of a perennial crop grower with perfect information and foresight, thus abstracting from problems of imperfect forecasting. Building on the age-structured perennial management model of Mitra et al. (1991), and adapting the age-structured forestry models of Salo and Tahvonen (2002, 2003), we use our model to find the optimal planting and replanting trajectories for a grower with age-structured perennial crops.[1] In two- and three-age-class infinite horizon models we find that cycles are generally optimal, unless the grower starts from a balanced orchard (i.e. even share of land allocated to each age-class).

Devadoss and Luckstead (2010) identify four key differences between annual and perennial crops:

> perennial crop supply is vastly different from the annual crop supply because (1) trees are a long-term investment; (2) perennial crops have long gestation intervals between initial plantings and first harvest; (3) once trees start yielding, there is an extended

[1] An age-structured capital model (including forestry and perennials) allows multiple stocks of capital within each period. The productivity of each stock is defined by the time since its installation or planting.

period of productivity, and then gradual decline in production; and (4) after trees reach their final productivity decline, they are removed.

From this list it is clear that, unlike annuals, a static theoretical model is inadequate for studying perennials. This sentiment is echoed by Just and Pope (2001, p. 706) in their review of agricultural production economics: "Specifically, lags and dynamic processes appear to be at the heart of understanding large-animal livestock and perennial crop production problems."

Having satisfactory models of the production decisions of perennial crop growers gives us insight into an important and valuable sector of the world's agricultural industry. Hence the study of perennial crops has been given considerable attention by agricultural economists, particularly because of their importance for the economies of tropical developing countries (Bellman and Hartley, 1985), but they also play an important role in the economies of developed countries.

In the US, the value of production of fruit and tree nuts was 30 billion USD, production mainly from perennial crops. Tree nut value is around $10b, led by almonds, walnuts, and pistachios (USDA, 2016). The fruit production value is led by grapes, apples, strawberries, and oranges (strawberries are grown as an annual crop in California (California Strawberry Commission, 2017), but can be grown as a perennial, a practice often done in colder climates. The other three crops are perennials). Perennial crops are also important to regional economies. For example, Apple production value in Washington state was greater than $1 billion in 2007 (Devadoss and Luckstead, 2010).

Perennial crops, such as sugarcane or *miscanthus*, are also a source of value through their conversion to biofuel. For example, Brazilian biofuel produced from sugarcane is a particularly low-carbon liquid fuel, with Crago et al. (2010) estimating that its life-cycle carbon emissions are about half those of corn ethanol. Furthermore, ethanol is an important component of Brazil's economy, accounting for around 2.3 percent of of its GDP in 2010 (Valdes, 2011).

Finally perennial crops can provide substantial environmental benefits, relative to their annual cousins, through increasing soil carbon, reducing erosion, and requiring fewer chemical inputs (Glover et al., 2010).

The optimal management of perennials under perfect information is a normative question, relating to what a grower or manager *should* do. The majority of analysis of perennial crops in agricultural economics has focused on the positive questions of how growers actually manage their perennial crops. Due to their positive focus, the majority of work in agricultural economics on the econometrics of perennial crop production has been on econometric studies (Rausser, 1971; French and Matthews, 1971; French et al., 1985; Trivedi, 1987; Knapp and Konyar, 1991; Kalaitzandonakes and Shonkwiler, 1992; Devadoss and Luckstead, 2010; Brady and Marsh, 2013).

There has been far less normative work, much of which has been based on simulation and computation, with very little analytical, theoretical work. Bellman and Hartley (1985) present a comprehensive dynamic programming framework to numerically simulate the supply of a perennial crop. They do not actually perform the simulation. Knapp (1987) presents a model and calibrated simulation for calculating the dynamic equilibrium in a market for perennial crops. He applies his model to the alfalfa industry in California. And Franklin (2012) presents a thorough age-structured dynamic programming computational model to simulate wine grape production in South Australia.

There are few purely analytic studies of the management of perennials, and those that have been done mostly focus on the management of a single tree, or an even-aged orchard. Kalaitzandonakes and Shonkwiler provide a possible reason for this in their 1992 paper:

> The theoretical underpinnings of perennial investment decisions have been elaborated by Bellman and Hartley, and Trivedi... These studies have demonstrated that due to the complex dynamics of perennial technologies and the heterogeneity of perennial capital, intertemporal profit maximization of the perennial firm admits analytical solutions only under rather restrictive technological assumptions.

However, the complexity of perennial crop production is not a compelling argument against the development and analysis of stylized, analytic models. As in other branches of economics, the development of analytic and empirical models are complements, not substitutes, creating a virtuous cycle in which new insights and knowledge are synthesized from their joint production. The analytic dimension of perennial crop production is far less developed than then empirical dimension, and it is not clear, *a priori*, that this imbalance is optimal. The value of the analytical approach to modeling perennials is that it can provide generalizable intuition about the behavior of perennials, identifying that which can be said in general, and that which depends on the specifics of the particular crop and context. The papers cited above (with the exception of Bellman and Hartley (1985)) are all rooted in particular contexts. Analyzing an analytical model that encompasses the details of these contexts can allow us to see whether the production and acreage cycles observed by these researchers are likely general features of perennial crops, or are products of the particular circumstances of their research environments.

Of the few analytic perennial crop management models, most are focused on the single tree management problem. The single-aged tree replacement problem is an example of the classic durable asset replacement problem (see Miranda and Fackler (2002, section 7.2.2) for a simple example, or Rust (1987) for a more complex one). Wesseler (1997) provides a framework for calculating the net present value of an even-aged orchard from an arbitrary starting age. Feinerman and Tsur (2014) analyzes the impacts of stochastic interruptions to an even-aged orchard production cycle (in particular, reduced water supply due to drought).

The only purely theoretical analyses of age-structured orchards that we are aware of are the studies by Mitra et al. (1991) and Wan (1993). The framework for their theoretical analysis is built on the multi-sector growth model literature (Majumdar, 1987; Brock and Dechert, 2008). The multi-sector growth approach is an intertemporal extension of general equilibrium theory, and is thus "analytically organized around existence of equilibrium, the core and equilibria, the two welfare theorems, as well as the 'anything goes' theorem of Sonnenschein, Mantel and Debreu (SMD)" (Brock and Dechert, 2008, p. 4). Agricultural economics, in contrast, is focused more on empirical applications and the behavior of farmers in response to changes in economic parameters, i.e. comparative statics.

Mitra et al. (1991) analyze an infinite horizon model of perennial production with arbitrarily many age-classes, fixed total land, and a positive discount factor. They characterize the set of optimal stationary orchards for both linear and strictly concave benefit functions. Their main results regard the convergence of an optimal program to the optimal stationary orchard under strictly concave benefit and positive discounting. They find that, by assuming the age-yield relationship is non-decreasing over the tree's lifespan, the optimal program never converges to the optimal stationary orchard, rather to a cycle in a neighborhood around it. They prove a neighborhood turnpike result—namely, that as the discount factor approaches 1, the amplitude of the optimal cycle approaches zero. Wan (1993) focuses on the existence and uniqueness of steady-states in a two age-class tree farm, as well as the possibility of cyclical tree planting in the long-run. In his remark in section 3, he notes that his framework is "quite general" and can be applied to the orchard model of Mitra et al. (1991). He states, without proof, the optimal control rule for 2-age-class perennials, which we re-derive here using a different analytical framework.

