International Environmental Agreements as an Equilibrium Choice in a Differential Game

Ken-Ichi Akao*, Kenko Uchida**, Yasuaki Wasa**

* Faculty of Social Sciences, Waseda University, Tokyo 169-8050, Japan

** Department of Electrical Engineering and Bioscience, Waseda University, Tokyo 169-8555, Japan

January 31, 2018

Abstract: In a differential game for international pollution control, Dockner and Long (Dockner, E.J. and N.V. Long (1993) "International Pollution Control: Cooperative versus Noncooperative Strategies," Journal of Environmental Economics and Management 25: 13-29) show the existence of multiple Markov Nash equilibria and suggest the interpretation that an international environmental negotiation may be a preplay communication to choose a better Nash equilibrium. Following their idea, we examine the payoff dominant equilibrium in Markov perfect Nash equilibria (MPNEs) and open-loop Nash equilibria (OLNEs). By allowing a discontinuous strategy, we reproduce the Dockner and Long's most conservative equilibrium as an MPNE (the DL MPNE), with the globally asymptotically stable steady state which converges to an efficient steady state as the discount rate goes to zero. We show that the DL MPNE is payoff dominant over the pollution levels greater than or equal to the steady state in the MPNEs. When an open-loop strategy is also available, there is a unique OLNE and it may dominate the DL MPNE if the pollution level is high. This indicates that a renegotiation to switch from the OLNE to the DL MPNE may occur when pollution is improved.

JEL Classification Code: C72, H23, Q58

Keywords: International environmental agreement, equilibriumranking, payoffdominant equilibrium, differential game

* Corresponding author: akao@waseda.jp, 1-6-1 Nishiwaseda Shinjuku, Tokyo 169-8050.

1 Introduction

In the absence of global government, international negotiation and agreement are the core for the solution of global environmental issues. Since the first international United Nations Conference on the Human Environment in 1972, several environmental conventions and protocols have been adopted. Some agree long term targets and some agree to the schedules of pollution control. The latest former example is the Paris Agreement of the United Nations Framework Convention on Climate Change (UNFCCC) which sets a long term target that holding the increase in the global average temperature to well below 2°C above pre-industrial levels. A prominent and successful latter example is the Montreal Protocol on Substances that Deplete the Ozone Layer, which has set the years to end the production and the consumption of the ozone depleting substances. The other latter examples include the Protocol on the Reduction of Sulphur Emission of the Convention on Long-range Trans-boundary Air Pollution and the Kyoto Protocol of UNFCCC, although in the Kyoto Protocol, the commitment was fragile and some countries seceded.

How can we understand these agreements? Dockner and Long (1993) model an international environmental negotiation as a differential game. Showing the existence of multiple Markov Nash equilibria, they propose a compelling interpretation about an international environmental negotiation: It may be a preplay communication to choose a better Nash equilibrium. This interpretation is supported theoretically by the notion of payoff dominance (Harsanyi and Selten, 1988). The negotiation and agreement can be a process which reduces the uncertainty about players' behaviors and prevents players from going to the other equilibrium.

By following their idea, this paper investigates what equilibrium is chosen as an international environmental agreement. Formally, this paper studies Dockner and Long's (1993) symmetric two player differential game model about pollution control. We consider symmetric stationary continuous and discontinuous MPNEs. We also consider symmetric open-loop Nash equilibria (OLNEs).[1] We rank their payoffs for each pollution level, with special interest in the payoff dominant equilibrium. Furthermore, we examine the efficiency of the steady state of the payoff dominant equilibrium.

[1] Since we only consider a symmetric equilibrium and a stationary Markov strategy, we omit the terms "symmetric" and "stationary" hereafter.

In the model, there is a continuum of MPNEs and a unique OLNE. An MPNE, whether it is continuous or discontinuous, has a unique steady state which is globally asymptotically stable. Thus the agreement of a long-term target such as the 2°C target in Paris Agreement can be interpreted as a choice of a steady state and, at the same time, as a choice of an MPNE. Another type of agreement which commits a schedule of actions like the Montreal Protocol corresponds to choosing an OLNE, the strategy of which is a function of time.

Dockner and Long (1993) show that, for the most conservative Markov equilibrium, its steady state, hereafter referred to as the DL steady state, converges to a Pareto efficient steady state as the discount rate approaches zero. Therefore, when the discount rate is very low, we may have an approximately efficient outcome in the long-run even in the noncooperative circumstances. This result has an important implication, because, in cooperative game theory, many studies find that a coalition by many countries is difficult in an international environmental agreement (Barrett, 2005).

The DL steady state, however, has been argued in two points. First, the equilibrium strategy is not defined over the state space.[2] Second, as Rubio and Casino (2002) argue, the steady state is not stable. With a small perturbation, the state variable moves away from the steady state and reaches to the state where the strategy is not defined.

In this paper, we reproduce the DL steady state as the globally asymptotically stable steady state of MPNE, by allowing a discontinuous strategy. Therefore we resolve the criticisms and strengthen the Dockner and Long's efficiency result.

Noncooperative dynamic game approach on international environmental agreement has become popular since about 1990s. Ploeg and de Zeeuw (1992) are one of the earliest works. They compare the cooperative solution, an OLNE and a Markov Nash equilibrium. But only specific equilibria are compared. Tsutsui and Mino (1990) showed that a linear quadratic differential game model has a continuum of nonlinear Markov Nash equilibria. Dockner and Long (1993) showed this multiplicity in a pollution control game. Besides the aforementioned results, they also showed that the nonlinear Markov Nash equilibria are more environmentally conservative than the linear Markov Nash equilibrium.[3] Rowat

[2] The problem was recognized by Tsutsui and Mino (1991) who showed the multiplicity of Markov Nash equilibria in a similar model to Dockner and Long (1993). Dockner and Wagener (2007) use the term "local Markov perfect Nash equilibrium" when a Markov equilibrium is locally subgame perfect.

[3] A property of the linear quadratic model as used by Dockner and Long (1993) is that the elasticity of marginal utility in control variable is increasing. Wirl (2007) shows that, if the elasticity of marginal utility is decreasing, the nonlinear

(2000) analyzes a more general linear quadratic models encompassing Tsutsui and Mino (1991) and Dockner and Long (1993) and studies MPNEs of n player asymmetric games in the context of global warming. A contribution of this paper is that we rank equilibrium payoffs over the state space, which, in our knowledge, has not been studied.

Our ranking results are summarized as follows. The DL steady state divides the state space into two intervals. In the larger interval, we show that the payoff ranking between two MPNEs is preserved. Therefore, there is an MPNE which is the payoff dominant equilibrium over the interval. The MPNE contains the Dockner and Long's most conservative Markov equilibrium and its globally asymptotically stable steady state is the DL steady state. For the payoff comparison between an MPNE and the OLNE, there may be a unique level of pollution stock such that if pollution is greater than it, the OLNE dominates the MPNE and vice versa. Even the payoff dominant MPNE may be dominated by the OLNE if the pollution stock level is high. This indicates that, if an international environmental negotiation is a process to choose the payoff dominant equilibrium, a renegotiation may occur to switch from the OLNE to the MPNE when pollution is improved.

In the interval less than the DL steady state, the payoff dominant strategy varies over the interval. However, if players keep to choose a better equilibrium through renegotiation, they eventually choose to stay at the DL steady state. This result reinforces the Dockner and Long's efficiency result.

The remainder of this paper is organized as follows. Section 2 shows the model and some basic results. Section 3 derives continuous and discontinuous MPNEs. Section 4 ranks their payoffs. Section 5 introduces the OLNE and shows the ranking results with MPNEs. Section 5 concludes.

2 Dockner-Long's pollution control game

2.1 Model

The game is a symmetric two player differential game. Time is continuous and denoted by $t \in [0, \infty)$. The players i, j are assumed to be identical. Let $P(t) \in \mathbb{R}_+$ be the pollution stock and $E_l(t) \in \mathbb{R}_+, l = i, j$

Markov strategies are less environmentally conservative than the linear Markov strategy.

be players' pollution emissions at t. The pollution stock evolves by

$$\dot{P}(t) = E_i(t) + E_j(t) - kP(t), \tag{2.1}$$

where $k > 0$ is the natural assimilation rate. With discount rate $r > 0$, player l's total payoff is given by

$$\int_0^{\infty} u(P(t), E_l(t))e^{-rt}dt, \quad l = i, j. \tag{2.2}$$

The instantaneous utility function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is quadratic:

$$u(P, E_l) = AE_l - (1/2)(E_l)^2 - (s/2)P^2, \quad A, s > 0, l = i, j. \tag{2.3}$$

The emission control space is given by $[0, A]$ since $u(P, E)$ is decreasing in E if $E > A$. Then, if $P \geq 2A/k$, $\dot{P} \leq 0$. Therefore, letting

$$\bar{P} = 2A/k, \tag{2.4}$$

we can restrict the state space for pollution stock to the finite interval $[0, \bar{P}]$.

Let

$$\mathcal{C}^P = \{\sigma : [0, \bar{P}] \to [0, A] | \sigma \text{ is a piecewise continuous function}\}. \tag{2.5}$$

The emission $E_l(t)$ for player $l = i, j$ is given by Markov strategy $\sigma_l \in \mathcal{C}^P$: $E_l(t) = \sigma_l(P(t))$. When σ_i and/or σ_j have a discontinuous point, the state equation (2.1) is not a standard differential equation but a differential equation with discontinuous right-hand side. Then we need to generalize the solution concept. In the Appendix, we show the definition of generalized solution. We restrict the strategy space, i.e. the set of available strategy pairs, to the following class. The strategy space Σ^2 is given by:

$$\Sigma^2 = \{(\sigma_i, \sigma_j) \in \mathcal{C}^P \times \mathcal{C}^P | \ \dot{P}(t) = \sigma_i(P(t)) + \sigma_j(P(t)) - kP(t) \text{ has a unique and absolutely continuous solution for any initial value } P(0) = P_0 \in [0, \bar{P}]\}. \tag{2.6}$$

This restriction is necessary to define a meaningful equilibrium Also it enables us to treat the above differential equation as if it were a standard differential equation. Although we do not characterize Σ^2, we will check for the candidates of equilibrium strategy pair whether it belongs to the strategy space.

Let us also define the σ-projection of Σ^2 by

$$\Sigma(\sigma') = \{\sigma \in \mathcal{C}^P \,|\, (\sigma, \sigma') \in \Sigma^2\}, \tag{2.7}$$

for each $\sigma' \in \mathcal{C}^P$.[4] When the opponent player uses $\sigma' \in \mathcal{C}^P$, the payoff function $W : [0, \bar{P}] \times \Sigma(\sigma') \to \mathbb{R}$ is defined by

$$W(P, \sigma; \sigma') = \int_0^\infty u(P(t), \sigma(P(t)))e^{-rt}dt, \tag{2.8}$$

where $P(t)$ is the solution for

$$\dot{P}(t) = \sigma(P(t)) + \sigma'(P(t)) - kP(t), \quad P(0) = P. \tag{2.9}$$

Given opponent strategy $\sigma' \in \mathcal{C}^P$, the value function $V_m : [0, \bar{P}] \to \mathbb{R}$ is defined by

$$V_m(P; \sigma') = \sup_{\sigma} W(P, \sigma; \sigma') \text{ subject to } \sigma \in \Sigma(\sigma'). \tag{2.10}$$

Now we define an MPNE. Following Dockner and Long (1993), we confine an equilibrium strategy pair (σ_i, σ_j) to the symmetric class $\sigma_i = \sigma_j$.

Definition: $(\sigma_m^*, \sigma_m^*) \in \Sigma^2$ is an MPNE if for any $P \in [0, \bar{P}]$, $\sigma_m^*(P)$ is a unique optimal policy control for the problem (2.10) with $\sigma' = \sigma_m^*$.[5]

2.2 Some basic results

The value function V_m and the equilibrium pollution path $P_m(t)$ induced by MPNE (σ_m^*, σ_m^*) have the following property. These properties are utilized to find and rank MPNEs.

