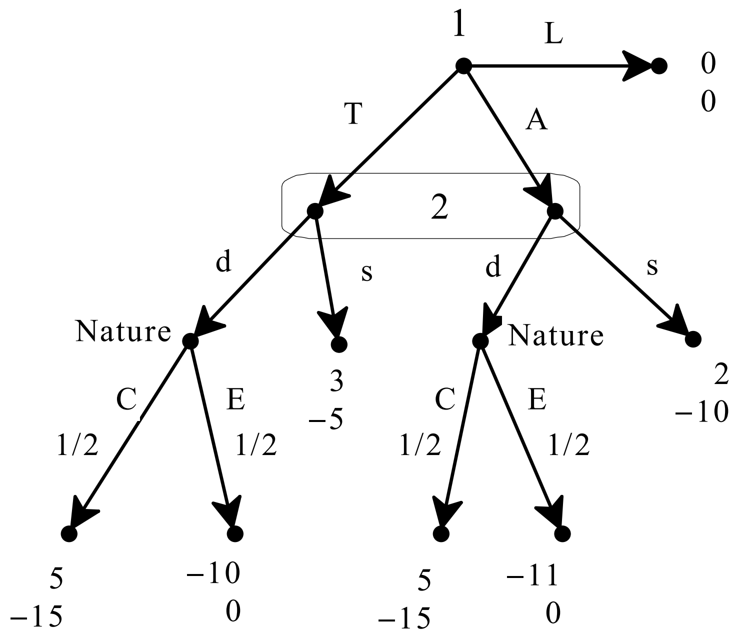
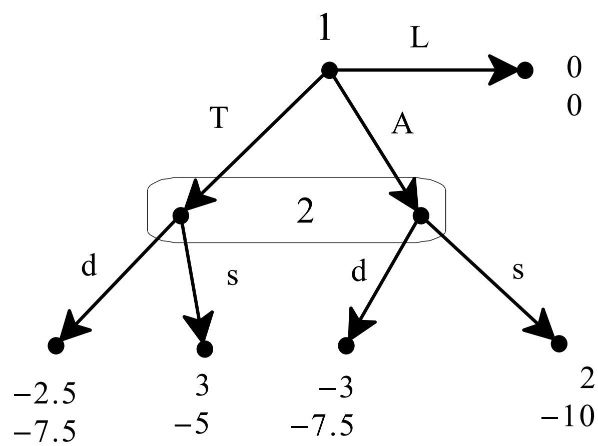
Micro Prelim August 2016 – ANSWER KEYS

QUESTION 1

(a.1) The game is as follows:



(a.2) First replace Nature's move with expected payoffs:



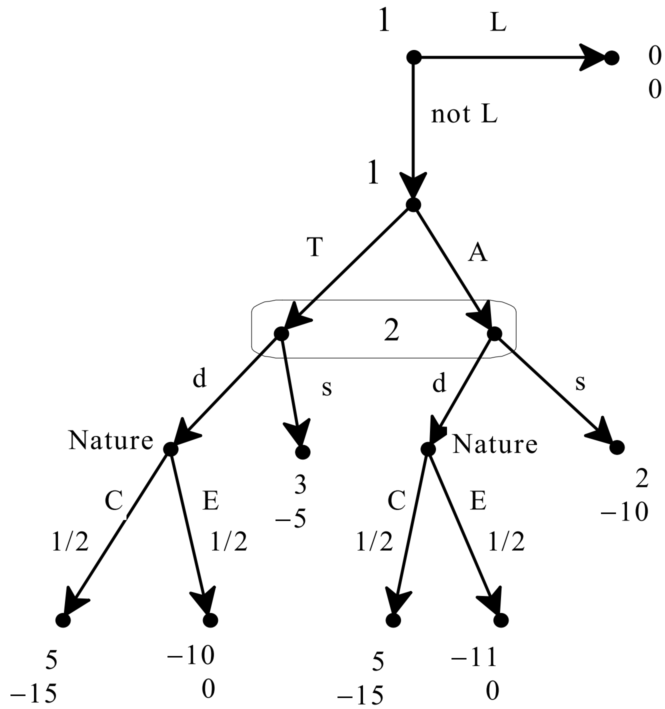
Since the game has no proper subgames, the set of subgame-perfect equilibria coincides with the set of Nash equilibria. The strategic form is as follows:

		Player 2			
		d		*s*	
	L	0	0	0	0
Player 1	*A*	–3	–7.5	2	–10
	T	–2.5	–7.5	3	–5

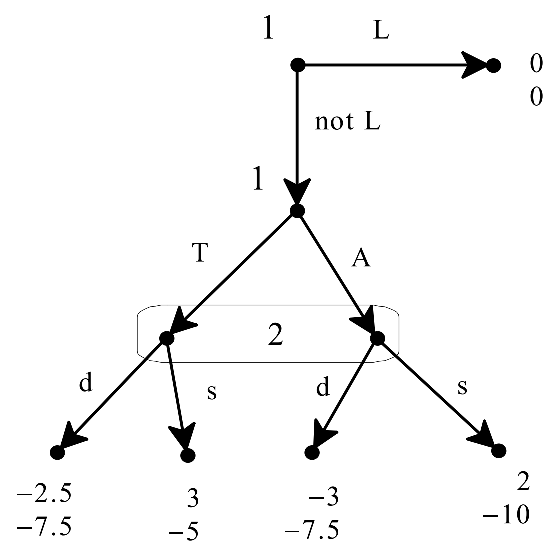
There are two pure-strategy Nash equilibria: (T,s) with expected payoffs $(3,-5)$ and (L,d) with expected payoffs (0, 0). There is also an infinite number of mixed-strategy equilibria:

$\begin{pmatrix} L & A & T & | & d & s \\ 1 & 0 & 0 & | & p & 1-p \end{pmatrix}$ for every $p \geq \frac{6}{11} = 0.545$, with expected payoffs (0,0).