A distinct, but related, strand of literature with a more agricultural and natural resource economics style is the series of four papers on the optimal management of age-structured forestry written by Salo and Tahvonen between 2002 and 2004. The forestry model is a relative of the orchard model, where forestry is an instance of *point input, point output* capital, while perennial crops are an example of *point input, flow output* (Mitra et al., 1991). Salo and Tahvonen develop their results using a dynamic Lagrangian approach (Chow, 1993, 1997), rather than the capital theoretic approach previously used in theoretical analyses of the economics of age-structured forestry models (Mitra and Wan, 1985, 1986), or orchard models (Mitra et al., 1991; Wan, 1993). Salo and Tahvonen bridge the gap between the capital theoretic approach, and the resource economics approach, beginning their analysis by focusing on the long-run equilibrium conditions of their age-structured forest, but then developing numerical transition paths, and supply curves, which have a more applied focus.

In this paper we develop an age-structured model of perennial crop planting decisions, using a similar model to Mitra et al. (1991) and Wan (1993), but analyzed using the dynamic Lagrangian approach of Salo and Tahvonen (2004), therefore allowing us to focus more on the intuition behind the stationary states, and develop comparative statics of the stationary states with respect to the parameters.

There are 2 key differences between our model and the earlier orchard models. First, like Wan (1993) but unlike Mitra et al. (1991), we assume two (or three) age-classes. This is an analytical convenience to help develop the modeling framework. In future research we intend to relax this assumption. Second, we use the dynamic Lagrangian approach to solving and analyzing the model.

This contrast to the capital theoretic support price approach used by Mitra et al. (1991) and the topological approach used by Wan (1993). The advantage of this approach is that it allows us to analyze age-yield relationships that decline towards the end of the tree's lifespan.

Similarly to Mitra et al. (1991) and Wan (1993), we choose a discrete time framework. Our choice of a discrete time framework for this analysis is twofold. First, the choice of a discrete time framework is realistic for the perennial crop setting. While timber may be harvested at any point during the year in general (especially in regions without strong seasonal variation in weather), many perennial crops are harvested on a yearly cycle due to the biology of the crop. Second, it is a far more throughly studied case, and the methods of analysis are more elementary, allowing us to focus on the applications and intuition of the model, and creating scope for reaching a wider audience. Fabbri et al. (2015) study the related Mitra and Wan forestry model in continuous time, a development pioneered by these authors. To analyze this problem they developed a new class of vintage models, allowing them to use measure valued controls, which allow for all members of a continuous time age-class to be cut at once. For their forestry model, they find that the majority of canonical results carry over to the continuous time case, with the notable exception of cyclical optimal solutions. Therefore it is likely that under continuous time cyclical solutions would also cease to exist in an orchard setting.

2 A two-age-class orchard model

In this section we present a two-age-class, infinite horizon model of a perennial orchard. We find optimal stationary solutions for the model under three cases: young and old trees have identical yield, young trees have higher yield, and old trees have higher yield. In the case where old trees have higher yield, we find that the solution is generally cyclical, with one exception. We provide intuition for the existence of optimal cycles, and examine how the maximum cycle radius changes as function of the discount factor and the yields of young and old trees.

2.1 The age-yield relationship

The age-yield relationship describes how the yield of the plant changes over its life cycle. We use a deterministic age-yield relationship, abstracting from the fact that observed age-yield relationships in empirical applications will be stochastic functions of multiple variables, including rainfall, temperature profile throughout the growing season, soil type, inputs applied (e.g. fertilizer, pesticide), hours of labor, etc.

Mitra et al. (1991) specify a general age-yield relationship with three integers P, Q, and T, with $0 < P \leq Q \leq T$, such that

$$R(0) \leq \ldots \leq R(P) = \ldots = R(Q) \geq \ldots \geq R(T)$$

and at least one inequality between $R(0)$ and $R(P)$. That is, the age-yield relationship is monotonically increasing, plateaus, then monotonically decreases until the end of the tree's lifespan. P corresponds with the beginning of the yield plateau, Q with the end of the yield plateau, and T with the end of the tree's lifespan.

They require only monotonic increasing and decreasing for the increasing and decreasing phases of the AY relationship. There could be arbitrarily many inflection points. There could also be other flat regions, and their model allows an initial non-bearing period.

In this paper we focus on a special case of their general relationship, adapted for a two-age-class model, where $R(0) = f_1$ and $R(1) = f_2$. In our analysis, we consider cases where $f_1 > f_2$, $f_1 = f_2$, and $f_1 < f_2$.

2.2 The optimal replacement age for a single two-age-class tree

Before studying the orchard problem, we need to know the optimal replacement age for a single tree, considered in isolation.

Assume a single perennial tree that lives for at most two periods, grown over an infinite horizon either on a 1-period or 2-period rotation. Let $\beta(< 1)$ be the discount factor. The net present value of the tree grown on a 1-period rotation is

$$\text{NPV}_1 = f_1 + \beta f_1 + \beta^2 f_1 + \beta^3 f_1 + \ldots = \frac{f_1}{1-\beta}$$

The net present value of the tree grown on a 2-period rotation is

$$\text{NPV}_2 = f_1 + \beta f_2 + \beta^2 f_1 + \beta^3 f_2 + \ldots = \frac{f_1 + \beta f_2}{1-\beta^2}$$

The difference between the two values is

$$\begin{aligned}
\text{NPV}_1 - \text{NPV}_2 &= \beta f_1 - \beta f_2 + \beta^3 f_1 - \beta^3 f_2 + \ldots \\
&= \beta(f_1 - f_2) + \beta^3(f_1 - f_2) + \ldots \\
&= \beta(f_1 - f_2)(1 + \beta^2 + \beta^4 + \ldots) \\
&= \frac{\beta(f_1 - f_2)}{1-\beta^2}
\end{aligned}$$

Hence if $f_1 > f_2$ the optimal replacement age is 1 (let N be the optimal replacement age, so $N = 1$). If $f_1 < f_2$ the optimal replacement age is 2 ($N = 2$). And if $f_1 = f_2$ ($N_1 = 1$; $N_2 = 2$).[2]

The optimal rotation age does not depend on discounting in this model. Discounting will affect the size of the net present value, but not the relative size of NPV_1 and NPV_2. This is a special result, due to the assumption of a two-age-class model. In a more general model, the optimal single tree replacement age is a function of the discount factor, as shown by Mitra et al. (1991) in proposition 3.1.

2.3 A two-age-class, infinite horizon orchard model

Salo and Tahvonen (2002) present a two-age-class, infinite horizon forestry model. The Salo and

[2]Theorem 3.1 in Mitra et al. (1991) states that there will either be one or two optimal cutting ages, N or N_1 and N_2. The age(s) will occur during the declining section of the age-yield relationship. If there are two optimal cutting ages, then they will be adjacent. That is $Q \leq N_1 \leq N_2 \leq N_1 + 1 \leq T$.