Lemma 2.1 *(i) V_m is bounded and continuous. (ii) $P_m(t)$ is monotone.*

[4] Note that for $\sigma' \in \mathcal{C}^P$, let $\sigma = -\sigma' + \sup\{\sigma'(P)\}$. Then $\sigma \in \mathcal{C}^P$ and $\dot{P} = \sigma(P) + \sigma'(P) - kP = \sup\{\sigma'(P)\} - kP$ is a standard differential equation. This shows $\Sigma(\sigma')$ is not an empty set for all $\sigma' \in \mathcal{C}^P$.

[5] We can show that an optimal path exists by applying Theorem 11.1 in Chapter 1 of Flemming and Soner (2006). See the Appendix.

Proof. (i) Since E and P are bounded and $r > 0$, V_m is bounded. The continuity follows from Ishii (2013, Proposition 1.2). (ii) If not, there is P such that $\dot{P} = 2\sigma_m^*(P) - kP$ takes both positive and negative signs. This is a contradiction since σ_m^* is a function. ■

For $\sigma \in \mathcal{C}^P$, define Hamiltonian H and maximized Hamiltonian H^* by

$$H(E, P, \lambda; \sigma) := [AE - (1/2)E^2 - (s/2)P^2] + \lambda[E + \sigma(P) - kP], \tag{2.11}$$

$$H^*(P, \lambda; \sigma) := \max\{H(E, P, \lambda; \sigma) | E \in [0, A]\}, \tag{2.12}$$

Lemma 2.2 *Given $(\sigma, \sigma') \in \Sigma^2$, if the payoff function W is differentiable with respect to P at P', then*

$$rW(P', \sigma; \sigma') = H(\sigma(P'), P', \partial W(P', \sigma; \sigma')/\partial P; \sigma'). \tag{2.13}$$

Proof. Let $P(t)$ be the solution for (2.9) with initial value $P(0) = P'$. If $\lim_{t\searrow 0} \sigma(P(t)) = \sigma(P')$, take $h > 0$. If $\lim_{t\nearrow 0} \sigma(P(t)) = \sigma(P')$, take $h < 0$. In both cases, the dynamic programing equation holds:

$$W(P', \sigma, \sigma') = \int_0^h u(P(t), \sigma(P(t)))e^{-rt}dt + W(P(t+h), \sigma, \sigma')e^{-rh}. \tag{2.14}$$

Then

$$-\frac{W(P(t+h), \sigma, \sigma')e^{-rh} - W(P, \sigma, \sigma')}{h} = \frac{1}{h}\int_0^h u(P(t), \sigma(P(t)))e^{-rt}dt$$

By taking limit $h \to 0$, we have

$$rW(P', \sigma, \sigma') = u(P', \sigma(P')) + \frac{\partial W(P', \sigma, \sigma')}{\partial P}\dot{P}(t), \tag{2.15}$$

which is equivalent to (2.13). ■

The following results characterize an MPNE.

Lemma 2.3 *(i) If (σ_m^*, σ_m^*) is an MPNE, and the value function V_m is differentiable at P, then the Hamilton-Jacobi-Bellman (HJB) equation holds:*

$$rV_m(P; \sigma_m^*) = H^*(P, \partial V_m(P; \sigma_m^*)/\partial P; \sigma_m^*). \tag{2.16}$$

(ii) If $(\sigma_m, \sigma_m) \in \Sigma^2$ and the associated payoff function $W(P, \sigma_m; \sigma_m)$ satisfies the HJB equation (2.16) for all $P \in [0, \bar{P}]$, then (σ_m, σ_m) is an MPNE and $W(P, \sigma_m; \sigma_m) = V_m(P; \sigma_m)$.

Proof. (i) Fix $P \in [0, \bar{P}]$. For any $\sigma \in \Sigma(\sigma_m)$, by the definitions of the value function (2.10) and the maximized Hamiltonian, and by Lemma 2.2, we have

$$
\begin{aligned}
rV_m(P; \sigma_m) &= \max_{\sigma \in \Sigma(\sigma_m)} rW(P, \sigma; \sigma_m) \\
&= \max_{\sigma \in \Sigma(\sigma_m)} H(\sigma(P), P, \partial W(P, \sigma, \sigma_m)/\partial P; \sigma_m) \\
&= H^*(P, \partial V_m(P; \sigma_m)/\partial P; \sigma_m).
\end{aligned}
$$

(ii) Fix $\sigma \in \Sigma(\sigma_m)$ and the initial value $P \in [0, \bar{P}]$. Let $P(t)$ be the solution for (2.9) with (σ, σ_m) and $P(0) = P$. By Bellman's principle of optimality, (2.16) holds for any $t \geq 0$, which implies that

$$
\begin{aligned}
rW(P(t), \sigma_m; \sigma_m) &\geq H(\sigma(P(t)), P(t), \partial W(P(t), \sigma_m, \sigma_m)/\partial P; \sigma_m) \\
&= u(P(t), \sigma(P(t))) + \frac{\partial W(P(t), \sigma_m, \sigma_m)}{\partial P} \dot{P}(t),
\end{aligned}
$$

and thus

$$
\frac{dW(P(t), \sigma_m; \sigma_m) e^{-rt}}{dt} \geq u(P(t), \sigma(P(t))) e^{-rt}. \tag{2.17}
$$

holds. Taking integral of (2.17) yields

$$
W(P, \sigma_m; \sigma_m) = W(P, \sigma_m; \sigma_m) - \lim_{t \to \infty} W(P(t), \sigma_m; \sigma_m) e^{-rt} \geq \int_0^\infty u(P(t), \sigma(P(t))) e^{-rt} dt.
$$

where the second equality follows from Lemma 2.1 (i). Since this inequality holds for all $\sigma \in \Sigma(\sigma_m)$ and all $P \in [0, \bar{P}]$, we conclude

$$
W(P, \sigma_m; \sigma_m) = V_m(P; \sigma_m).
$$

■

The following lemma is utilized to find an MPNE and to rank MPNEs.

Lemma 2.4 *Let (σ_m^*, σ_m^*) be an MPNE. For P at which $V_m(P; \sigma_m^*)$ is differentiable,*

$$\frac{\partial V_m(P;\sigma_m^*)}{\partial P}\begin{cases} \leq & -A \quad \text{if} \quad \sigma_m^*(P)=0 \\ = & \sigma_m^*(P)-A \quad \text{if} \quad \sigma_m^*(P)\in(0,A] \end{cases} \tag{2.18}$$

Proof. This follows from the Kuhn-Tucker conditions for the maximization of the Hamiltonian in (2.12). ■

The last lemma is utilized to construct a discontinuous MPNE and to rank MPNEs. The following function is a key:

$$\gamma_\infty(P) := (k/3)\,P + A/3. \tag{2.19}$$

Lemma 2.5 *Let (σ_m^*, σ_m^*) be an MPNE and fix $P \in [0, \bar{P}]$. (i) If $\sigma_m^*(P) \in (0, A)$ and $V_m(P; \sigma_m^*)$ is differentiable, then it holds that:*

$$V_m(P;\sigma_m^*) = r^{-1}\left[(3/2)\sigma_m^*(P)^2 - (A+kP)\,\sigma_m^*(P) - (s/2)P^2 + kAP\right] \tag{2.20}$$

(ii) Also let (σ'_m, σ'_m) be another MPNE and assume that $\sigma'_m(P) \in (0, A)$ and $V_m(P; \sigma'_m)$ is differentiable. If $\sigma_m^*(P) > \sigma'_m(P) \geq \gamma_\infty(P)$ or $\sigma_m^*(P) < \sigma'_m(P) \leq \gamma_\infty(P)$, then

$$V_m(P;\sigma_m^*) > V_m(P;\sigma'_m).$$

Proof. (i) By Lemmas 2.3 and 2.4,

$$\begin{aligned} V_m(P;\sigma_m^*) &= r^{-1}H^*(P, \sigma_m^*(P)-A; \sigma_m^*). \\ &= r^{-1}\left[\left(A\sigma_m^*(P) - (1/2)\sigma_m^*(P)^2 - (s/2)P^2\right) + \left(\sigma_m^*(P)-A\right)\left(2\sigma_m^*(P)-kP\right)\right] \\ &= r^{-1}\left[(3/2)\sigma_m^*(P)^2 - (A+kP)\,\sigma_m^*(P) - (s/2)P^2 + kAP\right] \end{aligned}$$

(ii) The right-hand of (2.20) as a function of $\sigma_m^*(P)$ is a strictly convex function and attains the global minimum at $\sigma_m^*(P) = \gamma_\infty(P)$. From this, the statement follows. ■

3 MPNEs

In this section, we find MPNEs by the following strategy. By Lemma 2.3, the solutions for the HJB equation are candidates of interior MPNE strategy. The other candidates are the corner strategy: $\sigma_m(P) = 0$. Their combinations are also the candidates. Using Lemma 2.1 and 2.3, we select MPNEs from these candidates. A candidate is indeed an MPNEs if the strategy pair belongs to Σ^2, the associated payoff function is continuous, and the HJB equation holds when the payoff function is differentiable. The $P-E$ plane is divided into two regions by $E = \gamma_\infty(P)$ in (2.19). By Lemma 2.5 (ii), at a discontinuous point of the strategy, the jump should be from one to another of the two regions, in order to keep the continuity of the value function.

3.1 Interior solutions

Assume that an MPNE strategy σ_m^* is differentiable. Differentiate the HJB equation (2.16). Then we have for $\sigma_m^* \in (0, A)$,

$$\frac{dE}{dP} = \frac{(k+r)E - A(k+r) + sP}{3E - A - kP}, \quad E = \sigma_m^*(P). \tag{3.1}$$

As Dockner and Long (1993) show,[6] this differential equation is solved as

$$K(E,P) := |-E - \alpha_a P + \beta_a|^{\xi_1} \, |-E - \alpha_b P + \beta_b|^{\xi_2} = K, \tag{3.2}$$

where K is the integral constant and

$$\alpha_h = -(k/3) - Z_h, \quad h = a, b \tag{3.3}$$

$$\beta_h = \frac{A}{3} - Z_h \frac{2A(k+r)}{rk + k^2 + 3s}, \quad h = a, b, \tag{3.4}$$

$$\xi_1 = \frac{Z_a}{Z_b - Z_a} < 0, \quad \xi_2 = \frac{-Z_b}{Z_b - Z_a} < 0, \tag{3.5}$$

with

$$Z_a = \frac{r}{6} + \frac{\sqrt{(r+2k)^2 + 12s}}{6} > 0, \quad Z_b = \frac{r}{6} - \frac{\sqrt{(r+2k)^2 + 12s}}{6} < 0. \tag{3.6}$$

[6] Also see Rowat (2000).

There are two singular solutions

$$E = -\alpha_h P + \beta_h, \quad h = a, b \tag{3.7}$$

and four families of non-singular solutions which are classified by the signs of $-E - \alpha_h P + \beta_h \neq 0$ $(h = a, b)$. We label them as

$$\text{I.} \ -E - \alpha_h P + \beta_h < 0 \quad h = a, b, \tag{3.8a}$$

$$\text{II.} \ -E - \alpha_a P + \beta_a > 0, \ -E - \alpha_b P + \beta_b < 0, \tag{3.8b}$$

$$\text{III.} \ -E - \alpha_h P + \beta_h > 0 \quad h = a, b, \tag{3.8c}$$

$$\text{IV.} \ -E - \alpha_a P + \beta_a < 0, \ -E - \alpha_b P + \beta_b > 0. \tag{3.8d}$$

Simple calculations show that

$$\alpha_a < 0, \ \alpha_b > 0; \ \beta_a \gtreqless 0 \text{ if } s \gtreqless (k+r)(k+2r), \ \beta_b \in (0, A) \tag{3.9}$$

<Figure 1>

Figure 1 illustrates these solutions as well as the two graphs, $E = \gamma_0(P)$ and $E = \gamma_\infty(P)$. They are defined by

$$E = \gamma_0(P) := -\frac{s}{k+r} P + A, \tag{3.10}$$

$$\text{and } E = \gamma_\infty(P) = \frac{k}{3} P + \frac{A}{3}. \tag{3.11}$$

$E = \gamma_0(P)$ is the locus of points that the numerator of (3.1) keeps zero. The nonsingular solution curves cross $E = \gamma_0(P)$ horizontally. On the other hand, $E = \gamma_\infty(P)$ is the locus of points that the denominator of (3.1) keeps zero and thus the solution curves cross the graph vertically. Note that $E = \gamma_\infty(P)$ has been introduced by (2.19). By Lemma 2.5 it provides information about the payoff ranking of MPNE.