(b.1) The game is as follows:

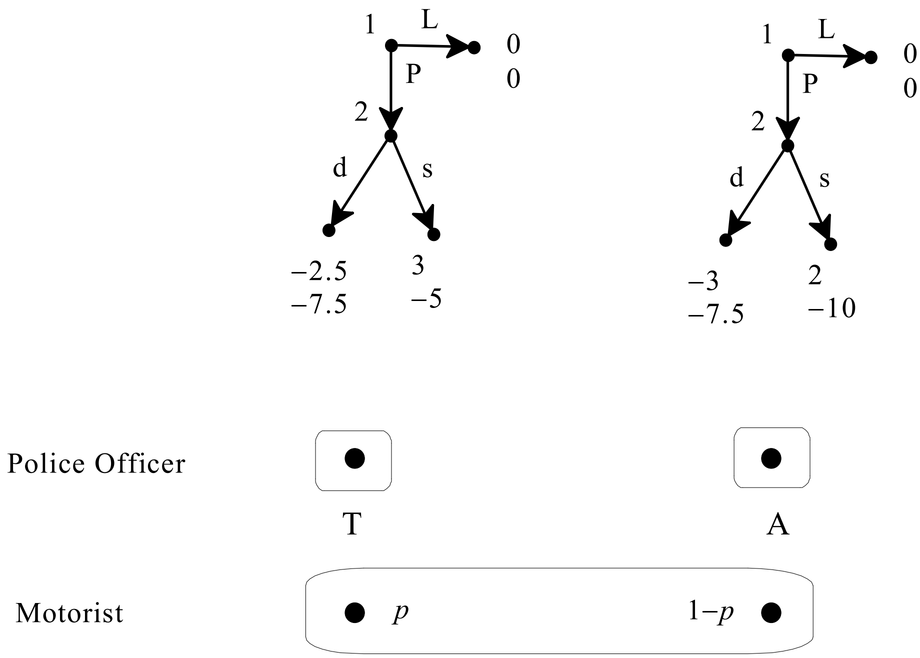


(b.2) First replace Nature's move with expected payoffs:

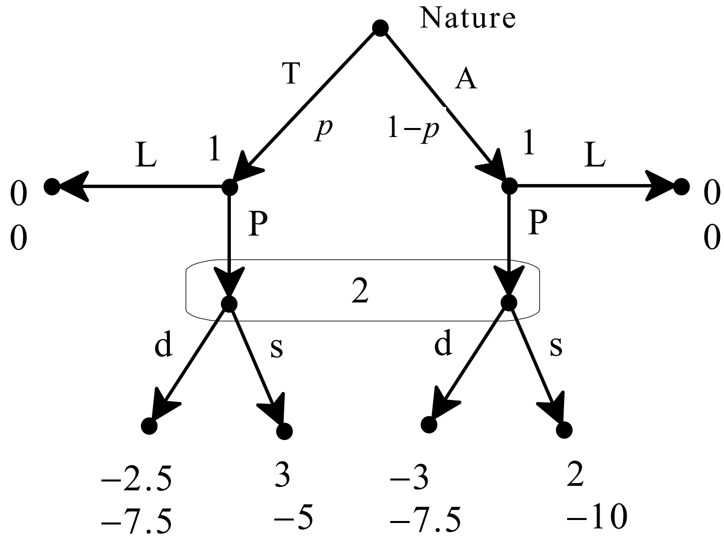


The proper subgame has a unique Nash equilibrium: (T,s) (because T strictly dominates A and against T the unique best response is s). Thus the unique subgame-perfect equilibrium is $\left((\neg L,T),s\right)$.

(c) Each of the two games is simplified by replacing Nature's move with expected payoffs.



(d) The game is as follows (again, we have replaced the second move by Nature with expected payoffs):



(e) When $p = \frac{3}{5}$ the strategic form is as follows:

		Player 2	
		d	s
	LL	$0,\ 0$	$0,\ 0$
Player	LP	$-\frac{6}{5},\ -3$	$\frac{4}{5},\ -4$
1	PL	$-\frac{3}{2},\ -\frac{9}{2}$	$\frac{9}{5},\ -3$
	PP	$-\frac{27}{10},\ -\frac{15}{2}$	$\frac{13}{5},\ -7$

There are two pure-strategy Nash equilibria: (LL,d) and (PP,s).

QUESTION 2

(a) Suppose that such an incentive compatible exists. In order for type M to choose e_M it must be that

$$w_M - \frac{e_M}{\theta_M} \geq w^* - \frac{e^*}{\theta_M} \qquad (1)$$

And in order for type L to choose w^* it must be that

$$w^* - \frac{e^*}{\theta_L} \geq w_M - \frac{e_M}{\theta_L} \qquad (2)$$

Multiplying (1) by θ_M and (2) by θ_L (recall that both θ_L and θ_M are positive) we get

$$\theta_M w_M - e_M \geq \theta_M w^* - e^*, \text{ that is, } \theta_M \left(w_M - w^*\right) \geq e_M - e^* \qquad (1a)$$

$$\theta_L w^* - e^* \geq \theta_L w_M - e_M, \text{ that is, } e_M - e^* \geq \theta_L \left(w_M - w^*\right) \qquad (2a)$$

It follows from (1a) and (2a) that $\theta_M \left(w_M - w^*\right) \geq \theta_L \left(w_M - w^*\right)$ and thus, since $\theta_M > \theta_L > 0$ it must be that $w_M - w^* > 0$. From this and the fact that $\theta_H > \theta_M$ it follows that $\theta_H \left(w_M - w^*\right) > \theta_M \left(w_M - w^*\right)$ and thus, by (1a), $\theta_H \left(w_M - w^*\right) > e_M - e^*$; that is, $w_M - w^* > \frac{e_M}{\theta_H} - \frac{e^*}{\theta_H}$, which can be written as $w_M - \frac{e_M}{\theta_H} > w^* - \frac{e^*}{\theta_H}$ which says that the H type prefers e_M to e^*, contradicting our hypothesis that we had an incentive compatible situation where both types θ_L and θ_H choose education level e^*, while type θ_M choose education e_M.

