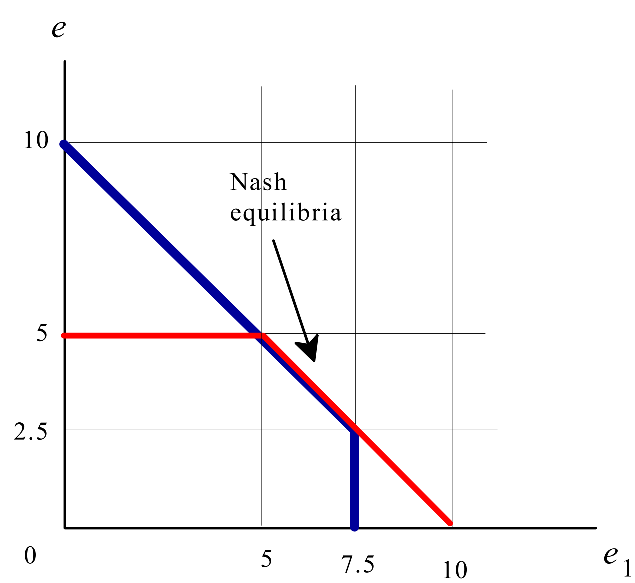
Micro Prelim June 2016 – ANSWER KEYS

QUESTION 1

(a) $u_1(e_1,e_2)=\begin{cases} e_1+e_2-\dfrac{(e_1)^2}{15} & \text{if } e_1+e_2\le 10 \\ 10-\dfrac{(e_1)^2}{15} & \text{if } e_1+e_2>10 \end{cases}$, $u_2(e_1,e_2)=\begin{cases} e_1+e_2-\dfrac{(e_2)^2}{10} & \text{if } e_1+e_2\le 10 \\ 10-\dfrac{(e_2)^2}{10} & \text{if } e_1+e_2>10 \end{cases}$

(b) The expression $e_1+e_2-\frac{(e_1)^2}{15}$ is maximized at $e_1=7.5$. If $e_2\le 2.5$ then $e_2+7.5\le 10$ and thus $e_1=7.5$ is indeed the optimal choice for Player 1. If $e_2>2.5$ then $e_1=7.5$ would take us into the range where $e_1+e_2>10$ and in that range u_1 is strictly decreasing in e_1, so that the optimal choice is $e_1=10-e_2$. Thus the best reply function of Player 1 is $b_1(e_2)=\begin{cases} 7.5 & \text{if } e_2\le 2.5 \\ 10-e_2 & \text{if } 2.5<e_2\le 10 \\ 0 & \text{if } e_2>10 \end{cases}$. Similar reasoning leads to $b_2(e_1)=\begin{cases} 5 & \text{if } e_1\le 5 \\ 10-e_1 & \text{if } 5<e_1\le 10 \\ 0 & \text{if } e_1>10 \end{cases}$

(c) See figure below. The Nash equilibria are given by the intersection of the two curves, that is all the points (e_1,e_2) with $e_1\in[5,7.5]$ and $e_2=10-e_1$.



(d) Player 2's choice as a function of Player 1's choice is given by 2's best reply function $b_2(e_1)=\begin{cases} 5 & \text{if } e_1\le 5 \\ 10-e_1 & \text{if } 5<e_1\le 10 \\ 0 & \text{if } e_1>10 \end{cases}$. Thus Player 1 chooses e_1 to maximize

$$U_1(e_1)=u_1\left(e_1,b_2(e_1)\right)=\begin{cases} e_1+5-\dfrac{(e_1)^2}{15} & \text{if } e_1\le 5 \\ 10-\dfrac{(e_1)^2}{15} & \text{if } 5<e_1\le 10 \\ e_1-\dfrac{(e_1)^2}{15} & \text{if } e_1>10 \end{cases}$$

. The solution is $e_1=5$ and thus the subgame-perfect equilibrium is $\left(5,b_2(e_1)\right)$ with actual choices being $e_1=e_2=5$.

(e) **(e.1)** A pure strategy for Player 1 specifies his initial effort level, as well as his additional effort level as a function of both his initial effort level and Player 2's effort level.

(e.2) A pure strategy for Player 2 specifies her effort level as a function of Player 1's initial effort level.