Their intersection is given by

$$(P_\gamma^m, E_\gamma^m) = \left(\frac{2A(k+r)}{k^2+rk+3s}, \frac{A\,(s+k(k+r))}{k^2+rk+3s}\right). \tag{3.12}$$

As is readily verified, this point is also the intersection of the two singular solutions (3.7). In Figure 1, we also depict the nullcline of $\dot{P}=0$ which is given by

$$E = \omega(P) = \frac{k}{2}P. \tag{3.13}$$

The arrows of solution curves in the figure follow from (3.13).

When we consider continuous MPNEs, among the solutions of (3.2), the candidates consist of the singular solution $E=\sigma_b(P)$ such that

$$\sigma_b(P) := -\alpha_b P + \beta_b, \tag{3.14}$$

and the Type III solutions $E=\sigma_{III}(P)$ which satisfy

$$-\sigma_{III}(P) - \alpha_h P + \beta_h > 0 \quad h=a,b. \tag{3.15}$$

As seen from Figure 1, the other solutions go out of the control space $[0,A]$ or can not be defined over the state space $[0,\bar{P}]$ without a jump.

The following lemma gives some properties of Type III solution. We denote by (P_{III}^L, P_{III}^U) the domain of a Type III solution.

Lemma 3.1 *(i) $P_{III}^L \in [0, A(k+r)/s)$ and $P_{III}^U \in (A(k+r)/s, \beta_b/\alpha_b)$, where $A(k+r)/s$ is the P-intercept of $E=\gamma_0(P)$. (ii) $P_{III}^L = 0$ only if $s>(k+r)(k+2r)$. (iii) A Type III solution crosses the nullcline $E=\omega(P)$ at most twice.*

Proof. (i) This follows from the facts that the solutions cross $E=\gamma_0(P)$ horizontally from left to right and two solutions never cross. (ii) $s>(k+r)(k+2r)$ is equivalent to $\beta_a>0$. Therefore, if $s\le(k+r)(k+2r)$, any Type III solution cannot have initial values $(0,E_0)$ with $E_0\ge 0$. (iii) See the

Appendix. ■

3.2 Corner solution

A continuous MPNE, if it exists, consists of a Type III solution and the corner solution $\underline{\sigma}(P) = 0$. When $(\underline{\sigma}, \underline{\sigma})$ is played, the associated pollution path is decreasing, given by $P(t) = P_0 e^{-kt}$ where $P_0 \in (0, \bar{P}]$ is the initial pollution level. The associated payoff is given by

$$W(P_0, \underline{\sigma}; \underline{\sigma}) = \int_0^\infty \left(-\frac{s}{2} e^{-2kt} P_0^2\right) e^{-rt} dt = -\frac{sP_0^2}{2(2k+r)}. \tag{3.16}$$

The following lemma shows that the corner solution can constitute an MPNE only if it is used in an interval that does not contain $P = 0$.

Lemma 3.2 *(i) $\underline{\sigma}(P) = 0$ on $[0, P')$ does not constitute a Markov equilibrium for any $P' \in (0, \bar{P}]$. (ii) The following strategy constitutes an MPNE.*

$$\sigma_m^* = \begin{cases} \sigma_m & \text{if } P \in [0, P^c] \\ \underline{\sigma} & \text{if } P \in (P^c, \bar{P}] \end{cases}, \tag{3.17}$$

where σ_m is a Markov equilibrium strategy on $[0, P^c]$ such that

$$\lim_{P \nearrow P^c} \sigma_m(P) = 0 \tag{3.18}$$

$$\text{and } P^c \in [A(k+r)/s, \bar{P}). \tag{3.19}$$

Proof. (i) The payoff function (3.16) is differentiable but

$$\frac{\partial W(P_0, \underline{\sigma}; \underline{\sigma})}{\partial P} = -\frac{sP_0}{(2k+r)} > -A.$$

if $P_0 < A(2k+r)/s$. Therefore the condition in Lemma 2.4 is not satisfied. (ii) σ_m is a Markov equilibrium strategy on $[0, P^c]$ and every equilibrium pollution path enters and stays in on $[0, P^c]$. Then the proof completes if we show that on $[P^c, \bar{P}]$, the associated payoff function is continuous and the

HJB equation (2.16) holds. Fix the initial pollution level $P_0 \in (P^c, \bar{P}]$. Let $V_m(P^c; \sigma_m)$ be the value function at P^c. Since the arrival time at P^c is

$$T(P_0) = (1/k)\ln(P_0/P^c), \tag{3.20}$$

the payoff on $(P^c, P_0]$ is given by

$$\begin{aligned} W(P, \underline{\sigma}; \underline{\sigma}) &= \int_0^{T(P)} \left(-\frac{s}{2} e^{-2kt} P^2\right) e^{-rt} dt + V_m(P^c; \sigma_m) e^{-rT(P)} \\ &= -\frac{sP^2}{2(2k+r)} \left[1 - \left(\frac{P^c}{P}\right)^{2+\frac{r}{k}}\right] + V_m(P^c; \sigma_m) \left(\frac{P^c}{P}\right)^{\frac{r}{k}}. \end{aligned} \tag{3.21}$$

(3.21) shows that W is continuous at P^c. (3.21) also shows that W is differentiable. We show that $\lim_{P \searrow P^c} \partial W(P,\underline{\sigma};\underline{\sigma})/\partial P \leq -A$ and, for all $P \in (P^c, P_0]$, $\partial^2 W/\partial P^2 \leq 0$. Then, for all $P \in (P^c, P_0]$, $\partial W/\partial P \leq -A$ and the HJB equation (2.16) is satisfied. By Lemma 2.2,

$$\frac{\partial W(P, \underline{\sigma}; \underline{\sigma})}{\partial P} = -\frac{sP}{2k} - \frac{r}{kP} W(P, \underline{\sigma}; \underline{\sigma}), \tag{3.22}$$

and

$$\lim_{P \searrow P^c} \frac{\partial W(P, \underline{\sigma}; \underline{\sigma})}{\partial P} = -\frac{1}{kP^c} \left\{ \frac{s\,(P^c)^2}{2} + rV_m(P^c) \right\}. \tag{3.23}$$

Since V_m is continuous by Lemma 2.1, the HJB equation (2.16) implies that

$$\begin{aligned} rV_m(P^c) &= \lim_{P \nearrow P^c} rV_m(P) \\ &= \lim_{P \nearrow P^c} \left\{ [A\sigma_m^*(P) - (1/2)\sigma_m^*(P)^2 - (s/2)P^2] + (\sigma_m^*(P) - A)\,(2\sigma_m^*(P) - kP) \right\} \\ &= -(s/2)\,(P^c)^2 + AkP^c, \end{aligned} \tag{3.24}$$

where the inequality in the second line follows from Lemma 2.4 and the third line uses $\lim_{P \nearrow P^c} \sigma_m^*(P) = 0$. Substitute (3.24) into (3.23) and we have

$$\lim_{P \searrow P^c} \frac{\partial W(P, \underline{\sigma}; \underline{\sigma})}{\partial P} = -A. \tag{3.25}$$

Next, differentiate (3.22):

$$\begin{aligned}\frac{\partial W^2(P,\underline{\sigma};\underline{\sigma})}{\partial P^2} &= \frac{-s}{2k+r} + \left(\frac{rs}{2} + \frac{2k+r}{(P^c)^2} r V_m(P^c;\sigma_m)\right) \frac{r+k}{(2k+r)k^2} \left(\frac{P^c}{P}\right)^{\frac{2k+r}{k}} \\ &= \frac{-s}{2k+r} + \left(-sP^c + A\,(2k+r)\right) \frac{(r+k)}{(2k+r)\,kP^c} \left(\frac{P^c}{P}\right)^{r/k+2}, \qquad (3.26)\end{aligned}$$

where the second line is obtained by substituting (3.24). If $P^c \geq A(2k+r)/s$, we have $\partial^2 W/\partial P^2 \leq 0$. When $P^c < A(2k+r)/s$, since the right-hand side of (3.26) is strictly decreasing, if $\lim_{P\searrow P^c} \partial^2 W/\partial P^2 \leq 0$, then $\partial^2 W/\partial P^2 \leq 0$ for all $P \in (P^c, \bar{P})$. This condition is equivalent to $P^c \geq A(k+r)/s$ since

$$\lim_{P\searrow P^c} \frac{\partial W^2(P,\underline{\sigma};\underline{\sigma})}{\partial P^2} = \frac{-sP^c + A\,(r+k)}{kP^c}.$$

■

3.3 Continuous and discontinuous MPNEs

First we consider continuous MPNEs. Denote by $\sigma_{III}(P;P_0,E_0)$ be a Type III solution with the initial value (P_0,E_0). Recall that $\sigma_b(P)$ is a singular solution defined in (3.14)

Proposition 3.1 *(i) The linear continuous strategy:*

$$\sigma_b^*(P) = \begin{cases} \sigma_b(P) & \text{if } P \in [0,\alpha_b/\beta_b] \\ 0 & \text{if } P \in (\alpha_b/\beta_b,\bar{P}] \end{cases}, \qquad (3.27)$$

constitutes an MPNE.

(ii) Assume that

$$s > (k+r)\,(k+2r)\,. \qquad (3.28)$$

For each $E \in [0,\beta_a)$, a nonlinear continuous strategy

$$\sigma_c^*(P;0,E) = \begin{cases} \sigma_{III}(P;0,E) & \text{if } P \in [0,P^U_{III}] \\ 0 & \text{if } P \in (P^U_{III},\bar{P}] \end{cases}, \qquad (3.29)$$

constitutes an MPNE.

Proof. They follow from Lemmas 3.1 (ii) and 3.2 (ii). ■

By Lemma 3.2 (i), in order for a Type III solution with $P^L_{III} > 0$ to constitute an MPNE, we need a discontinuous strategy. We consider a discontinuous strategy which jumps from a Type I solution to Type III solution. Denote by P^d be a discontinuous point. Also denote by $\sigma_I(P; P_0, E_0)$ be a Type I solution with the initial value (P_0, E_0). Define function $E^d_I : [0, \bar{P}] \times [0, A] \to \mathbb{R}$ by

$$E^d_I(P^d, E^d) := 2\gamma_\infty(P^d) - E^d, \tag{3.30}$$

which gives the initial value of Type I solution at the discontinuous point P^d, with which the associated payoff function is continuous at P. We prepare a lemma.