(b) If all three types choose the same education level $\bar{e}$ then all must be paid the same wage $\bar{w}$ equal to the expected productivity, that is, $\bar{w} = \frac{1}{3}\theta_H + \frac{1}{3}\theta_M + \frac{1}{3}\theta_L$. Assume that the employer believes that anybody who shows up with education level $e \neq \bar{e}$ must be of type *L* with probability 1 and thus offers him/her a wage equal to θ_L. The *L* type will be happy with choosing $\bar{e}$ if and only if $\bar{w} - \frac{\bar{e}}{\theta_L} \geq \theta_L - \frac{e}{\theta_L}$ for all $e \geq 0$. Since the RHS is decreasing in *e*, the inequality is satisfied if and only if $\boxed{\bar{w} - \frac{\bar{e}}{\theta_L} \geq \theta_L}$, which can also be written (since $\theta_L > 0$) as $\boxed{\theta_L (\bar{w} - \theta_L) \geq \bar{e}}$. Since $\theta_H > \theta_M > \theta_L$, this inequality implies also that $\bar{w} - \frac{\bar{e}}{\theta_M} \geq \theta_L$ and $\bar{w} - \frac{\bar{e}}{\theta_H} \geq \theta_L$, so that also the *M* and *H* types prefer $\bar{e}$ to any other level of education.

(c) Using what we found in part (b), when $\theta_L = 1,\ \theta_M = 2,\ \theta_H = 6$ at a pooling equilibrium it must be that $\bar{w} = \frac{1}{3}6 + \frac{1}{3}2 + \frac{1}{3}1 = 3$ and $\bar{e} \leq \theta_L (\bar{w} - \theta_L) = 1(3-1) = 2$. Thus any $\bar{e} \leq 2$ gives rise to a pooling equilibrium.

(d) **(d.1)** Let e_L^* be the education level chosen by the *L* type and e_H^* be the education level chosen by the *H* type and assume that $e_L^* \neq e_H^*$. The easiest way to construct an equilibrium is to assume that the employer would believe that anybody who shows up with an education level different from these two is of type *L* and is thus paid a wage equal to θ_L. Furthermore, in equilibrium those who show up with education e_L^* are paid θ_L and those who show up with education e_H^* are paid θ_H. The incentive compatibility constraints are:

For the *H* type: $\theta_H \theta_H - e_H^* \geq \theta_H \theta_L - e,\ \forall e \in [a,b]$ (1)

For the *L* type: $\begin{cases} \theta_L \theta_L - e_L^* \geq \theta_L \theta_L - e,\ \forall e \in [a,b] \setminus \{e_H^*\} & \text{(2a)} \\ \theta_L \theta_L - e_L^* \geq \theta_L \theta_H - e_H^* & \text{(2b)} \end{cases}$ (2)

From (2a) we get that $e_L^* = a$, so that (2b) becomes

$$\theta_L \theta_L - a \geq \theta_L \theta_H - e_H^* \qquad (3)$$

From (1) we get

$$\theta_H \theta_H - e_H^* \geq \theta_H \theta_L - a \qquad (4)$$

Thus necessary and sufficient conditions for a separating equilibrium are:
(i) $e_L^* = a$ and (ii) $\theta_L (\theta_H - \theta_L) + a \leq e_H^* \leq \min\{\theta_H (\theta_H - \theta_L) + a, b\}$.

(d.2) When $a = 6,\ b = 14,\ \theta_L = 3,\ \theta_H = 5$ there is a separating equilibrium with $e_L^* = a = 6$ and any e_H^* such that $\theta_L (\theta_H - \theta_L) + a \leq e_H^* \leq \min\{\theta_H (\theta_H - \theta_L) + a, b\}$, that is, $12 \leq e_H^* \leq 14$ (since $b = 14 < \theta_H (\theta_H - \theta_L) + a = 16$).

Answer Key - Question 3
August 2016

(a) P.O allocation must solve

$$\max \sum \alpha_i u^i(x^i, q^i, f(y,Q))$$

subject to

$$\sum_{i=1}^{I} q^i \le Q \qquad \mu$$

$$\sum_{i=1}^{I} x^i + \sum_{i=1}^{I} c(q^i) + y \le \sum_{i=1}^{I} w^i \qquad \rho$$

$$\mathcal{L} = \sum_i \alpha_i u^i(x^i, q^i, f(y,Q)) - \mu\left(\sum_i q_i - Q\right) - \rho\left(\sum_i x^i + \sum_i c(q^i) + y - \sum_i w^i\right)$$

(b) FOCs:

$$\alpha_i \frac{\partial u^i}{\partial x^i}(\;) = \rho \qquad i = 1 \ldots I \qquad \alpha_1 = 1 \qquad (1)$$

$$\alpha_i \frac{\partial u^i}{\partial q^i} - \mu - \rho c'(q^i) = 0 \qquad i = 1 \ldots I \qquad (2)$$

$$\sum_i \alpha_i \frac{\partial u^i}{\partial Y} \frac{\partial f}{\partial y} = \rho \qquad (3)$$

$$\sum_i \alpha_i \frac{\partial u^i}{\partial Y} \frac{\partial f}{\partial Q} + \mu = 0 \qquad (4)$$

$$\sum_i q^i = Q$$

$$\sum_i x^i + \sum_i c(q^i) + y = \sum_i w^i$$

(c) Eliminate the multipliers

(1) From (1) and (3)

$$\sum_i \rho \frac{\frac{\partial u^i}{\partial Y}}{\frac{\partial u^i}{\partial x^i}} \frac{\partial f}{\partial y} = \rho$$

or

$$\frac{\sum_i \frac{\partial u^i}{\partial y}\frac{\partial f}{\partial y}}{\frac{\partial u^i}{\partial x^i}} = 1$$

↑ marginal contribution that agent i is willing to make for one more unit of y.

↑ marginal cost of one unit of y

This is the form of the Samuelson condition for this model.

(ii) From ① and ②

$$p\left(\frac{\partial u^i/\partial q^i}{\partial u^i/\partial x^i} - c'(q^i)\right) = \mu$$

Given ④

$$p\left(\frac{\frac{\partial u^i}{\partial q^i}}{\frac{\partial u^i}{\partial x^i}} - c'(q^i)\right) = -\sum_j \alpha_j \frac{\partial u^j}{\partial y}\frac{\partial f}{\partial q}$$

Replacing α_j by its value in ① leads to

$$\frac{\frac{\partial u^i}{\partial q^i}}{\frac{\partial u^i}{\partial x^i}} - c'(q^i) = -\sum_j \frac{\frac{\partial u^j}{\partial y}}{\frac{\partial u^j}{\partial x^j}}\frac{\partial f}{\partial q}$$

or

$$\frac{\frac{\partial u^i}{\partial q^i}}{\frac{\partial u^i}{\partial x^i}} = c'(q^i) - \sum \frac{\frac{\partial u^j}{\partial y}}{\frac{\partial u^j}{\partial x^j}}\frac{\partial f}{\partial q}$$