(e.3) A pure strategy for Player 1 is: "my initial choice is $e_1 = 4$ and then (1) if my initial choice was 3 and Player 2 chooses 3 then I choose $\delta_1 = 4$ while if Player 2 chooses 4 then I choose $\delta_1 = 3$ and (2) if my initial choice was 4 and Player 2 chooses 3 then I choose $\delta_1 = 3$ while if Player 2 chooses 4 then I choose $\delta_1 = 4$. A pure strategy for Player 2 is: "if Player 1 chooses 3 then I choose 4 and if Player 1 chooses 4 then I choose 3".

(f) The payoff function of Player 1 is $\pi_1(e_1,\delta_1,e_2) = \begin{cases} e_1+\delta_1+e_2 - \dfrac{(e_1+\delta_1)^2}{15} & \text{if } e_1+e_2+\delta_1 \le 10 \\ 10 - \dfrac{(e_1+\delta_1)^2}{15} & \text{if } e_1+e_2+\delta_1 > 10 \end{cases}$.

We use backward induction. At the end of the game, as shown in part (b), it is optimal for Player 1 to target the sum of his efforts to $e_1+\delta_1 = 7.5$ if $e_2 \le 2.5$ and to target the total effort to $e_1+e_2+\delta_1 = 10$ if $e_2 > 2.5$. Hence, Player 1's optimal strategy at the end of the game is

$$\delta_1(e_1,e_2) = \begin{cases} 7.5-e_1 & \text{if } e_2 \le 2.5 \text{ and } e_1 \le 7.5 \\ 10-e_1-e_2 & \text{if } e_2 > 2.5 \text{ and } e_1+e_2 \le 10 \\ 0 & \text{if } e_2 \le 2.5 \text{ and } e_1 > 7.5 \text{ or } e_2 > 2.5 \text{ and } e_1+e_2 > 10 \end{cases}$$

In the middle of the game, Player 2 realizes that if she chooses $e_2 \ge 2.5$, Player 1's additional effort will bring the total effort up to at least 10. Thus player 2 should never choose an effort level higher than 2.5. If Player 1 chose an effort $e_1 \le 7.5$ in period 1, then this is exactly what Player 2 should do. But if Player 1 chose a higher effort level, then Player 2 need only exert enough effort to bring the total effort up to 10. Thus, Player 2's strategy in any subgame perfect equilibrium is described by

$$e_2(e_1) = \begin{cases} 2.5 & \text{if } e_1 \le 7.5 \\ 10-e_1 & \text{if } 7.5 < e_1 \le 10 \\ 0 & \text{if } e_1 > 10 \end{cases}$$

In the initial period, Player 1 should never choose an effort higher than 7.5, since choosing 7.5 is enough to guarantee that the total effort will be 10. But any $e_1 \in [0, 7.5]$ will cause Player 2 to choose $e_2 = 2.5$, which Player 1 will follow by choosing $\delta_1 = 7.5 - e_1$. Thus any $e_1 \in [0, 7.5]$ can be chosen initially in a subgame perfect equilibrium and there are many subgame-perfect equilibria.

QUESTION 2

(a) This is the standard case where Vickrey's theorem applies: $b_1 = v_1$ is a weakly dominant strategy.

(b) Recall that $p_1 < v_1 \le p_m$. No, it is not. We need to distinguish two cases. Case 1: $v_1 < p_m$; in this case bidding v_1 is not a dominant strategy: if Player 2 bids p_m then the outcome is $(2, v_1)$ and Player 1 would prefer bidding p_1, since – by benevolence – $(2, p_1) \succ_1 (2, v_1)$. Case 2: $v_1 = p_m$; also in this case bidding v_1 is not a dominant strategy: if Player 2 bids p_m then the outcome is $(1, v_1 = p_m)$ and Player 1 would prefer bidding any p_k with $k < m$ (thus letting Player 2 win), since – by benevolence – $(2, p_k) \succ_1 (2, p_m) \sim_1 (1, v_1)$.

(c) Recall that $p_1 < v_1 \le p_m$. Again we need to distinguish two cases. Case 1: $v_1 > p_2$; in this case bidding v_1 is not a dominant strategy: if Player 2 also bids v_1 then the outcome is $(1, v_1)$ and, since $(1, v_1) \sim_1 (2, p_1) \prec (2, p_2)$, Player 1 would prefer bidding p_2, inducing the outcome $(2, p_2)$. Case