Lemma 3.3 *The singular solution σ_b satisfies*

$$E^d_I(P, \sigma^*_b(P)) \in \begin{cases} (\sigma_a(P), \sigma_b(P)) & \text{if } P \in [0, P^m_\gamma) \\ (\sigma_b(P), \sigma_a(P)) & \text{if } P \in [0, P^m_\gamma) \end{cases}. \tag{3.31}$$

Proof. The graphs $E = \sigma_b(P)$, $E = E^d_I(P, \sigma^*_b(P))$ and $E = \sigma_a(P)$ cross at (P^m_γ, E^m_γ). For $P \in [0, \beta_b/\alpha_b)$,

$$\frac{dE^d_I(P, \sigma^*_b(P))}{dP} = \frac{2}{3}k + \alpha_b < -\alpha_a = \frac{d\sigma_a(P)}{dP} \tag{3.32}$$

These imply (3.31). ■

Proposition 3.2 *(i) Assume that Type III solution $\sigma_{III}(P; P^d, E^d)$ has a steady state P_{ss} such that $P_{ss} \geq P^d$ and the solution from P^d converges to P_{ss}. Also assume that Type I solution $\sigma_I(P; P^d, E^d_I(P^d, E^d))$ exists. Then the discontinuous strategy*

$$\sigma^*_d(P; P^d, E^d) = \begin{cases} \sigma_I(P; P^d, E^d_I(P^d, E^d)) & \text{if } P \in \left[0, P^d\right) \\ \sigma_{III}(P; P^d, E^d) & \text{if } P \in \left[P^d, P^U_{III}\right) \\ 0 & \text{if } P \in [P^U_{III}, \bar{P}] \end{cases}, \tag{3.33}$$

constitutes an MPNE. (ii) The conditions in (i) are satisfied with $(P^d, E^d) \in [0, P_\gamma^m) \times (\omega(P^d), \sigma_a(P^d))$. *(iii) For the singular solution* $\sigma_b(P)$*, there is no discontinuous MPNE strategy.*

Proof. (i) By Lemma 3.1 (iii) we can choose σ_{III} and $\left(P^d, E^d\right)$ as in the statement. The Appendix shows that the differential equation with the discontinuous right-hand side has a unique and absolutely continuous solution. When $P \geq P_{ss}$, the pollution level stays in $[P_{ss}^S, \bar{P}]$ and Proposition 3.1 is applied. When $P < P_{ss}$, the path satisfies the HJB equation by construction, except at $P = P^d$. Finally, the continuity of the value function at P^d follows from Lemma 2.5:

$$
\begin{aligned}
&\lim_{P \nearrow P^d} V_m(P; \sigma_d^*) - \lim_{P \searrow P^d} V_m(P; \sigma_d^*) \\
&= r^{-1}\left[(3/2)\left(\lim_{P \nearrow P^d} \sigma_d^*(P) + \lim_{P \searrow P^d} \sigma_d^*(P)\right) - (A + kP)\right]\left(\lim_{P \nearrow P^d} \sigma_d^*(P) - \lim_{P \searrow P^d} \sigma_d^*(P)\right) \\
&= r^{-1}\left[(3/2)\left(E_I^d - E_{III}^d\right) - \left(A + kP^d\right)\right]\left(E_I^d - E_{III}^d\right) \\
&= 3r^{-1}\left[\gamma_\infty(P^d) - \left(A + kP^d\right)/3\right]\left(E_I^d - E_{III}^d\right) \\
&= 0,
\end{aligned}
$$

(ii) The existence of a steady state and the convergence follow from $E^d > \omega(P^d)$. This inequality also implies that

$$
E_I^d(P^d, E^d) < E_I^d(P^d, \omega(P^d) = \frac{2}{3}A + \frac{k}{6}P < A. \tag{3.34}
$$

Finally, since $E_I^d(0, \sigma_a(0)) > \sigma_b(0)$ and $E_I^d(P_\gamma^m, \sigma_a(P_\gamma^m)) = \sigma_b(P_\gamma^m)$, we have for $P^d \in [0, P_\gamma^m)$, $E_I^d(P^d, \sigma_a(P^d)) > \sigma_b(P_\gamma^m)$ and thus

$$
E_I^d(P^d, E^d) > \sigma_b(P_\gamma^m) \tag{3.35}
$$

(3.34) and (3.35) ensure that the Type I solution exists. (iii) By (3.31) in Lemma 3.3, there is no Type I solution with initial value $\left(P, E_I^d(P, \sigma_b(P))\right)$, which proves the statement. ■

Proposition 3.2 (ii) shows the existence of a discontinuous MPNE. We have a continuum of MPNEs which have the same discontinuous point. We also have a continuum of MPNEs which share a Type III solution.

3.4 Dockner and Long's efficiency result

Now we look at a specific MPNE.

<Figure 2>

Figure 2 shows the strategy (3.33) whose discontinuous point is the tangent to the nullcline $E = \omega(P)$. Denote the strategy by σ^{DL} and refer to it as the DL strategy. The tangent point satisfies

$$\frac{(k+r)E - A(k+r) + sP}{3E - A - kP} = \frac{k}{2} \text{ and } 2E - kP = 0.$$

Therefore it is given by:

$$(P_{ss}^{DL}, E_{ss}^{DL}) = \left(\frac{2A(k+2r)}{k^2 + 2rk + 4s}, \frac{Ak(k+2r)}{k^2 + 2rk + 4s} \right), \tag{3.36}$$

which is the Dockner and Long's (1993) most conservative steady state. They showed that the steady state (3.36) coincides the one of the cooperative solution

$$(P_{ss}^{c}, E_{ss}^{c}) = \left(\frac{2A\,(r+k)}{k^2 + rk + 4s}, \frac{Ak\,(r+k)}{k^2 + rk + 4s} \right). \tag{3.37}$$

when the discount rate is zero.[7] They conclude that if the discount rate is sufficiently low, the use of nonlinear Markov strategies can be a "substitute" for fully coordinating environmental policies (Dockner and Long, 1993, p.24). However, as mentioned in the Introduction, their Markov equilibrium is not subgame perfect and the steady state is not stable.

Here we reproduce their most conservative Markov equilibrium as a discontinuous MPNE (3.33), referred to as the DL MPNE. The associated Type III solution is $\sigma_{III}(P; P_{ss}^{DL}, E_{ss}^{DL})$. The left limit of

[7] The cooperative problem is given by:

$$V^c(P) = \max \int_0^\infty u(P(t), E(t)) e^{-rt} dt$$
$$\text{subject to } \dot{P}(t) = 2E(t) - kP(t), P(0) = P \in [0, \bar{P}].$$

pollution emission at the discontinuous point E_I^{DL} satisfies

$$\lim_{P \nearrow P^d} \sigma^{DL}(P) := E_I^{DL} = E_I^d(P_{ss}^{DL}, E_{ss}^{DL}) = \frac{k^2 + 2rk + (8/3)\,s}{k^2 + 2rk + 4s} A. \tag{3.38}$$

We need a condition for the existence of Type I solution $\sigma_I(P; P^d, E_I^{DL})$.

Proposition 3.3 *If*

$$12s + (3k - 2r)(k + 4r) > 0, \tag{3.39}$$

the DL strategy

$$\sigma^{DL}(P) = \begin{cases} \sigma_I(P; P_{ss}^{DL}, E_I^{DL}) & \text{if } P \in \left[0, P_{ss}^{DL}\right) \\ \sigma_{III}(P; P_{ss}^{DL}, E_{ss}^{DL}) & \text{if } P \in \left[P_{ss}^{DL}, P_{III}^{U}\right) \\ 0 & \text{if } P \in [P_{III}^{U}, \bar{P}] \end{cases} \tag{3.40}$$

constitutes an MPNE.

Proof. We show that $\sigma_a\left(P_{ss}^{DL}\right) < E_I^{DL} \le A$. Then the statement follows from Proposition 3.2. $E_I^{DL} \le A$ follows from (3.38). For the other inequality, observe

$$\begin{aligned} & \sigma_a\left(P_{ss}^{DL}\right) < E_I^{DL}. \\ \Longleftrightarrow & \frac{k^2 + 2rk + (8/3)\,s}{k^2 + 2rk + 4s} A - \left(\frac{1}{3} \frac{A\,(2rs - ks)\left(r + \sqrt{(r+2k)^2 + 12s}\right)}{(3s + kr + k^2)\,(4s + 2kr + k^2)} + \frac{1}{3} \frac{A\left(4s + 6kr + 3k^2\right)}{(4s + 2kr + k^2)} \right) > 0. \\ \Longleftrightarrow & \frac{4}{3} A \frac{s}{k^2 + 2rk + 4s} - \frac{1}{3} \frac{As\,(2r - k)\left(r + \sqrt{(r+2k)^2 + 12s}\right)}{(3s + kr + k^2)\,(4s + 2kr + k^2)} > 0. \end{aligned}$$

If $2r - k \le 0$, then the inequality holds. Therefore assume $2r - k > 0$. Then the inequality is equivalent to

$$\begin{aligned} & \frac{\left(4k^2 + 4rk + 12s\right)}{(2r - k)} > r + \sqrt{(r+2k)^2 + 12s} \\ \Longleftrightarrow & \left(\frac{\left(4k^2 + 4rk + 12s\right)}{(2r - k)} - r \right)^2 - \left((r+2k)^2 + 12s\right) > 0 \\ \Longleftrightarrow & 12s + (3k - 2r)(k + 4r) > 0. \end{aligned}$$

Note that the equality holds if $2r - k \leq 0$. ■

If r is small, then (3.39) is satisfied. Therefore, we reproduce and strengthen the Dockner and Long's efficiency result as the result about an MPNE, where the equilibrium strategy σ^{DL} is defined over the state space, and the steady state $\left(P_{ss}^{DL}, E_{ss}^{DL}\right)$ is a globally asymptotically stable.

4 Payoff ranking in MPNEs

Corresponding to the Dockner and Long's interpretation of an international environmental negotiation, we examine the payoff ranking in equilibria. The following lemma gives the result in the corner solution part.

Lemma 4.1 *Let σ_1^* and σ_2^* be MPNE strategies and P_1^c and P_2^c, respectively, their kink points where the corner solution starts. If $P_1^c < P_2^c$, $V_m(P;\sigma_1^*) > V_m(P;\sigma_2^*)$ for $P \in [P_1^c, \bar{P}]$.*

Proof. Since $\sigma_1^*(P) = 0 < \sigma_2^*(P)$ for $P \in [P_1^c, P_2^c)$, $V_m(P;\sigma_1^*) > V_m(P;\sigma_2^*)$ on the interval. At $P = P_2^c$, by (3.21) and (3.24)

$$\begin{aligned} V_m(P_2^c;\sigma_2^*) &= \left(-(s/2)\left(P_2^c\right)^2 + AkP_2^c\right)/r, \\ V_m(P_2^c;\sigma_1^*) &= -\frac{s\left(P_2^c\right)^2}{2(2k+r)}\left[1-\left(\frac{P_1^c}{P_2^c}\right)^{2+\frac{r}{k}}\right] + \left(\frac{-(s/2)\left(P_1^c\right)^2 + AkP_1^c}{r}\right)\left(\frac{P_1^c}{P_2^c}\right)^{\frac{r}{k}}. \end{aligned}$$

From them, we have

$$V_m(P_2^c;\sigma_1^*) - V_m(P_2^c;\sigma_2^*) = \left[\left(\frac{sk\left(P_2^c\right)^2}{2k+r} - AkP_2^c\right)\left(P_2^c\right)^{\frac{r}{k}} - \left(\frac{sk\left(P_1^c\right)^2}{2k+r} - AkP_1^c\right)\left(P_1^c\right)^{\frac{r}{k}}\right]\frac{k}{r}\left(\frac{1}{P_2^c}\right)^{\frac{r}{k}} > 0, \tag{4.1}$$

since

$$\frac{d}{dP}\left(\frac{skP^2}{2k+r} - AkP\right)P^{\frac{r}{k}} = \left(sP - A\left(k+r\right)\right)P^{\frac{r}{k}}$$

and a kink point satisfies (3.19). For $P \in (P_2^c, \bar{P}]$, since the strategies are the same and the the inequality is preserved. ■

We begin the equilibrium ranking with the comparison of the linear MPNE and the other MPNEs.

Proposition 4.1 *The linear MPNE strategy σ_b^* defined in (3.27) is dominated by the other MPNE strategies defined in (3.29) and (3.33) over the state space.*

Proof. A nonlinear MPNE consists of Type I and Type III solutions. By Lemma 3.3, Type I and Type III solutions σ_I, σ_{III} satisfy

$$\sigma_I(P) > \max\{\sigma_a(P), \sigma_b(P), E_I^d(P, \sigma_b(P))\} > \gamma_\infty(P), \tag{4.2}$$

$$\sigma_{III}(P) < \min\{\sigma_a(P), \sigma_b(P), E_I^d(P, \sigma_b(P))\} < \gamma_\infty(P). \tag{4.3}$$

Then the statement follows from Lemma 2.5 (ii). ■

For the comparison in the nonlinear MPNEs, recall that there are a two dimensional continuum of equilibria: one stems from the arbitrary choice of discontinuous points with a fixed nonsingular solution and the other stems from the arbitrary choice of nonsingular solutions with a fixed discontinuous point. First we show that if two MPNEs share the same discontinuous point or both are continuous MPNEs, then their payoff order is preserved over the state space.