Tahvonen (2002) model adapted to perennials is

$$V(x_{10}, x_{20}) = \max_{\mathbf{x}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

subject to

$$c_t \equiv f_1 x_{1t} + f_2 x_{2t}$$

$$x_{1,t+1} = x_{2t} + z_t \tag{2}$$

$$x_{2,t+1} = x_{1t} - z_t \tag{3}$$

$$z_t \geq 0 \tag{4}$$

$$z_t - x_{1t} \leq 0 \tag{5}$$

where x_{1t} is the area of land allocated to young trees in period t, x_{2t} is the quantity of land allocated to old trees in period t, f_1 is the productivity of young trees, f_2 is the productivity of old trees,[3] $u(c_t)$ is the benefit to the grower from growing/consuming fruit in period t (we assume that the benefit function exhibits diminishing marginal returns: $u'(.) > 0$ and $u''(.) < 0$), c_t is the total quantity of fruit harvested in period t, and z_t is the area of young trees cut at the end of the period.

This model is identical to that of Salo and Tahvonen (2002) except that the definition of consumption, c_t, has been changed. In their forestry model consumption in period t was the sum of young trees replanted plus old trees replanted. In this model consumption in period t is the sum of the fruit harvested from young and old trees, which does not depend on the number of trees replanted.

Equations 2 and 3 are the *cross-vintage bound* constraints, using the terminology of Wan (1993). They state that the number of old trees in the next period cannot exceed the number of young trees in the current period. They also assume that all old trees are uprooted and replanted at the end of the period.

There is no need to add an explicit land constraint because the total land available each period is

[3] These productivities are net of planting costs.

determined by the initial land allocation. From then on the cross-vintage bound constraints prevent changes in the total quantity of land. Later in the paper we will normalize the total quantity of land to one unit.[4]

This is a convex optimization problem since the objective function is strictly concave and the constraints are linear. Therefore any solution to the Karush-Kuhn-Tucker conditions will also be a solution to the constrained optimization problem posed above.

The corresponding Lagrangian function is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(f_1 x_{1t} + f_2 x_{2t}) + \lambda_{1t}(x_{2t} + z_t - x_{1,t+1}) + \lambda_{2t}(x_{1t} - z_t - x_{2,t+1}) + \rho_t z_t + \theta_t(z_t - x_{1t}) \right]$$

where λ_{1t}, λ_{2t}, ρ_t, and θ_t are Lagrangian multipliers corresponding to the constraints in the initial problem. The first three multipliers are non-negative, but θ is non-positive since it corresponds to a less-than-or-equal-to constraint.[5]

There are three choice variables, and four constraints (c_t is considered a definition, not a constraint), two are equality constraints, and two are inequality constraints. Therefore there are five first order conditions (FOCs) and two complementary slackness conditions.

The five FOCs are

$$\beta^{-t}\frac{\partial \mathcal{L}}{\partial z_t} = \lambda_{1t} - \lambda_{2t} + \rho_t + \theta_t = 0 \tag{6}$$

$$\beta^{-t}\frac{\partial \mathcal{L}}{\partial x_{1,t+1}} = -\lambda_{1t} + \beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1}) f_1 + \beta \lambda_{2,t+1} - \beta \theta_{t+1} = 0 \tag{7}$$

$$\beta^{-t}\frac{\partial \mathcal{L}}{\partial x_{2,t+1}} = \beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1}) f_2 + \beta \lambda_{1,t+1} - \lambda_{2t} = 0 \tag{8}$$

$$\beta^{-t}\frac{\partial \mathcal{L}}{\partial \lambda_{1t}} = x_{2t} + z_t - x_{1,t+1} = 0,\ \lambda_{1t} \geq 0 \tag{9}$$

$$\beta^{-t}\frac{\partial \mathcal{L}}{\partial \lambda_{2t}} = x_{1t} - z_t - x_{2,t+1} = 0,\ \lambda_{2t} \geq 0 \tag{10}$$

[4]The current version of the paper uses this constraint by substituting it, rather than including it as an element of the Lagrangian function. This was done to facilitate comparison with Salo and Tahvonen (2002), but we intend to include this constraint explicitly in the Lagrangian in an update of this paper.

[5]This is a non-standard choice, which we use in this iteration of the paper for consistency with Salo and Tahvonen (2002). We intend to standardize this model by converting all multipliers to non-negative variables in a future iteration.

The two complementary slackness conditions are

$$\rho_t \geq 0;\ z_t \geq 0;\ \rho_t z_t = 0 \tag{11}$$

$$\theta_t \leq 0;\ z_t - x_{1t} \leq 0;\ \rho_t(z_t - x_{1t}) = 0 \tag{12}$$

We can eliminate x_{1t} and z_t to obtain a simpler statement of the KKT conditions. From 7 get

$$\lambda_{1t} = \beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1}) f_1 + \beta\lambda_{2,t+1} - \beta\theta_{t+1}$$

From 10 get

$$\lambda_{2t} = \beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1}) f_2 + \beta\lambda_{1,t+1}$$

Sub the expressions for λ_1 and λ_2 into 6

$$\beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1}) f_1 + \beta\lambda_{2,t+1} - \beta\theta_{t+1} - \beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1}) f_2 + \beta\lambda_{1,t+1} + \rho_t + \theta_t = 0$$

Rearranging gives

$$\beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1}) f_1 - \beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1}) f_2 + \beta(\lambda_{2,t+1} - \theta_{t+1} + \lambda_{1,t+1}) + \rho_t + \theta_t = 0$$

Shifting 6 forward by one period we get

$$\lambda_{1,t+1} - \theta_{t+1} - \lambda_{2,t+1} = \rho_{t+1}$$

So we get

$$\beta u'(f_1 x_{1,t+1} + f_2 x_{2,t+1})[f_1 - f_2] + \beta\rho_{t+1} + \rho_t + \theta_t = 0$$

Now eliminate $x_{1,t+1}$ by noting that $x_{1,t+1} = x_{2t} + z_t$, so

$$\beta u'(f_1(x_{2t} + z_t) + f_2 x_{2,t+1})[f_1 - f_2] + \beta\rho_{t+1} + \rho_t + \theta_t = 0$$

The total land is fixed by the initial allocation of land. The setup of the model does not allow land to be added or removed after period zero. Hence normalize the total quantity of land to 1, i.e.

$$\begin{aligned} & x_{1t} + x_{2t} = 1 \ \forall t \\ \Rightarrow & (z_t + x_{2,t+1}) + x_{2t} = 1 \\ \Rightarrow & z_t = 1 - x_{2t} - x_{2,t+1} \end{aligned}$$

Substituting into the previous equation

$$\beta u'(f_1(x_{2t} + 1 - x_{2t} - x_{2,t+1}) + f_2 x_{2,t+1})[f_1 - f_2] + \beta\rho_{t+1} + \rho_t + \theta_t = 0$$

which simplifies to

$$\beta u'(f_1 - x_{2,t+1}(f_1 - f_2))[f_1 - f_2] + \beta\rho_{t+1} + \rho_t + \theta_t = 0$$

Since x_{1t} has been eliminated from the KKTs, write $x_t \equiv x_{2t}$. The KKT equation becomes

$$\beta u'(f_1 + x_{t+1}(f_2 - f_1))[f_1 - f_2] + \beta\rho_{t+1} + \rho_t + \theta_t = 0 \tag{13}$$

This gives us the simplified KKT, a single equation that replaces the first five FOCs presented previously.

ρ_t is positive when the grower wants more old trees next period. A positive value of ρ_t implies that $z_t = 0$ by the complementary slackness condition. That means the $z_t \geq 0$ constraint is binding, so if it were possible the optimal z_t would be set below zero. A negative value of z_t implies that $x_{2,t+1} > 1 - x_{2t} = x_{1t}$. That is the grower would want to plant more old trees in the following period than there were young trees in the previous period.