↑ (marginal benefit for agent i of one more unit of q^i)

↑ (cost in resources spent)

↑ (cost of increasing congestion for all agents)

(d) (i) increase y by 1 (small) unit. Resource spent in private good: 1 unit. Agent i is ready to sacrifice

$$\Delta x_i = \frac{\frac{\partial u^i}{\partial Y}\frac{\partial f}{\partial y}}{\frac{\partial u^i}{\partial x^i}}$$

for this additional unit of public good, which leaves her at the same utility level. At P.O. the sum of the marginal contributions must be equal to the marginal cost

$$\sum_i \frac{\frac{\partial u^i}{\partial Y}\frac{\partial f}{\partial y}}{\frac{\partial u^i}{\partial x^i}} = 1$$

(ii) increase q^i by 1 (small) unit: it costs $c'(q^i)$ units of private good. Agent i is ready to sacrifice

$$\Delta x^i = \frac{\partial u^i/\partial q^i}{\partial u^i/\partial x^i} + \frac{\partial u^i/\partial Y \;\; \partial Y/\partial q}{\frac{\partial u^i}{\partial x^i}}$$

which leaves her at the same utility level. Other agents need to be compensated for the increase in congestion to stay at the same utility level. Agent j needs

$$\Delta x^j = -\frac{\frac{\partial u^j}{\partial Y}\;\partial Y/\partial q}{\frac{\partial u^j}{\partial x^j}}$$

as compensation (remember $\partial Y/\partial q < 0$)

At a P.O. allocation $\Delta x^i - \sum_{j\neq i} \Delta x^j - c'(q^i) = 0$

or
$$\frac{\frac{\partial u^i}{\partial q^i}}{\frac{\partial u^i}{\partial x^i}} + \sum_{j=1}^{J} \frac{\frac{\partial u^j}{\partial y}\frac{\partial y}{\partial Q}}{\frac{\partial u^j}{\partial x^j}} - c'(q^i) = 0.$$

(e) At an equilibrium, each agent chooses her optimal intensity of use, taking $\bar{\tau}$, y, Q as given.

$$\max_{q^i} \; u^i\left(w^i(1-\bar{\tau}) - c(q^i), q^i, \bar{y} \right)$$

FOC : $-\frac{\partial u^i}{\partial x^i} c'(\bar{q}^i) + \frac{\partial u^i}{\partial q^i} = 0, \quad i = 1 \dots I$

where the arguments of the functions are the equilibrium values of the relevant variables. This can be written as

$$c'(\bar{q}^i) - \frac{\frac{\partial u^i}{\partial q^i}}{\frac{\partial u^i}{\partial x^i}} = 0, \quad i = 1 \dots I$$

which can not coincide with the FOC c(ii) at a P.O if $\frac{\partial y}{\partial Q} < 0$, i.e. if the quality of the public good consumption is a decreasing function of congestion. The intensity of use of agent i imposes an externality on the other agents by increasing congestion, and this is not taken into account by agents in equilibrium who consider themselves to be negligible. Imposing a tax on q^i (e.g. a toll) to internalize the negative external effect would improve on the equilibrium, as long as the tax is not "too high".

Question 4 - Answer Key

We were all pretty relieved that Ali from our first prelim got such a simple consumption problem. Yet, we also realized that putting him into the Cobb-Douglas straightjacket missed some features of the story. Since the overall theme of the second prelim should be "We can do better!", let's also improve Ali's consumption model. As before, write $x_a, x_t, x_g \geq 0$ for the amounts of his consumption of almonds, toothpicks, and gifts, respectively. We assume for simplicity that these goods are infinitesimally divisible. Let $x = (x_a, x_t, x_g)$. Instead of assuming a utility function of the Cobb-Douglas form, we assume now that his utility function is given by

$$u(x_a, x_t, x_g) = \alpha \ln(x_a - b_a) + \theta \ln(x_t - b_t) + (1 - \alpha - \theta) \ln(x_g - b_g)$$

with $\alpha, \theta, b_i > 0$, $\alpha + \theta < 1$ and $x_i - b_i > 0$ for $i \in \{a, t, g\}$. As before, his income or wealth is denoted by $w > 0$. Finally, we denote by $p_a, p_t, p_g > 0$ the prices of almonds, toothpicks, and gifts, respectively, and let $p = (p_a, p_t, p_g)$.

a. Use the Kuhn-Tucker approach to derive step-by-step the Walrasian demand function $x(p, w)$. Verify also second-order conditions.

Our optimization problem is given by $\max_{x \in \mathbb{R}^3_+} u(x)$ s.t. $p \cdot x \leq w$. We set up the Lagrange function

$$L(x_a, x_t, x_g, \lambda) = \alpha \ln(x_a - b_a) + \theta \ln(x_t - b_t) + (1 - \alpha - \theta) \ln(x_g - b_g) - \lambda(p_a x_a + p_t x_t + p_g x_g - w) \quad (1)$$

The first-order conditions are

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_a} = \frac{\alpha}{x_a - b_a} - \lambda p_a \equiv 0 \quad (2)$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_t} = \frac{\theta}{x_a - b_a} - \lambda p_t \equiv 0 \quad (3)$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_g} = \frac{1 - \alpha - \theta}{x_g - b_g} - \lambda p_g \equiv 0 \quad (4)$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial \lambda} = -p_a x_a - p_t x_t - p_g x_g + w \equiv 0 \quad (5)$$

The second-order conditions are

$$\frac{\partial^2 L(x_a, x_t, x_g, \lambda)}{\partial x_a^2} = -\frac{\alpha}{(x_a - b_a)^2} < 0 \quad (6)$$

$$\frac{\partial^2 L(x_a, x_t, x_g, \lambda)}{\partial x_t^2} = -\frac{\theta}{(x_t - b_t)^2} < 0 \quad (7)$$

$$\frac{\partial^2 L(x_a, x_t, x_g, \lambda)}{\partial x_g^2} = -\frac{1 - \alpha - \theta}{(x_g - b_g)^2} < 0 \quad (8)$$

$$\frac{\partial^2 L(x_a, x_t, x_g, \lambda)}{\partial \lambda^2} = 0 \quad (9)$$

We conclude from second-order conditions show that first-order conditions are both necessary and sufficient.