Proposition 4.2 *Let σ_1^* and σ_2^* be MPNE strategies given by (3.29) or (3.33). Assume that they are both continuous MPNEs or both discontinuous MPNEs. In the latter case, also assume that they have the same discontinuous point P^d. Let $P^\# = P^d$ if P^d exists and $P^\# = 0$ if it does not exist. If there is a pollution level P that satisfies $P \in [P^\#, \bar{P}]$ and $\sigma_1^*(P) < \sigma_2^*(P)$ or $P \in [0, P^\#)$ and $\sigma_1^*(P) > \sigma_2^*(P)$, then σ_1^* dominates σ_2^*, i.e.*

$$V_m^*(P; \sigma_1^*) > V_m^*(P; \sigma_2^*) \text{ for all } P \in [0, \bar{P}].$$

Proof. In the part of the Type I and III solutions, the order is preserved and the result follows from Lemma 2.5 (ii). In the corner solution part, it follows from Lemma 4.1. ■

Next we fix a nonsingular solution and examine how a discontinuous point affects the payoff. We prepare a lemma.

Lemma 4.2 *Let σ_{d1}^* and σ_{d2}^* be the the discontinuous MPNE strategies defined in (3.30). These strategies are different only in their discontinuous points. Denote them, respectively, by P_1^d and P_2^d and* *assume that $P_1^d < P_2^d$. Then their value functions satisfies*

$$V_m(P;\sigma_{d1}^*) \geq V_m(P;\sigma_{d2}^*)$$

with strict inequality when $P \in [0, P_2^d)$.

Proof. Since the strategies are the same for $P \in [P_2^d, \bar{P}]$, $V_m(P;\sigma_{d1}^*) = V_m(P;\sigma_{d2}^*)$ on this interval. For $P \in [P_1^d, P_2^d)$, $\sigma_{d1}^*(P) < \sigma_{d2}^*(P)$, and we have

$$\begin{aligned} V_m(P;\sigma_{d1}^*) &= V_m(P_1^d;\sigma_{d1}^*) - \int_P^{P_1^d} \frac{\partial V_m(P;\sigma_{d1}^*)}{\partial P} dP \\ &= V_m(P_1^d;\sigma_{d1}^*) - \int_P^{P_1^d} \left(\sigma_{d1}^*(P) - A\right) dP \\ &> V_m(P_2^d;\sigma_{d2}^*) - \int_P^{P_1^d} \left(\sigma_{d2}^*(P) - A\right) dP \\ &= V_m(P_2^d;\sigma_{d2}^*) - \int_P^{P_1^d} \frac{\partial V_m(P;\sigma_{d2}^*)}{\partial P} dP = V_m(P;\sigma_{d2}^*) \end{aligned} \tag{4.4}$$

where we use $V_m(P_2^d;\sigma_{d1}^*) = V_m(P_2^d;\sigma_{d2}^*)$ and (2.18). Thus $V_m(P;\sigma_{d1}^*) > V_m(P;\sigma_{d2}^*)$ for $P \in [P_1^d, P_2^d)$. In particular, $V_m(P_1^d;\sigma_{d1}^*) > V_m(P_1^d;\sigma_{d2}^*)$. By Lemma 2.5 (ii), this implies $\left(P_1^d, \sigma_{d1}^*(P_1^d)\right) > \left(P_1^d, \sigma_{d2}^*(P_1^d)\right)$, and this inequality is preserved on $[0, P_1^d)$, since $\left(P, \sigma_{dh}^*(P)\right), h = 1, 2$ are the solutions of the differential equation (3.1). Then, again by Lemma 2.5 (ii), we have $V_m(P_1^d;\sigma_{d1}^*) > V_m(P_1^d;\sigma_{d2}^*)$ for $P \in [0, P_1^d)$. ■

Define the following discontinuous strategy.

Definition: For a Type III solution σ_{III} which has an unstable steady state $\left(P_{ss}^U, E_{ss}^U\right)$, the maximum MPNE is defined by

$$\sigma_d^M(P;\sigma_{III}) = \begin{cases} \sigma_I(P; P_{ss}^U, E_I^d\left(P_{ss}^U, E_{ss}^U\right)) & \text{if } P \in \left[0, P_{ss}^U\right] \\ \sigma_{III}(P; P_{ss}^U, E_{ss}^U) & \text{if } P \in (P_{ss}^U, P_{III}^U) \\ 0 & \text{if } P \in [P_{III}^U, \bar{P}] \end{cases} . \tag{4.5}$$

Proposition 4.3 *(i) Fix a Type III solution σ_{III} which has an unstable steady state $\left(P_{ss}^U, E_{ss}^U\right)$. Con-*

sider a family of discontinuous MPNEs $\sigma_d(P; P^d, \sigma_{III}(P^d))$. *Then, it holds that*

$$V_m(P; \sigma_d^M(P; \sigma_{III})) = \max_{P^d \in [P_{ss}^U, P_{III}^U)} V_m(P; \sigma_d(P; P^d, \sigma_{III}(P^d)) \text{ for } P \in [0, P^d). \tag{4.6}$$

(ii) Let σ_c^* *be a continuous MPNE (3.29) in Proposition 3.1. Any discontinuous MPNE (3.33) modified from* σ_c^* *is dominated by* σ_c^*.

Proof. (i) The result follows from Lemma 4.2. (ii) The result is immediate by applying the proof of Lemma 4.2 ■

Now we compare MPNEs that are different in the contained nonsingular solutions and in the discontinuous points. We focus on the maximum MPNEs, including the DL MPNE. By Lemma 2.5 (ii) and Lemma 4.1, the DL MPNE dominates any MPNE on the interval $[P^{DL}, \bar{P}]$. However, as shown in Proposition 4.4 below, it is dominated on $\left[0, P^{DL}\right]$ by another discontinuous MPNE, since the condition (4.7) below is satisfied with P_{ss}^{DL}.[8]

Proposition 4.4 *If maximum MPNE* $\sigma_d^M(P; \sigma_{III}) = \sigma_d(P; P_{ss}^U, E_{ss}^U)$ *satisfies*

$$P_{ss}^U > \underline{P} := \frac{2\,(3k+2r)\,A}{(3k+2r)\,k + 12s}, \tag{4.7}$$

then it is dominated by a maximum MPNE $\sigma_d^M(P; \sigma'_{III})$ *with* $P_{ss}^{U\prime} < P_{ss}^U$ *for* $\left[0, P_{ss}^{U\prime}\right]$.

Proof. At the left limit discontinuous point $\left(P_{ss}^U, E_I^d(P_{ss}^U, \omega\left(P_{ss}^U\right))\right)$, the gradient of the Type I solution is given by

$$\frac{dE}{dP} = \frac{(k+r)}{3} + \frac{2kP(k+r) + 6sP - 4A(k+r)}{3\,(kP+2A) - 6kP}.$$

[8] This follows from

$$P_{ss}^{DL} - \underline{P} = \frac{2A(k+2r)}{k^2+2rk+4s} - \frac{2\,(3k+2r)\,A}{(3k+2r)\,k+12s} = \frac{32Ars}{(4s+2kr+k^2)\,(12s+2kr+3k^2)}.$$

On the other hand, the gradient of $E = E_I^d(P, \omega(P))$ is $6/k$. Define their difference as function $\psi : \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned}\psi(P) &:= \frac{(k+r)}{3} + \frac{2kP(k+r) + 6sP - 4A(k+r)}{3(kP+2A) - 6kP} - \frac{k}{6} \\ &= \frac{1}{3}\left(-\frac{3k+2r}{2} - \frac{6s}{k} + \frac{12sA}{k(2A - Pk)}\right).\end{aligned}$$

The second line shows that ψ is strictly increasing. A simple calculation yields $\psi(\underline{P}) = 0$. Therefore, if (4.7) holds, $\sigma_d^M(P; \sigma_{III})$ crosses $E = E_I^d(P, \omega(P))$ from the south-west to the north-east. This implies that for a small $\varepsilon > 0$, $\sigma_d^M(P; \sigma'_{III}) := \sigma_d(P; P_{ss}^U - \varepsilon, \omega(P_{ss}^U - \varepsilon))$ satisfies $E_I^d(P_{ss}^U - \varepsilon, (k/2)(P_{ss}^U - \varepsilon)) > \sigma_d^M(P_{ss}^U - \varepsilon; \sigma_{III})$. Let $P_{ss}^{U\prime} = P_{ss}^U - \varepsilon$ and the proof finishes. ■

From this proposition, we also have:

Proposition 4.5 *The maximum MPNE strategy* $\sigma_d(P; \underline{P}, \omega(\underline{P}))$ *dominates any MPNE strategy on* $[0, \underline{P}]$*. But on* $(\underline{P}, \bar{P}]$*, it is dominated by another maximum MPNE strategy.*

Proof. The first part follows from Proposition 4.4. For the second part, for $P' \in (\underline{P}, P_{ss}^{DL}]$, it is dominated by the maximum MPNE $\sigma_d(P; P', \omega(P'))$ since $\sigma_d(P; P', \omega(P')) < \sigma_d(P'; \underline{P}, \omega(\underline{P})) < \gamma_\infty(P')$. ■

Propositions 4.1 - 4.5 provide the payoff ranking of the linear, continuous and discontinuous MPNEs defined in (3.27), (3.29) and (3.33).

Any continuous MPNE strategy, if it exists, dominates the linear MPNE strategy and the discontinuous MPNE strategies modified by the continuous MPNE strategy over the state space. The best continuous MPNE strategy is $\sigma_c^*(P; 0, 0)$. However, any discontinuous MPNE strategy $\sigma_d^*(P; P^d, E^d)$ such that its Type III solution part is below $\sigma_{III}(P; 0, 0)$ strictly dominates the best continuous MPNE $\sigma_c^*(P; 0, 0)$ on interval $[P^d, \bar{P}]$.

When (3.39) in Proposition 3.3 is satisfied, the DL MPNE strategy $\sigma^{DL} = \sigma_d^*\left(P_{ss}^{DL}, E_{ss}^{DL}\right)$ dominates all other MPNE strategies on $[P_{ss}^{DL}, \bar{P}]$. That is, the DL MPNE strategy is payoff dominant on $[P_{ss}^{DL}, \bar{P}]$. This indicates that if the strategy space contains discontinuous Markov strategies, and if the pollution level is high, an international environmental negotiation may agree to playing the DL MPNE.

The interval $[0, P_{ss}^{DL})$ is divided into two subintervals by $\underline{P}$ defined in (4.7). On $[0, \underline{P}]$, the discontinuous MPNE strategy $\sigma_d(P; \underline{P}, \omega(\underline{P}))$ is payoff dominant. For $P' \in (\underline{P}, P_{ss}^{DL})$, $\sigma_d(P; P', \omega(P'))$ is payoff dominant. Therefore, payoff dominant MPNE strategies vary on this interval. If renegotiation is costless, the players may keep on changing their equilibrium strategy, stay on $E = E_I^d(P, \omega(P))$, and eventually adopt the DL MPNE strategy σ^{DL} at its steady state $\left(P_{ss}^{DL}, E_{ss}^{DL}\right)$.

As the limit case of $r \to 0$, $\underline{P}$, P_{ss}^c (the efficient steady state) and P_{ss}^{DL} coincide. Then the DL MPNE strategy is the payoff dominant equilibrium strategy not only on $[P_{ss}^{DL}, \bar{P}]$ but also on $[0, P_{ss}^{DL})$.