θ_t is positive when the grower wants more young trees next period. A positive value of θ_t implies that $z_t - x_{1t} = 0$ by the complementary slackness condition. This implies that the $z_t \leq x_{1t}$ condition is binding, so in the absence of this constraint, the optimal value of z_t is greater than x_{1t} (think of z_t as the choice variable since the x_t's are pre-determined). This means that the optimal number of young trees in the next period would be greater than the the number of young trees and the number of old trees in the previous period.

In both cases the the constraints prevent the total number of trees from increasing beyond the initial number of trees endowed at the beginning of the problem. Within a single period, the two constraints are mutually exclusive. Together the constraints constrain $0 \leq z_t \leq x_{1t}$. Even if $x_{1t} = 0$, the two constraints cannot bind simultaneously. So assuming that at least one constraint is binding, the other cannot be, that is, $\rho_t \theta_t = 0$.[6]

There is a difference in the KKTs between the forestry and the perennial problem. In forestry, the age-structure adjustment comes before consumption. In perennials, the age-structure adjustment comes after consumption. For perennials the change in age-structure this year only affects consumption next year, it does not have an impact on consumption or utility this year.

Forestry:

Inherit trees → grow → remove/replant → consume → bequeath trees to next period

Perennials:

Inherit trees → grow → consume → remove/replant → bequeath trees to next period

2.4 Stationary solutions to the model

We will identify stationary solutions to the two-age-class, infinite horizon orchard model. A stationary solution is one where the land allocation is constant over time, or is cyclical, returning to the same land allocation after finite time.

[6]It is possible that neither constraint is binding, for example if the orchard were in a steady state and it were not optimal to adjust the age-structure, so $z_t^* = 0$. In this case, the complementary slackness condition $\rho_t \theta_t = 0$ still holds with $\rho_t = \theta_t = 0$.

2.4.1 Case 1: $f_1 = f_2$

First consider the case where the grower is indifferent between replacing trees after one or two periods ($N_1 = 1$, $N_2 = 2$). This implies $f_1 = f_2$.

When $f_1 = f_2$, equation 13 becomes

$$\beta\rho_{t+1} + \rho_t + \theta_t = 0$$

Only $\rho_t = \theta_t = 0$ is consistent with this equation. Since $x_t\ \forall t$ is not present in the KKT equation, we infer that all feasible choices of x_t are optimal. This conclusion is clear from examining the objective function

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t u(f_1(1-x_t) + f_2 x_t) &= \sum_{t=0}^{\infty} \beta^t u(f_1 - f_1 x_t + f_2 x_t) \\
&= \sum_{t=0}^{\infty} \beta^t u(f_1 + x_t(f_2 - f_1)) \\
&= \sum_{t=0}^{\infty} \beta^t u(f_1) && \text{(Since } f_1 = f_2\text{)}
\end{aligned}$$

So the objective function no longer depends on the choice of x_t. All sequences of x_t are equally good, so long as they are feasible. This includes all stationary states $x_t \in [0, \frac{1}{2}]\ \forall t$. This also includes all feasible cycles.

2.5 Case 2: $f_1 > f_2$

Second, consider the case where $N = 1$, that is $f_1 > f_2$. Take the basic KKT, equation 13, and assume $f_1 > f_2$. This implies the first term is always positive

$$\underbrace{\beta u'(f_1 + x_{t+1}(f_2 - f_1))[f_1 - f_2]}_{>0} + \beta\rho_{t+1} + \rho_t + \theta_t = 0$$

Assuming $\rho_t \geq 0$ and $\theta_t = 0$ clearly violates the equation. If $\theta_t \leq 0$ and $\rho_t = 0$, $\theta_t = -\beta u'(f_1 + x_{t+1}(f_2 - f_1))[f_1 - f_2]$, which is the only solution consistent with the KKT conditions.

If $\theta_t < 0$, $z_t + x_t = 1$, which implies $x_{t+1} = 0$. Hence if $\theta_t < 0\ \forall t$, $x_{t+1} = 0\ \forall t$ is the only feasible trajectory. An initial condition with $x_0 > 0$ is still consistent with the KKTs, but the optimal trajectory sets $x_1 = 1$ and leaves it there.

This is a stronger result than Salo and Tahvonen (2002) for forestry. They prove that if young trees are more productive than old trees, then the stationary state with no old trees must be reached in finite time. Here we show that, with perennials, this stationary state must be reached in one period.

2.6 Case 3: $f_1 < f_2$

Consider the case where $N = 2$, that is $f_1 < f_2$.

Proposition 1. *If $f_1 < f_2$ and $\beta < 1$, then there exist two-period optimal cycles.*

Take the basic KKT, equation 13, and assume $f_1 < f_2$. This implies the first term is always negative

$$\underbrace{\beta u'(f_1 + x_{t+1}(f_2 - f_1))[f_1 - f_2]}_{<0} + \beta\rho_{t+1} + \rho_t + \theta_t = 0$$

Assuming $\theta_t \leq 0$ and $\rho_t = 0$ is inconsistent with this equation. So the only feasible solution required $\rho_t \geq 0$ and $\theta_t = 0$. Write the KKT with $\theta_t = 0$ in terms of consumption for clarity

$$\beta u'(c_{t+1})[f_1 - f_2] + \beta\rho_{t+1} + \rho_t = 0$$

Hence

$$\rho_{t+1} = u'(c_{t+1})[f_2 - f_1] - \frac{1}{\beta}\rho_t$$

$\rho_t = 0\ \forall t$ is not a solution to this equation. Hence any solution must have $\rho_t > 0$, implying $z_t = 1 - x_t - x_{t+1} = 0$.

There are two cases to consider: $x_t = x_{t+1}$ and $x_t \neq x_{t+1}$.

Combining $1 - x_t - x_{t+1} = 0$ and $x_t = x_{t+1}$ gives $x_t = \frac{1}{2}\ \forall t$, which further implies constant consumption, $c_t = f_1 + \frac{1}{2}(f_2 - f_1)$ Solving the KKT equation with constant consumption gives

$$\rho_t = \rho_{t+1} = \frac{1}{1+\beta} u'(c_t)[f_2 - f_1]\ (> 0)$$

Hence the only stationary state consistent with the KKTs is $x_t = \frac{1}{2}$. This is a "normal orchard", where $\frac{1}{n}$ of the land is allocated to each of the n age-classes. It is analogous to the "normal forest" concept in forestry.

Now consider the case $x_t \neq x_{t+1}$. Since $z_t = 0$, $x_{t+1} = 1 - x_t$. Iterating this forward by one period gives $x_{t+2} = 1 - x_{t+1} = 1 - (1 - x_t) = x_t$. So if $x_t \neq x_{t+1}$ there must be a two period cycle.

Are there solutions to the KKTs that are consistent with two period cycles? We show that $x_t = x_{t+2}$ and $\rho_t = \rho_{t+2}$ are consistent with the KKTs.

The result for x_t is shown above.