Next, we want to solve for Walrasian demands. As usual "many ways lead to Rome". I guess the simplest one is to use the fact that $\alpha + \theta + (1 - \alpha - \theta) = 1$. Rearrange equations (2) to (4) to obtain

$$\frac{\alpha}{x_a - b_a} = \lambda p_a \tag{10}$$
$$\frac{\theta}{x_t - b_t} = \lambda p_t \tag{11}$$
$$\frac{1 - \alpha - \theta}{x_g - b_g} = \lambda p_g \tag{12}$$

and further

$$\alpha = \lambda p_a(x_a - b_a) \tag{13}$$
$$\theta = \lambda p_t(x_t - b_t) \tag{14}$$
$$1 - \alpha - \theta = \lambda p_g(x_g - b_g) \tag{15}$$

Sum the last three equations to obtain

$$\alpha + \theta + (1 - \alpha - \theta) = \lambda p_a(x_a - b_a) + \lambda p_t(x_t - b_t) + \lambda p_g(x_g - b_g) \tag{16}$$
$$1 = \lambda((p_a x_a + p_t x_t + p_g x_g) - (p_a b_a + p_t b_t + p_g b_g)) \tag{17}$$
$$\lambda = \frac{1}{(p_a x_a + p_t x_t + p_g x_g) - (p_a b_a + p_t b_t + p_g b_g)} \tag{18}$$

Using equation (5), we can write

$$\lambda = \frac{1}{w - (p_a b_a + p_t b_t + p_g b_g)}. \tag{19}$$

Rearrange equations (10) to (12), respectively, to obtain

$$x_a - b_a = \frac{\alpha}{\lambda p_a} \tag{20}$$
$$x_t - b_t = \frac{\theta}{\lambda p_t} \tag{21}$$
$$x_g - b_g = \frac{1 - \alpha - \theta}{\lambda p_g} \tag{22}$$

and further

$$x_a = b_a + \frac{\alpha}{\lambda p_a} \tag{23}$$
$$x_t = b_t + \frac{\theta}{\lambda p_t} \tag{24}$$
$$x_g = b_g + \frac{1 - \alpha - \theta}{\lambda p_g} \tag{25}$$

Substituting (19) into these equations yields

$$x_a = b_a + \frac{\alpha}{p_a\left(\frac{1}{w-(p_a b_a + p_t b_t + p_g b_g)}\right)} \tag{26}$$

$$x_t = b_t + \frac{\theta}{p_t\left(\frac{1}{w-(p_a b_a + p_t b_t + p_g b_g)}\right)} \tag{27}$$

$$x_g = b_g + \frac{1-\alpha-\theta}{p_g\left(\frac{1}{w-(p_a b_a + p_t b_t + p_g b_g)}\right)} \tag{28}$$

which simply further to

$$x_a(p,w) = b_a + \frac{\alpha}{p_a}\left(w-(p_a b_a + p_t b_t + p_g b_g)\right) \tag{29}$$

$$x_t(p,w) = b_t + \frac{\theta}{p_t}\left(w-(p_a b_a + p_t b_t + p_g b_g)\right) \tag{30}$$

$$x_g(p,w) = b_g + \frac{1-\alpha-\theta}{p_g}\left(w-(p_a b_a + p_t b_t + p_g b_g)\right) \tag{31}$$

b. Verify that the demand function is homogenous of degree zero and satisfies Walras' Law.

For homogeneity of degree zero, we need to show $x_i(\gamma w, \gamma p) = x_i(w,p)$ for all $\gamma > 0$, $i \in \{a,t,g\}$. But this is obvious, e.g., $x_a(\gamma w, \gamma p) = b_a + \frac{\alpha}{\gamma p_a}\left(\gamma w - (\gamma p_a b_a + \gamma p_t b_t + \gamma p_g b_g)\right) = b_a + \frac{\alpha}{\gamma p_a}\left(\gamma w - \gamma(p_a b_a + p_t b_t + p_g b_g)\right) = b_a + \frac{\alpha}{\not{\gamma} p_a}\left(\not{\gamma} w - \not{\gamma}(p_a b_a + p_t b_t + p_g b_g)\right) = b_a + \frac{\alpha}{p_a}\left(w - (p_a b_a + p_t b_t + p_g b_g)\right) = x_a(p,w)$ (analogously for toothpicks and gifts).

To verify Walras' Law, we need to verify

$$p_a x_a(p,w) + p_t x_t(p,w) + p_g x_g(p,w) = w.$$

To this end, it will be helpful to write Walrasian demand functions in "expenditure form", i.e.,

$$p_a x_a = p_a b_a + \alpha\left(w-(p_a b_a + p_t b_t + p_g b_g)\right) \tag{32}$$

$$p_t x_t = p_t b_t + \theta\left(w-(p_a b_a + p_t b_t + p_g b_g)\right) \tag{33}$$

$$p_g x_g = p_g b_g + (1-\alpha-\theta)\left(w-(p_a b_a + p_t b_t + p_g b_g)\right) \tag{34}$$

Now we see that the expenditures are linear in price and wealth. For that reason this demand system is called the Linear Expenditure System (LES).

Summing over the last equations yields

$$p_a x_a + p_t x_t + p_g x_g = p_a b_a + p_t b_t + p_g b_g + (\alpha + \theta + (1-\alpha+\theta))(w - (p_a b_a + p_t b_t + p_g b_g)) \tag{35}$$

$$p_a x_a + p_t x_t + p_g x_g = p_a b_a + p_t b_t + p_g b_g + (w - (p_a b_a + p_t b_t + p_g b_g)) \tag{36}$$

$$p_a x_a + p_t x_t + p_g x_g = w \tag{37}$$

c. Provide a verbal interpretation of this demand system. (It will be helpful to consider the demand system in its "expenditure form" by multiplying both sides of each demand equation by its respective price.)

From the solution to the previous answer, we saw that expenditures are linear in price and wealth. For that reason this demand system is called the Linear Expenditure System (LES). Ali behaves as if he is committed to necessarily buy quantities (b_a, b_t, b_g) of almonds, toothpicks, and gifts, respectively, and divide his remaining expenditure $w - p_a b_a - p_t b_t - p_g b_g$ among almonds, toothpicks, and gifts in fixed proportions α, θ, and $1 - \alpha - \theta$, respectively.

d. You would expect that the more almonds Ali eats, the more they get stuck in his teeth and the more toothpicks he purchases. In light of such considerations, does it make sense to assume Ali has the utility function above? (Consider changes in the demand for almonds and toothpicks caused by changes in the price of almonds and changes in the parameters b_a and α, respectively.)