5 OLNE

5.1 Equilibrium and value function

This section introduces open-loop strategies into the strategy space, assuming that the players can commit the time path of pollution emissions. The strategy is denoted by $\sigma_o : [0, \bar{P}] \times [0, \infty) \to [0, A]$. Fix the initial pollution level $P_0 \in [0, \bar{P}]$. (σ_o^*, σ_o^*) is an OLNE if

$$\sigma_o^*(t; P_0) \in \arg \max_{E(t) \in [0,A]} \int_0^\infty [AE(t) - (1/2)(E(t))^2 - (s/2)P(t)^2] e^{-rt} dt \tag{5.1}$$

$$\text{subject to } \dot{P}(t) = E(t) + \sigma_o(t; P_0) - kP(t), \quad P(0) = P_0.$$

The associated Hamiltonian H_o and maximized Hamiltonian H_o^* are defined by

$$H_o(E, P, \lambda_o; \sigma_o^*) := [AE - (1/2)E^2 - (s/2)P^2] + \lambda_o[E + \sigma_o(t) - kP], \tag{5.2}$$

$$H_o^*(P, \lambda_o; \sigma_o^*) := \max\{H_o(E, P, \lambda_o; \sigma_o^*) | E \in [0, A]\} = H_o(E_o^*, P, \lambda_o), \tag{5.3}$$

where E_o^* is the maximizer that satisfies

$$E_o^* = \begin{cases} A + \lambda_o & \text{if } A + \lambda_o > 0 \\ 0 & \text{if } A + \lambda_o \leq 0 \end{cases}. \tag{5.4}$$

The costate variable λ_o satisfies

$$\dot{\lambda}_o(t) = r\lambda_o(t) - \frac{\partial H_o^*(P(t), \lambda_o(t); \sigma_o^*)}{\partial P} = (r+k)\,\lambda_o(t) + sP(t). \tag{5.5}$$

Then an interior optimal control satisfies the canonical system of differential equations:

$$\begin{pmatrix} dP/dt \\ dE_o/dt \end{pmatrix} = \Phi_o \begin{pmatrix} P \\ E_o \end{pmatrix} + \psi_o. \tag{5.6}$$

where

$$\Phi_o = \begin{pmatrix} -k & 2 \\ s & r+k \end{pmatrix}, \psi_o = \begin{pmatrix} 0 \\ -(r+k)A \end{pmatrix} \tag{5.7}$$

The steady state is given by

$$\begin{pmatrix} P_o^{ss} \\ E_o^{ss} \end{pmatrix} = \frac{(r+k)\,A}{k^2+rk+2s} \begin{pmatrix} 2 \\ k \end{pmatrix}. \tag{5.8}$$

The steady state is on the graph of $E = \gamma_0(P)$. We remark it as a lemma:

Lemma 5.1

$$E_o^{ss} = \gamma_0\left(P_o^{ss}\right) \tag{5.9}$$

Proof.

$$\gamma_0\left(P_o^{ss}\right) = -\frac{s}{k+r}\frac{2\,(r+k)\,A}{k^2+rk+2s} + A = \frac{(r+k)\,Ak}{k^2+rk+2s} = E_o^{ss}.$$

■

The eigenvector associated with the negative eigenvalue of Φ_o is $(\xi_o, 1)$ where

$$\xi_o = -\frac{1}{2s}\left(r + 2k + \sqrt{(r+2k)^2 + 8s}\right). \tag{5.10}$$

Since the canonical system (5.6) is a system of linear differential equations, an interior optimal policy control should be given by:

$$E_o(P) = -\alpha_o P + \beta_o. \tag{5.11}$$

where

$$\alpha_o = \frac{-2k + \left(-r + \sqrt{(r+2k)^2 + 8s}\right)}{4}, \tag{5.12}$$

$$\beta_o = \frac{(k+r)\left(-r + \sqrt{(2k+r)^2 + 8s}\right)}{2\left(k^2 + kr + 2s\right)} A. \tag{5.13}$$

Indeed it is a unique optimal policy control for $P \in [0, \beta_o/\alpha_o)$ by the Arrow's sufficiency theorem, since the maximized Hamiltonian (5.3) with opponent strategy (5.11) is strictly concave in P. The Arrow's sufficiency theorem is also applied to show that the boundary solution $E_o(P) = 0$ is an optimal policy control on $\left[\beta_o/\alpha_o, \bar{P}\right]$. Note that from (5.11), the OLNE strategy is expressed as a function of current pollution level. Thus we write the OLNE strategy as a function of P: $\sigma_o^*(P)$. These results are summarized as:

Proposition 5.1 *There is a unique symmetric OLNE. The equilibrium strategy $\sigma_o^*(P)$ is given by*

$$\sigma_o^*(P_0) = \begin{cases} -\alpha_o P + \beta_o & \text{if } P \in [0, \beta_o/\alpha_o) \\ 0 & \text{if } P \in [\beta_o/\alpha_o, \infty) \end{cases}. \tag{5.14}$$

The steady state (5.8) is globally asymptotically stable.

Let $V_o : [0, \bar{P}] \to \mathbb{R}$ be the value function associated with the OLNE. The HJB equation is the same as the one for MPNEs.

$$rV_o(P) = [AE - (1/2)E^2 - (s/2)P^2] + V_o'(P)[E + \sigma_o^*(P) - kP]. \tag{5.15}$$

In particular, for $P \in [0, \beta_o/\alpha_o)$,

$$rV_o(P) = u(P, E) + (E - A)(2E - kP), E = \sigma_o^*(P) \tag{5.16}$$

holds. It is the same as (2.16). Therefore, we can compare the payoff with the ones of MPNEs by applying Lemma 2.5.

5.2 Ranking results

In this section we examine the payoff ranking. We first compare the OLNE σ_o^* with the linear MPNE σ_b^*. Recall that $\left(P_\gamma^m, E_\gamma^m\right)$ in (3.12) is the intersection of $E = \sigma_b^*(P)$ and $E = \gamma_\infty(P)$. Denote by $\left(P_\gamma^o, E_\gamma^o\right)$ the intersection of $E = \sigma_o^*(P)$ and $E = \gamma_\infty(P)$.

Proposition 5.2 *There is pollution level $P_L^s \in \left(P_\gamma^o, P_\gamma^m\right)$ such that for $P < P_L^s$, the linear MPNE strategy σ_b^* dominates the OLNE strategy σ_o^*, whereas for $P > P_L^s$, σ_o^* dominates σ_b^*. That is,*

$$V_o(P) \lesseqgtr V_m(P; \sigma_b^*) \text{ if } P \lesseqgtr P_L^s. \tag{5.17}$$

Proof. From (3.4) and (5.13),

$$\begin{aligned}
\frac{\beta_b}{\beta_o} &= \frac{1}{3}\frac{r + \sqrt{(2k+r)^2 + 8s}}{r + (2k+r)} + \frac{1}{6}\frac{\left(\sqrt{(2k+r)^2 + 12s} - r\right)\left(\sqrt{(2k+r)^2 + 8s} + r\right)}{(k^2 + rk + 3s)} \\
&> \frac{1}{3}\frac{r + \sqrt{(2k+r)^2 + 8s}}{r + (2k+r)} + \frac{1}{6}\frac{\left(\sqrt{(2k+r)^2 + 8s} - r\right)\left(r + \sqrt{(2k+r)^2 + 8s}\right)}{(k^2 + rk + 3s)} \\
&= \frac{1}{3}\frac{r + \sqrt{(2k+r)^2 + 8s}}{r + (2k+r)} + \frac{2}{3}\frac{k^2 + rk + 4s}{k^2 + rk + 3s} \\
&> 1,
\end{aligned}$$

and thus $\beta_b > \beta_o$. From (3.3) and (5.12),

$$\begin{aligned}
\frac{\alpha_b}{\alpha_o} &= \frac{2}{3}\frac{(r+2k) - \sqrt{(r+2k)^2 + 12s}}{(r+2k) - \sqrt{(r+2k)^2 + 8s}} \\
&= \frac{1}{12s}\left(-(r+2k) + \sqrt{(r+2k)^2 + 12s}\right)\left((r+2k) + \sqrt{(r+2k)^2 + 8s}\right) \\
&< \frac{1}{12s}\left(-(r+2k) + \sqrt{(r+2k)^2 + 12s}\right)\left((r+2k) + \sqrt{(r+2k)^2 + 12s}\right) \\
&= 1,
\end{aligned}$$

and thus $\alpha_b < \alpha_o$. These inequalities imply that $\sigma_b^*(P) > \sigma_o^*(P)$ for all $[0, \beta_b/\alpha_b]$. Then Lemmas 2.5 and 4.1 imply that $V_o^*(P) < V_m(P; \sigma_b^*)$ if $P \in [0, P_\gamma^o]$ and $V_o(P) > V_m(P; \sigma_b^*)$ if $P \in [P_\gamma^m, \bar{P}]$. By continuity of value functions, there is P' at which $V_o(P') - V_m(P';\sigma_b^*) = 0$. If P' is not unique, then $V_o(P) - V_m(P;\sigma_b^*)$ has a decreasing interval and two increasing intervals on $[0, P_\gamma^m]$. However, this is not possible because, as seen in (2.16) and (5.16), $V_o(P) - V_m(P;\sigma_b^*)$ is a quadratic function. Therefore, P' is unique with $P' = P_L^s$ and (5.17) holds. ■

Second, we compare the payoffs of the OLNE and a nonlinear MPNE. We prepare a lemma. Recall that the upper bound of the domain of a Type III solution is denoted by P_{III}^U. Also recall that β_o/α_o is the P-intercept of the OLNE strategy.

Lemma 5.2 *A Type III solution $\sigma_{III}(P; P_{III}^U, 0)$ crosses the OLNE strategy σ_o^* once if $P_{III}^U \geq \beta_o/\alpha_o$. If $P_{III}^U < \beta_o/\alpha_o$, there is no intersection.*

Proof. The gradient of a Type III solution (3.1) at the intersection with the OLNE strategy is given by

$$\frac{d\sigma_{III}}{dP} = \frac{(k+r)(-\alpha_o P + \beta_o) - A(k+r) + sP}{3(-\alpha_o P + \beta_o) - A - kP} \tag{5.18}$$

The gradient of the OLNE strategy is given by $d\sigma_o/dp = (-\alpha_o)$. Substitute (5.12) and (5.13), we have

$$\frac{d\sigma_{III}}{dP} - (-\alpha_o) = \frac{\left(\frac{A}{2r+2\sqrt{8s+(2k+r)^2}} + \frac{1}{8}P\right)\left((2k+r)\sqrt{8s+(2k+r)^2} - (2k+r)^2 - 4s\right)}{3(-\alpha_o P + \beta_o) - A - kP}. \tag{5.19}$$

The denominator is negative since the intersection is located below $E = \gamma_\infty(P)$. The numerator is also negative, since

$$\left(\sqrt{8s+(2k+r)^2}\right)^2 < \left((2k+r) + \frac{4s}{2k+r}\right)^2 = 8s + (2k+r)^2 + \left(\frac{4s}{2k+r}\right)^2. \tag{5.20}$$

Therefore,

$$d\sigma_{III}/dP > d\sigma_o/dp. \tag{5.21}$$

Then if $P_{III}^U < \beta_o/\alpha_o$, it is impossible to have an intersection. If $P_{III}^U = \beta_o/\alpha_o$, it is a unique intersection. If $P_{III}^U > \beta_o/\alpha_o$, letting $P' = P_{III}^L$ or 0, we have $\sigma_{III}(P'; P_{III}^U, 0) < \sigma_o(P')$ and $\sigma_{III}(\beta_o/\alpha_o; P_{III}^U, 0) > \sigma_o(\beta_o/\alpha_o)$. This implies that there is an intersection, and (5.21) ensures that it is a unique intersection.

■

Proposition 5.3 *(i) Any MPNE strategy dominates the OLNE strategy σ_o^* on $[0, P_L^s]$., where P_L^s is the pollution level defined in Proposition 5.2. (ii) If an MPNE strategy σ_m^* has an interior intersection $\left(\tilde{P}, \tilde{E}\right)$ with σ_o^*, then σ_o^* dominates σ_m^* for $P \in (\tilde{P}, \bar{P}]$ whereas σ_m^* dominates σ_o for $P \in [\tilde{P}', \tilde{P}]$ with some pollution level $\tilde{P}' \in [0, \tilde{P})$.*

Proof. (i) By Proposition 4.1, any nonlinear MPNE strategy dominates the linear MPNE strategy over the state space, and by Proposition 5.2, the linear MPNE strategy dominates the OLNE strategy on $[0, P_L^s]$. (ii) The result follows from a similar proof of Proposition 4.2. ■

In Figure 2, the relationship between the DL MPNE strategy and the OLNE strategy is illustrated. They have an interior intersection. If the pollution level is greater than it and an open-loop strategy is available, the players choose to play the OLNE, and when the pollution level is improved to the intersection level, they renegotiate and adopt the DL MPNE strategy. The expectation of renegotiation does not affect the initial equilibrium choice, since as far as the OLNE is chosen, it is the payoff dominant equilibrium at the pollution level.