For ρ_t begin by assuming $x_t = x_{t+2} \Rightarrow c_t = c_{t+2}$.

$$\rho_{t+1} = u'(c_{t+1})[f_2 - f_1] - \frac{1}{\beta}\rho_t$$

$$\begin{aligned} p_{t+2} &= u'(c_{t+2})[f_2 - f_1] - \frac{1}{\beta}\rho_{t+1} \\ &= u'(c_t)[f_2 - f_1] - \frac{1}{\beta}\rho_{t+1} \qquad \text{(Since } c_t = c_{t+2}\text{)} \end{aligned}$$

Substituting to get the equation for p_{t+2} in terms of ρ_t

$$\begin{aligned} p_{t+2} &= u'(c_t)[f_2 - f_1] - \frac{1}{\beta}(u'(c_{t+1})[f_2 - f_1] - \frac{1}{\beta}\rho_t) \\ &= u'(c_t)[f_2 - f_1] - \frac{1}{\beta}u'(c_{t+1})[f_2 - f_1] + \frac{1}{\beta^2}\rho_t \end{aligned}$$

Assume $\rho_t = p_{t+2}$ and solve for ρ_t.

$$\rho_t = \frac{f_2 - f_1}{1-\beta^2}\left[u'(c_t) - \frac{1}{\beta}u'(c_{t+1})\right]$$

The first term is always positive, so for $\rho_t \geq 0$ we must have

$$u'(c_t) - \frac{1}{\beta}u'(c_{t+1}) \geq 0$$
$$\Rightarrow \frac{u'(c_{t+1})}{u'(c_t)} \geq \beta$$

Similarly assume $\rho_{t+1} = p_{t+3}$ which implies

$$\rho_{t+1} = \frac{f_2 - f_1}{1-\beta^2}\left[u'(c_{t+1}) - \frac{1}{\beta}u'(c_t)\right]$$

Similarly,

$$u'(c_{t+1}) - \frac{1}{\beta}u'(c_t) \geq 0$$
$$\Rightarrow \frac{u'(c_{t+1})}{u'(c_t)} \leq \frac{1}{\beta}$$

So for both $\rho_t \geq 0$ and $\rho_{t+1} \geq 0$ to hold simultaneously we must have

$$\beta \leq \frac{u'(c_{t+1})}{u'(c_t)} \leq \frac{1}{\beta}$$

If $c_t = c_{t+1}$, then this inequality holds, confirming that a constant sequence is consistent with the KKTs. If $\beta < 1$, then there are sequences with $c_t \neq c_{t+1}$ which are compatible with these inequalities. Hence there are sequences with $x_t \neq x_{t+1}$, and $x_t = x_{t+2}$ that are compatible with the KKTs. Hence for all $\beta < 1$ and all $f_1 < f_2$ there will exist optimal cyclical sequences with a period of 2.

2.7 Intuition for the existence of cycles

To understand the optimality of cycles, consider what happens when adjusting a cycle closer to a normal orchard. For this discussion assume that $x_t < \frac{1}{2}$ This means that the grower will have $1 - x_t > \frac{1}{2}$ old trees next year.

To get closer to the cycle, the grower needs to harvest some young trees at the end of period t, so that there are more young trees and fewer old trees in period $t+1$. To make this adjustment, $z_t > 0$, since the adjustment means $x_{t+1} < 1 - x_t$. Recall $z_t = 1 - x_t - x_{t+1}$. Hence $1 - x_t - x_{t+1} - z_t = 0$, and $x_{t+1} = 1 - x_t - z_t$, and $x_{t+2} = x_t + z_t$.

The total utility from this altered program is

$$
\begin{aligned}
V_t &= u(f_1 + x_t(f_2 - f_1)) + \beta u(f_1 + (1 - x_t - z_t)(f_2 - f_1)) + \beta^2 u(f_1 + (x_t + z_t)(f_2 - f_1)) + \dots \\
&= u(f_1 + x_t(f_2 - f_1)) + \frac{\beta u(f_1 + (1 - x_t - z_t)(f_2 - f_1))}{1 - \beta^2} + \frac{\beta^2 u(f_1 + (x_t + z_t)(f_2 - f_1))}{1 - \beta^2}
\end{aligned}
$$

The marginal utility from a change in the adjustment z_t is

$$
\begin{aligned}
\frac{\partial V_t}{\partial z_t} &= 0 - \beta(f_2 - f_1)u'(f_1 + (1 - x_t - z_t)(f_2 - f_1)) + \beta^2(f_2 - f_1)u'(f_1 + (x_t + z_t)(f_2 - f_1)) + \dots \\
&= \frac{\beta(f_2 - f_1)}{1 - \beta^2}\left[\beta u'(f_1 + (x_t + z_t)(f_2 - f_1)) - u'(f_1 + (1 - x_t - z_t)(f_2 - f_1))\right]
\end{aligned}
$$

In a stationary or cyclical solution, $z_t = 0$. Hence a stationary or cyclical solution will be optimal if

$$
\begin{aligned}
&\left.\frac{\partial V_t}{\partial z_t}\right|_{z_t=0} \leq 0 \\
\Rightarrow &\beta u'(f_1 + x_t(f_2 - f_1)) - u'(f_1 + (1 - x_t)(f_2 - f_1)) \leq 0
\end{aligned}
$$

which is the same inequality we derived from the KKTs.

For an adjustment that takes the orchard closer to the normal orchard, there is no immediate effect, followed by a marginal reduction in utility because the adjustment creates a period with fewer old trees than there would have otherwise been ($1 - x_t - z_t < 1 - x_t$), followed by a period with an increase in marginal utility due to the addition of more old trees. For this adjustment to be worthwhile, the increase in utility from the addition of old trees in even periods $(t+2, 4, 6\ldots)$ must be greater than the loss of utility from the reduction in old trees in the odd periods $(t+1, 3, 5\ldots)$.

Although the marginal utility from periods with fewer old trees is higher than the marginal utility for periods with more old trees, the first benefit from old trees occurs after the loss. With a discount factor less than 1, there will always be some threshold $x_t < \frac{1}{2}$, where the marginal benefits of more smoothing are dominated by the marginal losses.

In sum, to make an adjustment, the grower is cutting trees too early, losing the fruit the old tree would give in the next period in favor of more fruit from old trees in future periods.

2.8 Comparative statics of cycle radius

How does the size of the region in which cycles are optimal change with the parameters of the problem: β, f_1, and f_2. First we need to identify this region, and then calculate comparative statics to examine how it changes with parameters.

Write the inequality out with respect to land allocations rather than consumption.

$$\beta \leq \frac{u'(f_1 + x_{t+1}(f_2 - f_1))}{u'(f_1 + x_t(f_2 - f_1))} \leq \frac{1}{\beta}$$

Apply the rule that total land allocation is equal to 1

$$\beta \leq \frac{u'(f_1 + (1 - x_t)(f_2 - f_1))}{u'(f_1 + x_t(f_2 - f_1))} \leq \frac{1}{\beta}$$

Since we only have one variable, we only need to keep one of the inequalities. Assume WLOG that $x_t < \frac{1}{2}$, so the fraction is less than 1.

$$\beta \leq \frac{u'(f_1 + (1 - x_t)(f_2 - f_1))}{u'(f_1 + x_t(f_2 - f_1))}$$

We can now define the maximum cycle radius $\phi = \frac{1}{2} - x_t$, which is implicitly defined by the function

$g(\phi;\ \beta, f_1, f_2)$.

$$g(\phi;\ \beta, f_1, f_2) = \frac{u'(f_1 + (\frac{1}{2} + \phi)(f_2 - f_1))}{u'(f_1 + (\frac{1}{2} - \phi)(f_2 - f_1))} - \beta = 0 \tag{14}$$

Using equation 14, which has one endogenous variable and three exogenous variables, we can use the implicit function theorem to study the comparative statics of ϕ with respect to the exogenous parameters β, f_1, and f_2.