If the price of almonds goes up, then it follows from $x_a(p, w)$ that his demand for almonds goes down. Moreover, $\frac{\partial x_t(p,w)}{\partial p_a} = -\frac{\theta}{p_t} b_a < 0$. The demand of toothpicks decreases as well, which makes at least qualitatively some sense.

If the "necessary" consumption of almonds b_a goes up, then $x_a(p, w)$ also shifts up since $\frac{\partial x_a(p,w)}{\partial b_a} = 1 - \frac{\alpha}{p_a} p_a = 1 - \alpha > 0$. Yet, the consumption of toothpicks goes down because $\frac{\partial x_t(p,w)}{\partial b_a} = -\frac{\theta}{p_t} p_a < 0$, which does not make sense given the story.

Similarly, if the fixed proportion of wealth that is not bound by "necessary consumption" increases slightly, then $x_a(p, w)$ increases while $x_t(p, w)$ stays constant since $\frac{\partial x_a(p,w)}{\partial \alpha} = \frac{1}{p_a} (w - p_a b_a - p_t b_t - p_g b_g) > 0$ and $x_t(p, w)$ does not depend on α. So again, this does not make sense given the story.

e. Consider now a utility function given by

$$\hat{u}(x_a, x_t, x_g) = (x_a - b_a)^{\alpha}(x_t - b_t)^{\theta}(x_g - b_g)^{1-\alpha-\theta},$$

with $\alpha, \theta, b_i > 0$, $\alpha + \theta < 1$, and $x_i - b_i > 0$ for all $i \in \{a, t, g\}$. How is this utility function related to the one given at the beginning of Question 4?

Clearly, the utility function given at the beginning of question 4 is a monotone transformation of the utility function given here using the logarithmus naturalis.

f. Remember that when Professor Schipper interviewed Ali about how exactly he arrives at his optimal consumption bundle, Ali expressed ignorance about maximizing utility subject to his budget constraint. Instead, he seemed to minimize his expenditure on consumption such that he reaches a certain level of utility. A smart undergraduate student walked by and claimed that this is clear evidence against the assumption of utility maximization in economics. Since Professor Schipper likes Linear Expenditure Systems as much as Cobb-Douglas utility functions, he conveniently sent the student to you so that you can show him how expenditure minimization works. Again, use the Kuhn-Tucker approach to derive the Hicksian demand function but use the utility function in part e instead.

Simplification: Let's not write our fingers to the bone. Assume from now on (for all parts f. to ℓ) that Ali got rid of his girlfriend. Sure you must be sad about it but there is clearly a tradeoff between having a girlfriend and completing successfully and on time a prelim exam. Most important to Ali: No more gifts! Thus, we can consider now the case of two goods, almonds and toothpicks, only. Set $\theta = 1 - \alpha$ to economize on parameters.

For the answer key, I will keep all three goods in case it is of any interest.

The expenditure minimization problem is $\min_{x \in \mathbb{R}^3_+} p \cdot x$ subject to $u(x) \geq \bar{u}$ where $\bar{u} > u(0)$. First, we set up the Lagrange function

$$L(x_a, x_t, x_g, \lambda) = p_a x_a + p_t x_t + p_g x_g - \lambda \left((x_a - b_a)^{\alpha} (x_t - b_t)^{\theta} (x_g - b_g)^{1-\alpha-\theta} - \bar{u} \right) \tag{38}$$

We derive first-order conditions

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_a} = p_a - \lambda \alpha \left((x_a - b_a)^{\alpha-1} (x_t - b_t)^{\theta} (x_g - b_g)^{1-\alpha-\theta} \right) \equiv 0 \tag{39}$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_t} = p_t - \lambda \theta \left((x_a - b_a)^{\alpha} (x_t - b_t)^{\theta-1} (x_g - b_g)^{1-\alpha-\theta} \right) \equiv 0 \tag{40}$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_g} \tag{41}$$

$$= p_g - \lambda (1 - \alpha - \theta) \left((x_a - b_a)^{\alpha} (x_t - b_t)^{\theta} (x_g - b_g)^{-\alpha-\theta} \right) \equiv 0$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial \lambda} = - \left((x_a - b_a)^{\alpha} (x_t - b_t)^{\theta} (x_g - b_g)^{1-\alpha-\theta} \right) + \bar{u} \equiv 0 \tag{42}$$

Taking the ratio of (39) and (40) as well as (39) and (41), respectively, we obtain

$$\frac{p_a}{p_t} = \frac{\alpha (x_t - b_t)}{\theta (x_a - b_a)} \tag{43}$$

$$\frac{p_a}{p_g} = \frac{\alpha (x_g - b_g)}{(1 - \alpha - \theta)(x_a - b_a)} \tag{44}$$

We solve for x_t and x_g, respectively, as a function of x_a to obtain

$$x_t = \frac{\theta p_a (x_a - b_a)}{\alpha p_t} + b_t \tag{45}$$

$$x_g = \frac{(1 - \alpha - \theta) p_a (x_a - b_a)}{\alpha p_g} + b_g \tag{46}$$

Substituting into the utility function and factoring out any variable and parameter pertaining to almonds,

$$
\begin{aligned}
u(x_a, x_t, x_g) &= (x_a - b_a)^{\alpha} \left(\frac{\theta p_a (x_a - b_a)}{\alpha p_t} + b_t - b_t \right)^{\theta} \left(\frac{(1-\alpha-\theta) p_a (x_a - b_a)}{\alpha p_g} + b_g - b_g \right)^{1-\alpha-\theta} && (47) \\
&= (x_a - b_a)^{\alpha} \left(\frac{\theta p_a (x_a - b_a)}{\alpha p_t} \right)^{\theta} \left(\frac{(1-\alpha-\theta) p_a (x_a - b_a)}{\alpha p_g} \right)^{1-\alpha-\theta} && (48) \\
&= \left(\frac{\alpha p_a (x_a - b_a)}{\alpha p_a} \right)^{\alpha} \left(\frac{\theta p_a (x_a - b_a)}{\alpha p_t} \right)^{\theta} \left(\frac{(1-\alpha-\theta) p_a (x_a - b_a)}{\alpha p_g} \right)^{1-\alpha-\theta} && (49) \\
&= \left(\frac{p_a (x_a - b_a)}{\alpha} \right) \left(\frac{\alpha}{p_a} \right)^{\alpha} \left(\frac{\theta}{p_t} \right)^{\theta} \left(\frac{1-\alpha-\theta}{p_g} \right)^{1-\alpha-\theta} && (50)
\end{aligned}
$$