6 Concluding Remark

Any MPNE converges to a certain steady state, because the state space is bounded and an equilibrium pollution path is monotone. As we show, a steady state has a one to one relationship with the associated MPNE in the pollution levels greater than the steady state. In the lower interval, the situation is different: there is a continuum of MPNEs which converge to the same steady state. This indeterminacy is caused by the arbitrariness of discontinuous points and nonsingular solutions that is contained in the MPNE strategy. The property of payoff dominant equilibrium also differs depending on whether the pollution level is higher than the DL steady state. When the pollution is higher, the payoff dominant equilibrium is either the DL MPNE and the OLNE. If the latter is the payoff dominant equilibrium, switching to the DL MPNE occurs just once when the pollution is improved. When the pollution level is lower than the DL steady state but not very clean, the payoff dominant strategy is different in each pollution level. This makes it difficult that the players reach an agreement. These differences could explain why an international environmental negotiation is initiated when an environmental problem becomes serious.

From the viewpoint of differential game theory, this paper is a study of discontinuous MPNEs. Discontinuous strategies are considered by Dutta and Sundaram (1993), Dockner and Sorger (1996) and Sorger (1998). Using a discrete time model, Dutta and Sundaram (1993) show an MPNE whose steady state is larger than the efficient steady state, which indicates the possibility of excessive conservation rather than excessive exploitation. In the present model we do not have the result. It might be specific to a discrete time model. Dockner and Sorger (1996) have the same efficiency result to Dockner and Long's (1993) efficiency result. Sorger (1998) obtains that, unless discount rate is very high, there is a discontinuous MPNE whose payoff is almost the same as the efficient one when the state variable is sufficiently near the efficient steady state. We can have the same result if we choose the discontinuous point sufficiently near the cooperative steady state (3.37). For the multiplicity of MPNEs, Dockner and Sorger (1996) show a continuum of discontinuous MPNEs by parameterizing the discontinuous point, whereas Sorger (1998) shows it by parameterizing the jumps with a fixed discontinuous point. In this paper, we parameterize both and show a two dimensional continuum of MPNEs. Recently, Dockner and Wagener (2014) generalize the Tsutsui and Mino's technique (1991) to find a continuum of Markov Nash equilibria. They covers asymmetric models, asymmetric equilibria and discontinuous strategies. For the study of international environmental agreements, an asymmetric game is more appropriate than a symmetric game. An extension to an asymmetric game is an important future research.

A Appendix

A.1 The solution for a differential equation with the discontinuous right hand side.

A.1.1 Definition

Define $g(P) := \sigma_i(t) + \sigma_j(t) - kP(t)$ and assume that g has a unique discontinuous point $P^d \in (0, \bar{P})$. Let

$$a = \min\left\{ \lim_{P \nearrow P^d} g(P), \lim_{P \searrow P^d} g(P) \right\}, b = \max\left\{ \lim_{P \nearrow P^d} g(P), \lim_{P \searrow P^d} g(P) \right\}.$$

Following Filippov (1988), we define the associated differential inclusion $\dot{P} = G(P)$ where G is a set valued function such that

$$G = \begin{cases} g(P) & P \in [0, \bar{P}] \backslash \{P^d\} \\ [a, b] & P = P^d \end{cases}$$

Definition: $P(t)$ is a solution of the initial value problem $\dot{P} = g(P)$ with $P(0) = P_0 \in [0, \bar{P}]$ if $P(t)$ is absolutely continuous and $\dot{P}(t)$ satisfies $\dot{P}(t) \in G(P(t))$ almost everywhere.

As seen in the main text, we only have to consider the case that g has a unique discontinuous point in $(0, \bar{P})$ and this definition is enough for us. But in a general case that g has countably many discontinuous points, we can define the generalized solution in the same way.

A.1.2 Existence and uniqueness of a solution induced by (3.33)

Let us express the differential equation as reduced form $\dot{P} = g(P)$. $g : [0, \bar{P}] \to \mathbb{R}$ is a discontinuous function at $P^d \in (0, \bar{P})$ and has following properties: (i) There is M such that $|g(P)| < M$ for $P \in [0, \bar{P}]$; (ii) $g \in C^1$ on $[0, P^d)$ and $(P^d, \bar{P}]$; (iii) $g(P) > 0$ if $P \in [0, P^d)$, $g(P) = 0$ if $P = P^d$, and $g(P) < 0$ if $P \in (P^d, \bar{P}]$; (iii) $\lim_{P \nearrow P^d} g(P) := a > 0$ and $\lim_{P \searrow P^d} g(P) = 0$. The associated differential inclusion $\dot{P} \in G(P)$ is given by

$$G(P) = \begin{cases} g(P) & \text{if } P \in [0, P^d) \cup (P^d, \bar{P}] \\ [0, a] & \text{if } P = P^d \end{cases}.$$

A solution with initial value $P_0 \in (P^d, \bar{P}]$ keeps staying in the interval and $g \in C^1$ on $(P^d, \bar{P}]$. This ensures the existence and uniqueness of the solution. When initial value P_0 is in $[0, P^d]$, consider a path $P(t)$ such that $\dot{P}(t) = g(P(t))$ if $P(t) \in [P_0, P^d)$ and $\dot{P}(t) = 0$ if $P(t) = P^d$. At $P = P^d$, $\dot{P} = 0 \in G(P^d)$. We can show that this path is absolutely continuous and it is a solution. Consider a path $\tilde{P}(t)$ such that

$$\tilde{P}(t) = \begin{cases} P(t) & \text{if } t < T^d \\ P^d + a(t - T^d) & \text{if } t \geq T^d \end{cases}$$

where $T^d > 0$ is the reaching time to P^d. Note that $\tilde{P}(t)$ is continuously differentiable and thus Lipschitz continuous. Denote the Lipschitz coefficient by L. Since $a > 0$, for any $t \in [0, \infty)$ and $t' \in (t, \infty)$, we have

$$P(t') - P(t) \leq \tilde{P}(t') - \tilde{P}(t) \leq L(t' - t).$$

This shows that $P(t)$ is Lipschitz continuous and thus is absolutely continuous.

We show the uniqueness of the solution by applying Filippov (1988, Theorem 1 in Chapter 2, Section 10). It follows from the theorem that a solution $P(t)$ is unique on $t \in [0, T]$ with $T < \infty$ if there is $l > 0$ such that for $x, y \in [0, \bar{P}]$,

$$(x - y)(g(x) - g(y)) \leq l(x - y)^2. \tag{A.1}$$

Since Lipschitz condition $|g(x) - g(y)| \leq l|x - y|$ implies

$$(x - y)(g(x) - g(y)) \leq |x - y||g(x) - g(y)| \leq l(x - y)^2,$$

(A.1) holds if $x, y \in [0, P^d)$ or $(P^d, \bar{P}]$. Assume that $x \in [0, P^d]$, $y \in [P^d, \bar{P}]$ and $x \neq y$. Since $x < y$ and $g(x) > g(y)$, (A.1) holds. Obviously we have the same result for $y \in [0, P^d]$, $x \in [P^d, \bar{P}]$ and $x = y$.

A.2 Existence of an optimal control to Problem (2.10)

Given opponent strategy $\sigma_m \in \Sigma$ that is a stationary Markov strategy, the value function $V_m : [0, \bar{P}] \to \mathbb{R}$ is defined by

$$V_m(P) = \sup_{E(\cdot)} \int_0^\infty u(P(t), E(t)) e^{-rt} dt \tag{A.2}$$

$$\text{subject to } \dot{P}(t) = E(t) + \sigma_m(P(t)) - kP(t), \quad E(t) \in [0, A], \quad P(t) \geq 0$$

$$P(0) = P \in [0, \bar{P}] \text{ given.}$$

By the principle of optimality, V_m satisfies

$$V_m(P) = \sup_{E(\cdot)} \int_0^T u(P(t), E(t))e^{-rt}dt + V_m(P(T))e^{-rT} \tag{A.3}$$

$$\text{subject to } \dot{P}(t) = E(t) + \sigma_m(P(t)) - kP(t), \quad E(t) \in [0, A], \quad P(t) \geq 0$$

$$P(0) = P \in [0, \bar{P}] \text{ given.}$$

for all $T > 0$. For the proof of existence, we prepare two lemmas:

Lemma A.1 *If Problem (A.3) has an optimal control for any $T > 0$, then there is an optimal control to Problem (A.2).*

Proof. Denote an optimal path $(P^*(\cdot; T, P), E^*(\cdot; T, P))$ when the end time is $T > 0$ and the initial state is $P \in [0, \bar{P}]$. Fix T and P. Take $t > 0$. Then for $t \in [0, T)$, the principle of optimality

$$(P^*(t; T, P), E^*(t; T, P)) = (P^*(t; T + T', P), E^*(t; T + t, P))$$

holds. Otherwise, we have the following contradiction.

$$\begin{aligned} V_m(P) &= \int_0^T u(P^*(t; T, P), E^*(t; T, P))e^{-rt}dt + V_m(P^*(T; T, P))e^{-rT} \\ &> \int_0^T u(P^*(t; T + T', P), E^*(t; T + T', P))e^{-rt}dt + V_m(P^*(T; T + T', P))e^{-rT} \\ &\geq \int_0^{T+T'} u(P^*(t; T + tT', P), E^*(t; T + T', P))e^{-rt}dt + V_m(P^*(T + T'; T + T', P))e^{-rT+T'} \\ &= V_m(P) \end{aligned}$$

Then for any$T > 0$ and any feasible path $(P(t), E(t))$ that satisfies the initial value problem in Problem (A.2),

$$V_m(P) = \int_0^T u(P^*(t; T, P), E^*(t; T, P))e^{-rt}dt + V_m(P^*(T; T, P))e^{-rT} \geq \int_0^T u\left(P(t), E(t)\right) e^{-rt}dt + V_m(P(T))e^{-rT}$$

holds with equality if $(P(t), E(t)) = (P^*(t; T, P), E^*(t; T, P))$ for $t \in [0, T)$. As $T \to \infty$, since V_m is bounded by Lemma 3.1, we have for any feasible path

$$\int_0^\infty u(P^*(t; \infty, P), E^*(t; \infty, P))e^{-rt}dt \geq \int_0^\infty u\left(P(t), E(t)\right) e^{-rt}dt$$

with equality if $(P(t), E(t)) = (P^*(t; \infty, P), E^*(t; \infty, P))$ for $t \in [0, \infty)$. This shows that $(P^*(t; \infty, P), E^*(t; \infty, P))$ is an optimal control and thus its existence. ■

Lemma A.2 *For any $T > 0$, Problem (A.3) has an optimal path.*

Proof. Fix T and P. Let $(P_n(\cdot), E_n(\cdot))$ be a sequence of feasible paths from $P(0) = P$ such that

$$V_m(P) = \lim_{n\to\infty} \int_0^T u(P_n(t), E_n(t))e^{-rt}dt + V_m(P_n(T))e^{-rT}.$$