Proposition 2. *The comparative statics of the maximum cycle radius with respect to the parameters* β*,* f_1*, and* f_2 *are presented in table 1.*

α	$\frac{\partial \phi}{\partial \alpha}$
β	(< 0)
f_1	$(> 0) \Leftrightarrow \frac{A(c(\frac{1}{2}+\phi))}{A(c(\frac{1}{2}-\phi))} < \frac{(\frac{1}{2}+\phi)}{(\frac{1}{2}-\phi)}$
f_2	$(< 0) \Leftrightarrow \frac{A(c(\frac{1}{2}+\phi))}{A(c(\frac{1}{2}-\phi))} < \frac{(\frac{1}{2}-\phi)}{(\frac{1}{2}+\phi)}$

Table 1: Signs of comparative statics of cycle radius, ϕ, with respect to β, f_1, and f_2.

Using the implicit function theorem, the partial derivative of ϕ with respect to any parameter α is

$$\frac{\partial \phi}{\partial \alpha} = \frac{-\frac{\partial g}{\partial \alpha}}{\frac{\partial g}{\partial \phi}}$$

Let $c(x) = x(f_2 - f_1) + f_1$.

$$\frac{\partial g}{\partial \phi} = \frac{(f_2 - f_1)\left[u'(c(\frac{1}{2} + \phi))u''(c(\frac{1}{2} - \phi)) + u'(c(\frac{1}{2} - \phi))u''(c(\frac{1}{2} + \phi))\right]}{u'(c(\frac{1}{2} - \phi))^2}$$

$\frac{\partial g}{\partial \phi} < 0$ since $u''(.) < 0$. So $\text{sign}(\frac{\partial \phi}{\partial \alpha}) = \text{sign}(\frac{\partial g}{\partial \alpha})$.

Computing the sign of $\frac{\partial \phi}{\partial \beta}$ gives

$$\text{sign}(\frac{\partial \phi}{\partial \beta}) = \text{sign}(-1) \quad (< 0 \; \forall \beta)$$

Computing the sign of $\frac{\partial \phi}{\partial f_1}$ gives

$$\text{sign}\left(\frac{\partial \phi}{\partial f_1}\right) = \text{sign}\left(\frac{(\frac{1}{2}-\phi)u'(c(\frac{1}{2}-\phi))u''(c(\frac{1}{2}+\phi)) - (\frac{1}{2}+\phi)u'(c(\frac{1}{2}+\phi))u''(c(\frac{1}{2}-\phi))}{u'(c(\frac{1}{2}-\phi))^2}\right)$$

$$\begin{aligned}
\text{sign}\left(\frac{\partial \phi}{\partial f_1}\right) > 0 \Leftrightarrow & (\frac{1}{2}-\phi)u'(c(\frac{1}{2}-\phi))u''(c(\frac{1}{2}+\phi)) - (\frac{1}{2}+\phi)u'(c(\frac{1}{2}+\phi))u''(c(\frac{1}{2}-\phi)) > 0 \\
& u'(c(\frac{1}{2}-\phi))u''(c(\frac{1}{2}+\phi)) > \frac{(\frac{1}{2}+\phi)}{(\frac{1}{2}-\phi)}u'(c(\frac{1}{2}+\phi))u''(c(\frac{1}{2}-\phi)) \\
& \frac{u'(c(\frac{1}{2}-\phi))u''(c(\frac{1}{2}+\phi))}{u'(c(\frac{1}{2}+\phi))u''(c(\frac{1}{2}-\phi))} < \frac{(\frac{1}{2}+\phi)}{(\frac{1}{2}-\phi)} \qquad \text{(Since } u''(.) < 0\text{)}
\end{aligned}$$

$$\frac{A(c(\frac{1}{2}+\phi))}{A(c(\frac{1}{2}-\phi))} < \frac{(\frac{1}{2}+\phi)}{(\frac{1}{2}-\phi)}$$

Where $A(x) = \frac{-u''(x)}{u'(x)}$, the Arrow-Pratt measure of absolute risk aversion.

Computing the sign of $\frac{\partial \phi}{\partial f_2}$ gives

$$\text{sign}\left(\frac{\partial \phi}{\partial f_2}\right) = \text{sign}\left(\frac{(\frac{1}{2}+\phi)u'(c(\frac{1}{2}-\phi))u''(c(\frac{1}{2}+\phi)) - (\frac{1}{2}-\phi)u'(c(\frac{1}{2}+\phi))u''(c(\frac{1}{2}-\phi))}{u'(c(\frac{1}{2}-\phi))^2}\right)$$

$$\text{sign}\left(\frac{\partial\phi}{\partial f_2}\right) > 0 \Leftrightarrow (\frac{1}{2}+\phi)u'(c(\frac{1}{2}-\phi))u''(c(\frac{1}{2}+\phi)) - (\frac{1}{2}-\phi)u'(c(\frac{1}{2}+\phi))u''(c(\frac{1}{2}-\phi)) > 0$$

$$u'(c(\frac{1}{2}-\phi))u''(c(\frac{1}{2}+\phi)) > \frac{(\frac{1}{2}-\phi)}{(\frac{1}{2}+\phi)}u'(c(\frac{1}{2}+\phi))u''(c(\frac{1}{2}-\phi))$$

$$\frac{u'(c(\frac{1}{2}-\phi))u''(c(\frac{1}{2}+\phi))}{u'(c(\frac{1}{2}+\phi))u''(c(\frac{1}{2}-\phi))} < \frac{(\frac{1}{2}-\phi)}{(\frac{1}{2}+\phi)} \qquad \text{(Since } u''(.) < 0\text{)}$$

$$\frac{A(c(\frac{1}{2}+\phi))}{A(c(\frac{1}{2}-\phi))} < \frac{(\frac{1}{2}-\phi)}{(\frac{1}{2}+\phi)}$$

Where $A(x) = \frac{-u''(x)}{u'(x)}$, the Arrow-Pratt measure of absolute risk aversion.

The level of absolute risk aversion in the utility function affects the comparative statics of the maximum cycle radius. For example, for a function with constant absolute risk aversion, maximum cycle radius would always increase with f_1 and decrease with f_2.

Example [Logarithmic Utility]: Let $u(x) = \ln(x)$. Equation 14 becomes

$$\beta = \frac{f_1 + (\frac{1}{2}-\phi)(f_2 - f_1)}{f_1 + (\frac{1}{2}+\phi)(f_2 - f_1)}$$

Solving for ϕ gives

$$\phi = \frac{(1-\beta)(f_1+f_2)}{2(1+\beta)(f_2-f_1)}$$

There are three comparative statics to consider here

$$\frac{\partial\phi}{\partial\beta} = \frac{-(f_1+f_2)}{(1+\beta)^2(f_2-f_1)} \quad (< 0)$$

$$\frac{\partial\phi}{\partial f_1} = \frac{f_2(1-\beta)}{(1+\beta)(f_2-f_1)^2} \quad (> 0)$$

$$\frac{\partial\phi}{\partial f_2} = \frac{-f_1(1-\beta)}{(1+\beta)(f_2-f_1)^2} \quad (< 0)$$

Cycle radius is increasing in f_1, and decreasing in β and f_2 so long as $f_1 < f_2$.