Let $\bar{u} = u(x_a, x_t, x_g)$ and solve for x_a as a function of $\bar{u}$ and p. This is the Hicksian demand function for almonds. So may want to denote it like in class by $h_a(p, \bar{u})$. It is helpful to focus first on the difference $x_a - b_a$. Divide both sides of the last equation by $\left(\frac{\alpha}{p_a}\right)^{\alpha} \left(\frac{\theta}{p_t}\right)^{\theta} \left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}$ to obtain

$$
\begin{aligned}
\frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha} \left(\frac{\theta}{p_t}\right)^{\theta} \left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} &= \frac{p_a}{\alpha} && (51) \\
x_a - b_a &= \frac{\alpha}{p_a} \frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha} \left(\frac{\theta}{p_t}\right)^{\theta} \left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} && (52)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
x_t - b_t &= \frac{\theta}{p_t} \frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha} \left(\frac{\theta}{p_t}\right)^{\theta} \left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} && (53) \\
x_g - b_g &= \frac{1-\alpha-\theta}{p_g} \frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha} \left(\frac{\theta}{p_t}\right)^{\theta} \left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} && (54)
\end{aligned}
$$

Therefore, the Hicksian demand functions are

$$
\begin{aligned}
h_a(p, \bar{u}) &= \bar{u} \frac{\alpha}{p_a} \left(\frac{p_a}{\alpha}\right)^{\alpha} \left(\frac{p_t}{\theta}\right)^{\theta} \left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta} + b_a && (55) \\
h_t(p, \bar{u}) &= \bar{u} \frac{\theta}{p_t} \left(\frac{p_a}{\alpha}\right)^{\alpha} \left(\frac{p_t}{\theta}\right)^{\theta} \left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta} + b_t && (56) \\
h_g(p, \bar{u}) &= \bar{u} \frac{1-\alpha-\theta}{p_g} \left(\frac{p_a}{\alpha}\right)^{\alpha} \left(\frac{p_t}{\theta}\right)^{\theta} \left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta} + b_g && (57)
\end{aligned}
$$

g. Derive the expenditure function.

$$
\begin{aligned}
e(p,\bar{u}) &= p_a h_a(p,\bar{u}) + p_t h_t(p,\bar{u}) + p_g h_g(p,\bar{u}) && (58)\\
&= p_a\left(\bar{u}\frac{\alpha}{p_a}\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}+b_a\right)\\
&\quad +p_t\left(\bar{u}\frac{\theta}{p_t}\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}+b_t\right)\\
&\quad +p_g\left(\bar{u}\frac{1-\alpha-\theta}{p_g}\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}+b_g\right) && (59)\\
&= \bar{u}\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}(\alpha+\theta+(1-\alpha-\theta))\\
&\quad +p_a b_a + p_t b_t + p_g b_g && (60)\\
&= \bar{u}\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}+p_a b_a + p_t b_t + p_g b_g && (61)
\end{aligned}
$$

h. Show that the expenditure function is homogeneous of degree 1 in prices, strictly increasing in $\bar{u}$ as well as nondecreasing and concave in the price of each good taken separately.

It is easy to see that it is homogenous of degree 1 in prices, strictly increasing in $\bar{u}$, and nondecreasing in prices. For concavity in prices, differentiate twice the expenditure function w.r.t. p_a,

$$
\frac{\partial^2 e(p,\bar{u})}{\partial p_a^2} = (\alpha-1)\bar{u}\left(\frac{p_a}{\alpha}\right)^{\alpha-2}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta} < 0 \qquad (62)
$$

which is clearly negative since $\bar{u} > u(0)$ and $\alpha - 1 < 0$ (analogous w.r.t. to p_t and p_g, respectively).

i. Derive Ali's indirect utility function using the expenditure function just derived.

We know that the expenditure function and indirect utility function are inverse to each other. Thus, from equation (61) we obtain

$$
v(w,p) = \frac{w - p_a b_a - p_t b_t - p_g b_g}{\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}} \qquad (63)
$$

j. Verify that Ali satisfies Roy's identity with respect to almonds.

Roy's identity with respect to almonds reads

$$
x_a(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_a}}{\frac{\partial v(p,w)}{\partial w}} \qquad (64)
$$

$$\frac{\partial v(p,w)}{\partial p_a} = \tag{65}$$

$$\frac{-b_a\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right) - (w - p_a b_a - p_t b_t - p_g b_g)\alpha\left(\frac{p_a}{\alpha}\right)^{\alpha-1}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\frac{1}{\alpha}}{\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right)^2}$$

$$= \frac{-b_a\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right) - (w - p_a b_a - p_t b_t - p_g b_g)\left(\frac{p_a}{\alpha}\right)^{-1}\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}}{\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right)^2}$$

$$= \frac{-b_a - (w - p_a b_a - p_t b_t - p_g b_g)\left(\frac{p_a}{\alpha}\right)^{-1}}{\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}} \tag{66}$$

$$\frac{\partial v(p,w)}{\partial w} = \frac{1}{\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}} \tag{67}$$

Hence,

$$-\frac{\frac{\partial v(p,w)}{\partial p_a}}{\frac{\partial v(p,w)}{\partial w}} = b_a + \frac{\alpha}{p_a}(w - p_a b_a - p_t b_t - p_g b_g) \tag{68}$$

$$= x_a(p,w) \tag{69}$$

k. Verify the (own price) Slutsky equation for the example of almonds.