Since the set of feasible controls $E(\cdot) \in [0, T] \times [0, A]$ is a subset of the L^2 space, we can take a subsequence of $E_n(t)$ which weakly converges. The associate pollution stock paths with this subsequence has a subsequence which weakly converges, since the set of feasible pollution paths $P(\cdot) \in [0, T] \times [0, \bar{P}]$ is also a subset of the L^2 space. Therefore we can have a pair of sequences $(P_n(\cdot), E_n(\cdot))$ that weakly converge. Denote the limit by $(P^*(\cdot), E^*(\cdot))$. If we show that this limit is a feasible path, the proof completes. It is obvious that $P^*(0) = P$. Also $(P^*(t), E^*(t)) \in [0, \bar{P}] \times [0, A]$ for almost every $t \in [0, T]$, since we can take a subsequence of a weakly convergent sequence that converges almost everywhere. Finally the state equation $\dot{P}^*(t) = E^*(t) + \sigma_m(P^*(t)) - kP^*(t)$ is satisfied. This is verified as follows: for each $t \in [0, T]$,

$$\begin{aligned}
P_n(t) &= P + \int_0^t E_n(t) + \sigma_m(P_n(t)) - kP_n(t)dt \\
&= P + \int_0^t \left(E_n(t) - E^*(t)\right) + \left(\sigma_m(P_n(t)) - \sigma_m(P^*(t))\right) - k\left(P(t) - kP^*(t)\right)dt \\
&+ \int_0^t E^*(t) + \sigma_m(P^*(t)) - kP^*(t)dt \\
&\to P + \int_0^t E^*(t) + \sigma_m(P^*(t)) - kP^*(t)dt \ (n \to \infty),
\end{aligned}$$

since σ_m is Lipschitz continuous we have

$$0 \leq \int_0^t |\sigma_m(P_n(t)) - \sigma_m(P^*(t))|dt \leq \int_0^t L|P_n(t) - P^*(t))|dt \to 0 \ (n \to \infty),$$

where L is the Lipschitz constant. Therefore

$$P^*(t) = P + \int_0^t E^*(t) + \sigma_m(P^*(t)) - kP^*(t)dt,$$

which is the integrated form of the state equation. We then have

$$V_m(P) = \int_0^T u(P^*(t), E^*(t))e^{-rt}dt + V_m(P^*(T))e^{-rT},$$

i.e. an optimal path $(P^*(t), E^*(t))$ exists. ■

Proposition A.1 *There exists an optimal path* $(P^*(t), E^*(t))$ *to (A.2)*

Proof. Combine Lemmas A.3 and A.4. ■

A.3 Proof of Lemma 3.1 (iii)

First recall each solution of Type III is characterized by $K = K(P, E) \in [K(A(k+r)/s, \infty)$ in (3.2). Then denote a Type III solution by $\sigma_{III}(P; K)$. We show that a Type III solution $E = \sigma_{III}(P; K)$ is a concave function in the increasing interval. If this is true, then a solution crosses $E = \omega(P)$ at most twice, and if there are two intersections in the interval, then there is no intersection in the decreasing interval because after the second crossing, $d\sigma_{III}/dP < 0$ whereas $d\omega/dP > 0$. $\sigma_{III}(P; K)$ satisfies

$$\frac{d\sigma_{III}(P; K)}{dP} = \frac{(k+r)E - A(k+r) + sP}{3E - A - kP}. \tag{A.4}$$

Define

$$N = (k+r)E - A(k+r) + Ps, \tag{A.5}$$

$$D = 3E - A - Pk. \tag{A.6}$$

They are, respectively, the numerator and the denominator of (A.4). We take a set

$$\mathcal{S} := \{(P, E) \in [0, \bar{P}] \times [0, A] | N \in (-A(k+r), 0), D \in (-A(1-(k+r)k/s), 0)\,, E < \sigma_a(P)\}.$$

$\mathcal{S}$ is the graph of the Type III solutions and their increasing intervals. Take a derivative of (A.4) and express it with N and D.

$$\frac{d^2\sigma_{III}}{dP^2} = \frac{(2k+r)\,ND + sD^2 - 3N^2}{D^3}, \tag{A.7}$$

Since $D < 0$, if we show the numerator is nonnegative on $\mathcal{S}$, we have $d^2\sigma_{III}/dP^2 \leq 0$ and the proof completes. Fix N and see D as a function of K. Then the numerator in (A.7) is written as a function of K:

$$f(D(K); N) := (2k+r)\,ND(K) + sD(K)^2 - 3N^2. \tag{A.8}$$

A constant N means that (P, E) moves on a line which is parallel to the graph of $E = \gamma_0(P)$. As K becomes large, (P, E) approaches the singular solution σ_a along the line. This implies that, along the line, $D(K) < 0$ monotonically converges to 0 and thus $dD/dK > 0$. Differentiate the function f to have

$$\frac{df(D(K); N)}{dK} = ((2k+r)\,N + 2sD)\,\frac{dD}{dK} < 0. \tag{A.9}$$

holds. Then for each $\bar{N} \in (-A(k+r), 0)$,

$$f(D(K); \bar{N}) \geq \inf_{(\bar{N}, D) \in \mathcal{S}} f(D(K); \bar{N}) = \lim_{K \to \infty} f(D(K); \bar{N}) = 0.$$

Therefore, we have $d^2\sigma_{III}/dP^2 \leq 0$.

References

[1] Barrett, S. : The theory of international agreements. Mäler, K.-G. and J.R. Vincent (eds.) Handbook of Environmental Economics, Vol. 3: Elsevier (2005)

[2] Dockner, E. J. and Long, N. V.: International pollution control: Cooperative versus noncooperative strategies. Journal of Environmental Economics and Management 25, 3-29 (1993).

[3] Dockner, E. J. and Sorger, G.: Existence and properties of equilibria for a dynamic game on productive assets. Journal of Economic Theory 71, 209-227 (1996).

[4] Dockner, E. J. and Wagener, F.: Markov perfect Nash equilibria in models with a single capital stock. Economic Theory 56: 585–625 (2014).

[5] Filippov, A.F.: Differential Equations with Discontinuous Righthand Sides. Springer (1988)

[6] Harsanyi, J.C. and R. Selten: A General Theory of Equilibrium Selection in Games. MIT Press (1988)

[7] Ishii, H.: A short introduction to viscosity solutions and the large time behavior of solutions of Hamilton–Jacobi Equations. Achdou, Y. et al. (eds.) Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications, Springer, 111-249 (2013)

[8] Rowat, C.: Additive Externality Games. Ph.D. Thesis King's College, Cambridge University.

[9] Rubio, S.J. and B. Casino: A note on cooperative versus non-cooperative strategies in international pollution control. Resource and Energy Economics 24: 251-261 (2002)

[10] Sorger, G.: Markov-perfect Nash equilibria in a class of resources games. Econ Theory **11**, 78–100 (1998)

[11] Tsutsui, S. and Mino, K.: Nonlinear strategies in dynamic duopolistic competition with sticky prices. J Econ Theory **52**, 136–161 (1990)

[12] Van der Ploeg, F. and A. J. de Zeeuw: International aspects of pollution control. Environmental and Resource Economics 2: 117-139 (1992)

[13] Wirl, F.: Do multiple Nash equilibria in Markov strategies mitigate the tragedy of the commons? Journal of Economic Dynamics and Control 31: 3723-3740 (2007)

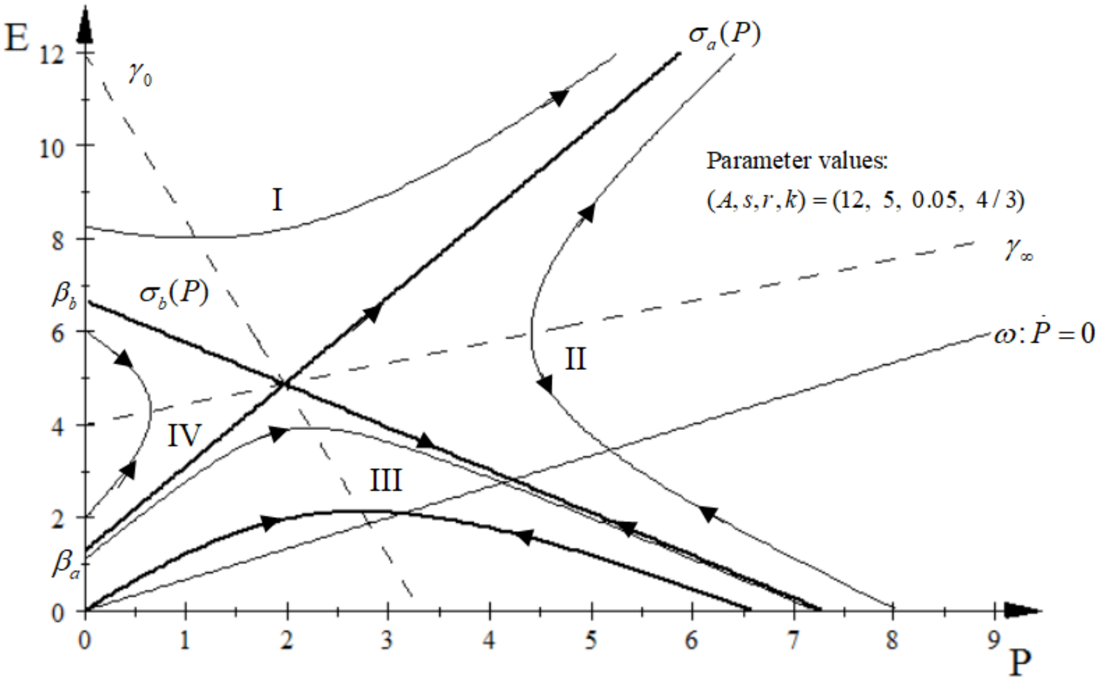
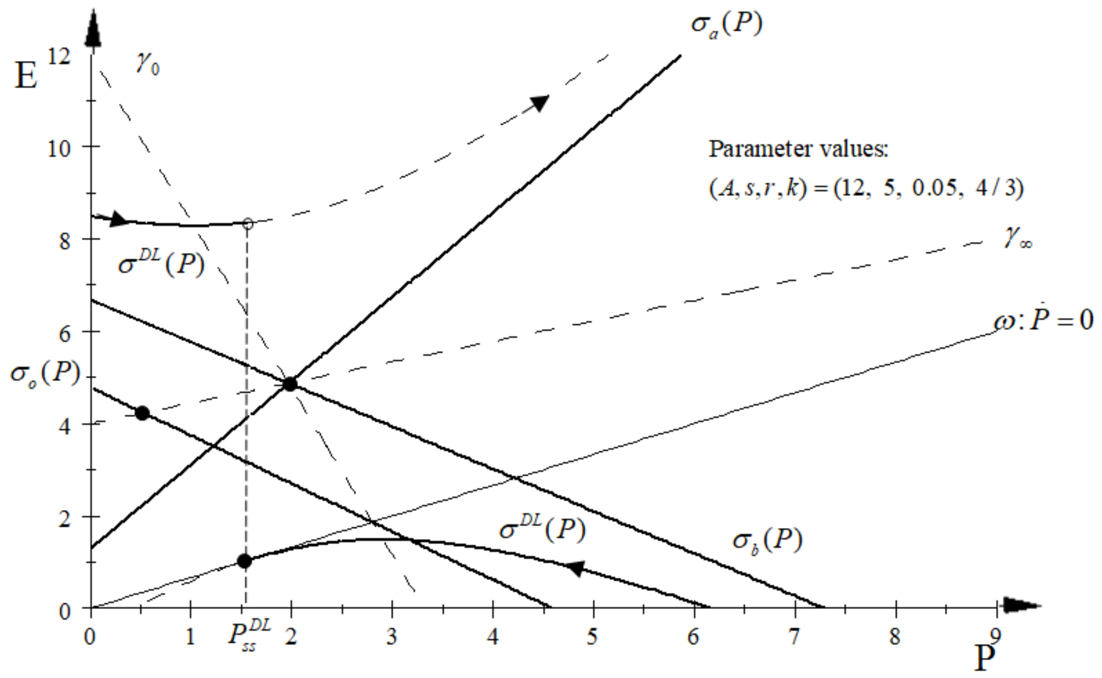

E
P
$\sigma_a(P)$
$\sigma_b(P)$
γ_0
γ_∞
$\omega: \dot{P} = 0$
β_b
β_a
I
II
III
IV
Parameter values:
$(A, s, r, k) = (12,\ 5,\ 0.05,\ 4/3)$
0
2
4
6
8
10
12
0
1
2
3
4
5
6
7
8
9

Figure 1



E
P
$\sigma_a(P)$
γ_0
Parameter values:
$(A, s, r, k) = (12,\ 5,\ 0.05,\ 4/3)$
γ_∞
$\omega : \dot{P} = 0$
$\sigma^{DL}(P)$
$\sigma_o(P)$
$\sigma^{DL}(P)$
$\sigma_b(P)$
P_{ss}^{DL}
0
2
4
6
8
10
12
0
1
2
3
4
5
6
7
8
9

Figure 2