Increasing f_1 has two effects, boosting total yield and reducing the relative yield of old to young trees. The expression for ϕ shows that cycle radius is increasing in total productivity of trees

$(f_1 + f_2)$, and decreasing in relative productivity $(f_2 - f_1)$.

With logarithmic utility, increasing total production lowers the marginal utility of an extra unit of old and young trees (unit of land). As total production increases, the utility function approaches an affine function. Mitra et al. (1991) shows that there is no incentive to smooth with affine functions.

This scale effect suggests that larger growers are less likely to smooth, and are more likely to exhibit cycles.

2.9 Discussion of two-age-class model

Mitra et al., Wan, and the analysis in the previous section make an unrealistic assumption about the age-yield relationship—that it is monotonically increasing. The interesting case in the previous section and in Wan (1993)—both two-age-class models—is when the old age-class has a higher yield. In this case there is an incentive for the grower to keep each tree for two periods, and thus the potential for cycles. Mitra et al. (1991) develop their main theorem—that the optimal stationary state in an T-age-class model is a cycle for all initial orchards except the balanced orchard (proposition 5.2)—assuming that the age-yield relationship is monotonically increasing ($Q = T$). However, perennial tree yields generally decline before the end of the tree's life (e.g. Haworth and Vincent (1977)). Furthermore, proposition 3.1 of Mitra et al. (1991) states that the optimal replacement age of a single perennial tree occurs in the declining portion of the tree's age-yield relationship. Their proof of proposition 5.2 relies on collapsing the declining portion of the age-yield relationship, so that $Q = T$. Hence, the existence of cycles in an orchard management model of trees with declining yields has, to our knowledge, not yet been analyzed (i.e. $Q < T$). Our next step is to extend the analysis in this paper to a three-age-class model and identify the set of optimal stationary trajectories in the case where $f_1 < f_2 > f_3$ ($Q = 2$, $T = 3$).

3 A three-age-class orchard model

We now extend the model to include a third age-class. In this model, trees live for three periods before dying. The three periods are labeled: young, mature, and old trees. As before, the grower's objective is to maximize the discounted benefits from the stream of harvests from each type of tree over an infinite time horizon.

The statement of the problem is:

$$V(x_{10}, x_{20}, x_{30}) = \max_{\mathbf{x}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{15}$$

subject to

$$c_t \equiv f_1 x_{1t} + f_2 x_{2t} + f_3 x_{3t}$$

$$x_{1t} + x_{2t} + x_{3t} = 1 \tag{16}$$

$$x_{1,t+1} = z_{1t} + z_{2t} + x_{3t} \tag{17}$$

$$x_{2,t+1} = x_{1t} - z_{1t} \tag{18}$$

$$x_{3,t+1} = x_{2t} - z_{2t} \tag{19}$$

$$z_{st} \geq 0 \tag{20}$$

$$x_{st} \geq 0 \tag{21}$$

To analyze this model we adopt the approach of Salo and Tahvonen (2003), but adapt it to the orchard model context. To simplify the statement of the Lagrangian, several variables can be eliminated without changing the structure of the problem. First, eliminate x_{1t} by using the land constraint (eq 16). Next eliminate the z_{st}'s using equations 17, 18, and 19. This reduces the problem to:

$$V(x_{10}, x_{20}, x_{30}) = \max_{\mathbf{x}_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{22}$$

subject to

$$c_t \equiv f_1(1 - x_{2t} - x_{3t}) + f_2 x_{2t} + f_3 x_{3t}$$

$$x_{2,t+1} \leq 1 - x_{2t} - x_{3t} \tag{23}$$

$$x_{3,t+1} \leq x_{2t} \tag{24}$$

$$x_{st} \geq 0 \tag{25}$$

The Lagrangian of the reduced problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_{2t}(1 - x_{2t} - x_{3t} - x_{2,t+1}) + \lambda_{3t}(x_{2t} - x_{3,t+1}) + \rho_t z_t + \theta_t(z_t - x_{1t})\right]$$

This leads to the KKT conditions

$$\beta^{-t}\frac{\partial \mathcal{L}}{\partial x_{2,t+1}} = -\lambda_{2t} + \beta[(f_2 - f_1)u'(c_{t+1}) - \lambda_{2,t+1} + \lambda_{3,t+1}] \leq 0 \tag{26}$$

$$\beta^{-t}\frac{\partial \mathcal{L}}{\partial x_{3,t+1}} = -\lambda_{3t} + \beta[(f_3 - f_1)u'(c_{t+1}) - \lambda_{2,t+1}] \leq 0 \tag{27}$$

$$x_{s,t+1} \geq 0;\; x_{s,t+1}\frac{\partial \mathcal{L}}{\partial x_{2,t+1}} = 0 \tag{28}$$

$$\lambda_{2,t+1} \geq 0;\; \lambda_{2,t+1}(1 - x_{2t} - x_{3t} - x_{2,t+1}) = 0 \tag{29}$$

$$\lambda_{3,t+1} \geq 0;\; \lambda_{3,t+1}(x_{2t} - x_{3,t+1}) = 0 \tag{30}$$

The focus of the following analysis will be on the existence and comparative statics of cyclical stationary optimal trajectories. Three periods is the longest possible cycle, since there are only three age-classes. In a three period cycle, the cross-vintage bounds hold with equality, and the land allocation in period $t + 3$ is equal to the land allocation in period t.

Assuming the derivatives with respect to the state variables hold with equality (interior optimum), we can use equations 26 and 27 to derive expressions for the Lagrangian multipliers for each period in the cycle.

$$\begin{bmatrix} \lambda_{2t} \\ \lambda_{2,t+1} \\ \lambda_{2,t+2} \end{bmatrix} = \frac{\beta}{1-\beta^3} \begin{bmatrix} \beta^2(f_1 - f_3) & (f_2 - f_1) & \beta(f_3 - f_2) \\ \beta(f_3 - f_2) & \beta^2(f_1 - f_3) & (f_2 - f_1) \\ (f_2 - f_1) & \beta(f_3 - f_2) & \beta^2(f_1 - f_3) \end{bmatrix} \begin{bmatrix} u'(c_t) \\ u'(c_{t+1}) \\ u'(c_{t+2}) \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} \lambda_{3t} \\ \lambda_{3,t+1} \\ \lambda_{3,t+2} \end{bmatrix} = \frac{\beta}{1-\beta^3} \begin{bmatrix} \beta^2(f_2 - f_3) & (f_3 - f_1) & \beta(f_1 - f_2) \\ \beta(f_1 - f_2) & \beta^2(f_2 - f_3) & (f_3 - f_1) \\ (f_3 - f_1) & \beta(f_1 - f_2) & \beta^2(f_2 - f_3) \end{bmatrix} \begin{bmatrix} u'(c_t) \\ u'(c_{t+1}) \\ u'(c_{t+2}) \end{bmatrix} \tag{32}$$

A sufficient condition for a three period interior cycle to be optimal is that all six Lagrange multipliers must be non-negative. A sufficient condition for this is that the Faustmann age for the related single tree replacement problem must be 3.