We need to verify

$$\frac{\partial h_a(p,\bar{u})}{\partial p_a} = \frac{\partial x_a(p,w)}{\partial p_a} + \frac{\partial x_a(p,w)}{\partial w} x_a(p,w) \tag{70}$$

for $\bar{u} = v(p,w)$. Focus first on the left-hand side. Taking the partial derivative of

Hicksian demand for almonds w.r.t. to p_a yields

$$\frac{\partial h_a(p,\bar{u})}{\partial p_a} = (-1)\bar{u}\frac{\alpha}{p_a^2}\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right) \tag{71}$$

$$+\bar{u}\frac{\alpha}{p_a}\alpha\frac{1}{\alpha}\left(\left(\frac{p_a}{\alpha}\right)^{\alpha-1}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right) \tag{72}$$

$$= (-1)\bar{u}\frac{\alpha}{p_a^2}\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right) \tag{73}$$

$$+\bar{u}\frac{\alpha}{p_a}\frac{\alpha}{p_a}\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right) \tag{74}$$

$$= \bar{u}\frac{\alpha}{p_a^2}(\alpha-1)\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right) \tag{75}$$

Since $\bar{u} = v(p,w)$,

$$\frac{\partial h_a(p,\bar{u})}{\partial p_a} = \tag{76}$$

$$\frac{w - p_a b_a - p_t x_t - p_g x_g}{\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}}\frac{\alpha}{p_a^2}(\alpha-1)\left(\left(\frac{p_a}{\alpha}\right)^{\alpha}\left(\frac{p_t}{\theta}\right)^{\theta}\left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}\right) \tag{77}$$

$$= (w - p_a b_a - p_t b_t - p_g b_g)\frac{\alpha}{p_a^2}(\alpha-1) \tag{78}$$

Now turn to the right-hand side.

$$\frac{\partial x_a(p,w)}{\partial p_a} = -\frac{\alpha}{p_a^2}(w - p_a b_a - p_t b_t - p_g b_g) - \frac{\alpha}{p_a}b_a \tag{79}$$

$$\frac{\partial x_a(p,w)}{\partial w} = \frac{\alpha}{p_a} \tag{80}$$

Thus,

$$\frac{\partial x_a(p,w)}{\partial w}x_a(p,w) = \frac{\alpha}{p_a^2}\left(b_a + \frac{\alpha}{p_a}(w - p_a b_a - p_t b_t - p_g b_g)\right) \tag{81}$$

$$= \frac{\alpha}{p_a}b_a + \frac{\alpha^2}{p_a^2}(w - p_a b_a - p_t b_t - p_g b_g) \tag{82}$$

Adding (79) and (82) yields

$$\frac{\partial x_a(p,w)}{\partial p_a} + \frac{\partial x_a(p,w)}{\partial w}x_a(p,w) = -\frac{\alpha}{p_a^2}(w - p_a b_a - p_t b_t - p_g b_g) - \frac{\alpha}{p_a}b_a \tag{83}$$

$$+\frac{\alpha}{p_a}b_a + \frac{\alpha^2}{p_a^2}(w - p_a b_a - p_t b_t - p_g b_g)$$

$$= \frac{\alpha^2-\alpha}{p_a^2}(w - p_a b_a - p_t b_t - p_g b_g) \tag{84}$$

This finishes verifying the Slutsky equation.

ℓ. Because of the drought, the price of almonds changes from p_a^0 to p_a^1. The newly hired Senior Vice Provost for Crocodile Welfare at UC Davis is eager to prove his usefulness by offering Ali to sacrifice part of his income in return to continue offering almonds from campus at the old price p_a^0. Ali is not too excited but likes to find out more about the proposal, in particular about how much income he would be willing to sacrifice in return. He plans to turn to Professor Schipper for advice on the exact income deduction that would make him indifferent between accepting this deduction or the higher price of almonds. Unfortunately, Professor Schipper is away in Europe. Luckily the arboretum is buzzing with first year econ PhD students rehearsing microeconomics. So he approaches you to calculate the amount. Help him!

Essentially the problem asks for the Equivalent Variation. It is a measure of change of welfare of a consumer due to a price change. It measures in units of wealth the amount Ali would just be willing to sacrifice in lieu of the price increase.

Let $p^0 = (p_a^0, p_t, p_g)$, $p^1 = (p_a^1, p_t, p_g)$, $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$. The Equivalent Variation is defined by

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) \tag{85}$$

Since Ali spends his entire wealth in the expenditure minimizing consumption bundle, i.e., $e(p^0, u^0) = w$, we have

$$EV(p^0, p^1, w) = e(p^0, u^1) - w \tag{86}$$

We compute

$$\begin{aligned}
e(p^0, u^1) &= u^1 \left(\frac{p_a^0}{\alpha}\right)^{\alpha} \left(\frac{p_t}{\theta}\right)^{\theta} \left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta} + p_a^0 b_a + p_t b_t + p_g b_g && (87)\\
&= v(p^1, w) \left(\frac{p_a^0}{\alpha}\right)^{\alpha} \left(\frac{p_t}{\theta}\right)^{\theta} \left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta} + p_a^0 b_a + p_t b_t + p_g b_g && (88)\\
&= \left(\frac{w - p_a^1 b_a - p_t b_t - p_g b_g}{\left(\frac{p_a^1}{\alpha}\right)^{\alpha} \left(\frac{p_t}{\theta}\right)^{\theta} \left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta}}\right) \left(\frac{p_a^0}{\alpha}\right)^{\alpha} \left(\frac{p_t}{\theta}\right)^{\theta} \left(\frac{p_g}{1-\alpha-\theta}\right)^{1-\alpha-\theta} \\
&\quad + p_a^0 b_a + p_t b_t + p_g b_g && (89)\\
&= \frac{\left(\frac{p_a^0}{\alpha}\right)^{\alpha}}{\left(\frac{p_a^1}{\alpha}\right)^{\alpha}} (w - p_a^1 b_a - p_t b_t - p_g b_g) + p_a^0 b_a + p_t b_t + p_g b_g && (90)\\
&= \left(\frac{p_a^0}{p_a^1}\right)^{\alpha} (w - p_a^1 b_a - p_t b_t - p_g b_g) + p_a^0 b_a + p_t b_t + p_g b_g && (91)
\end{aligned}$$

Thus,

$$EV(p^0, p^1, w) = \left(\frac{p_a^0}{p_a^1}\right)^{\alpha} (w - p_a^1 b_a - p_t b_t - p_g b_g) - (w - p_a^0 b_a - p_t b_t - p_g b_g) \tag{92}$$