For a perennial crop, the Faustmann age, m, is defined by

$$\frac{\sum_{s=1}^{m} \beta^{s-1} f_s}{1-\beta^m} \geq \frac{\sum_{s=1}^{j} \beta^{s-1} f_s}{1-\beta^j}$$

The Faustmann age is unique if this condition is satisfied with strict inequality.

For a three-age-class model, a unique Faustmann age of three implies two inequalities

$$\begin{aligned} \frac{f_1}{1-\beta} &< \frac{f_1 + \beta f_2 + \beta^2 f_3}{1-\beta^3} \quad \Leftrightarrow \quad f_1 < \frac{f_2 + \beta f_3}{1+\beta} \\ \frac{f_1 + \beta f_2}{1-\beta^2} &< \frac{f_1 + \beta f_2 + \beta^2 f_3}{1-\beta^3} \quad \Leftrightarrow \quad f_1 < f_3(1+\beta) - \beta f_2 \end{aligned}$$

Assuming $m = 3$, in a balanced (non-cyclical) stationary orchard, one third of the land is allocated to each of the three age classes each period. A constant land allocation leads to constant utility and to constant marginal utility. The λ's are therefore constant, reducing the sufficiency conditions to

$$\lambda_2 = (f_2 - f_1) + \beta(f_3 - f_2) + \beta^2(f_1 - f_3) \geq 0 \quad \Leftrightarrow \quad f_1 \leq \frac{f_2 + \beta f_3}{1+\beta}$$

$$\lambda_3 = (f_3 - f_1) + \beta(f_1 - f_2) + \beta^2(f_2 - f_3) \geq 0 \quad \Leftrightarrow \quad f_1 \leq f_3(1+\beta) - f_2$$

Therefore we can see how assuming a unique Faustmann age of three guarantees the optimality of the balanced stationary orchard

$$f_1 < \frac{f_2 + \beta f_3}{1+\beta} \Rightarrow \lambda_2 > 0$$

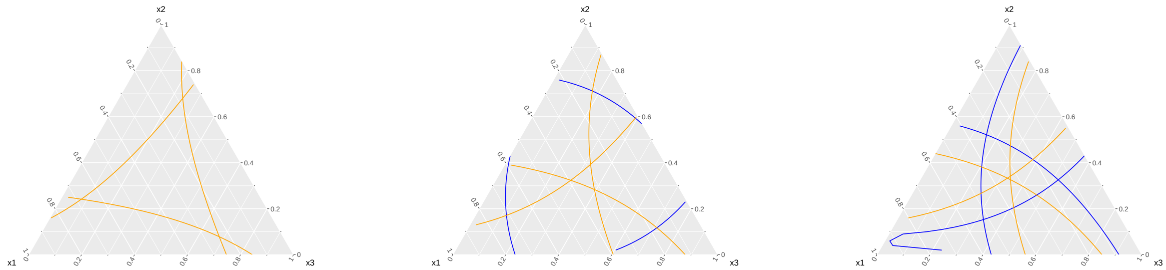
$$f_1 < f_3(1+\beta) - \beta f_2 < f_3(1+\beta) - f_2 \Rightarrow \lambda_3 > 0$$

Since both multipliers are strictly positive at the balanced stationary orchard, by the continuity of 31 and 32 there exists some set of deviations ϕ_1, ϕ_2 and ϕ_3, where $\phi_1 + \phi_2 + \phi_3 = 0$ and at least one ϕ_s is non-zero, such that the cycle defined by $\mathcal{O} = \{\frac{1}{3} + \phi_1, \frac{1}{3} + \phi_2, \frac{1}{3} + \phi_3\}$ is also optimal.

Therefore when $m = 3$ and for all $\beta \in (0, 1)$ there exist cyclical trajectories of period 3 that are optimal.

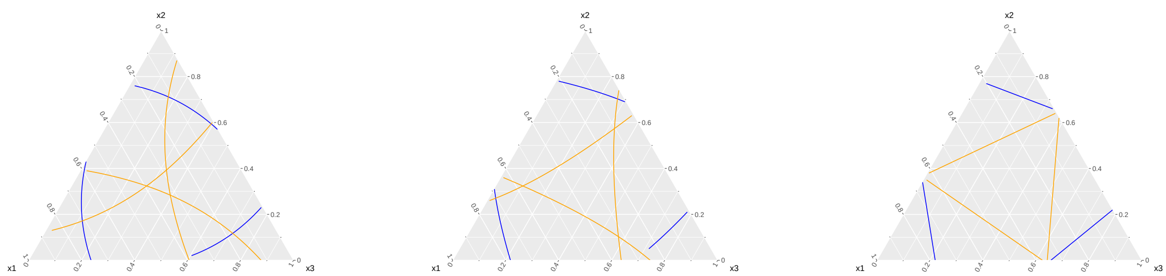
3.1 Numerical comparative statics for three-age-class model

Numerical comparative statics of the optimal stationary region are presented in figures 1–3. Note there are numerical issues in the region boundaries for several of the figures. These will be corrected in a future draft. Moreover, they do not appear to affect the overall pattern.



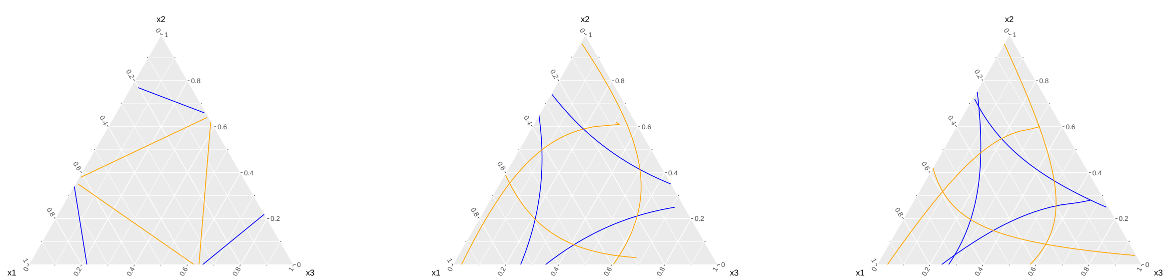
(a) $\beta = 0.5$ (b) $\beta = 0.75$ (c) $\beta = 0.9$

Figure 1: The optimal stationary region is the intersection of the two colored 'triangles'. When increasing the discount factor (more patient), the optimal stationary region shrinks. Adjusting β appears to affect the 'distance' between the two types of constraint. $f_1 = \frac{1}{3}$; $f_2 = 1$; $f_3 = \frac{2}{3}$



(a) $f_3 = \frac{2}{3}$ (b) $f_3 = \frac{5}{6}$ (c) $f_3 = 1$

Figure 2: When old trees are less productive, the λ_3 constraint set (orange) is binding. Increasing the productivity of old trees increases the size of λ_3 constraint set. $\beta = 0.75$, $f_1 = \frac{1}{3}$; $f_2 = 1$



(a) $f_3 = 1$ (b) $f_3 = 2$ (c) $f_3 = 3$

Figure 3: Further increasing the yield of old trees shrinks both constraint sets, but now the λ_2 constraint set (blue) is also binding. The intersection becomes a 'hexagon'. $\beta = 0.75$, $f_1 = \frac{1}{3}$; $f_2 = 1$

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