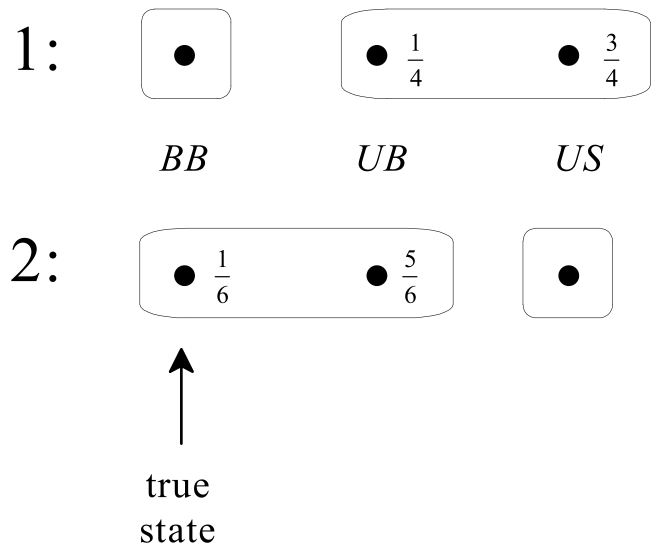
2: $v_1 \le p_2$; also in this case bidding v_1 is not a dominant strategy: if Player 2 bids p_4 (recall that $m > 3$) then the outcome is $(2, v_1)$ and Player 1 would prefer bidding p_3, inducing the outcome $(2, p_3)$ which she prefers to $(2, p_2)$ which, in turn, is at least as good as $(2, v_1)$ (since $v_1 \le p_2$).

(d) The assumption is that both players are selfish and **uncaring** and $B = \{1, 2, 3 = v_1, 4, 5 = v_2\}$. Since bidding one's own true value is a weakly dominant strategy, (3,5) is a Nash equilibrium; however it is not the only Nash equilibrium. All of the following are Nash equilibria: (1,5), (2,5), (3,5), (4,5), (1,4), (2,4), (3,4), (1,3), (2,3), (5,1), (5,2) and (5,3). Thus a total of 12 equilibria.

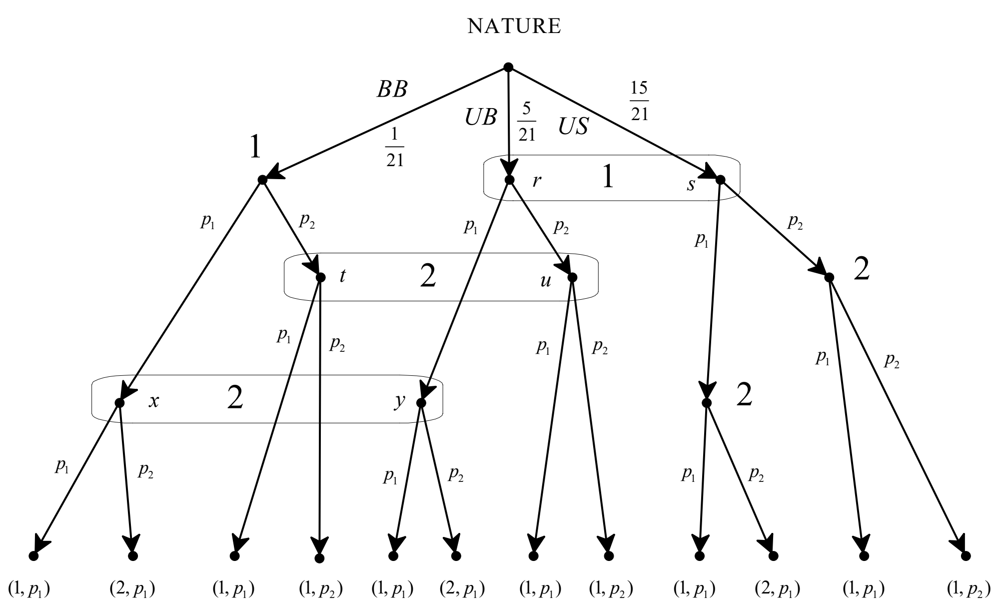
(e) The assumption is that both players are selfish and **benevolent** and $B = \{1, 2, 3 = v_1, 4, 5 = v_2\}$. The Nash equilibria are (1,3), (1,4), (1,5) and (5,1). A total of 4.

(f) The assumption is that both players are selfish and **spiteful** and $B = \{1, 2, 3 = v_1, 4, 5 = v_2\}$. The Nash equilibria are: (4,5), (3,4), (2,3). A total of 3.

(g) In the following figure the first element of the double label refers to Player 1 and the second to Player 2; thus, for example, *UB* means that Player 1 is *U* and Player 2 is *B*.

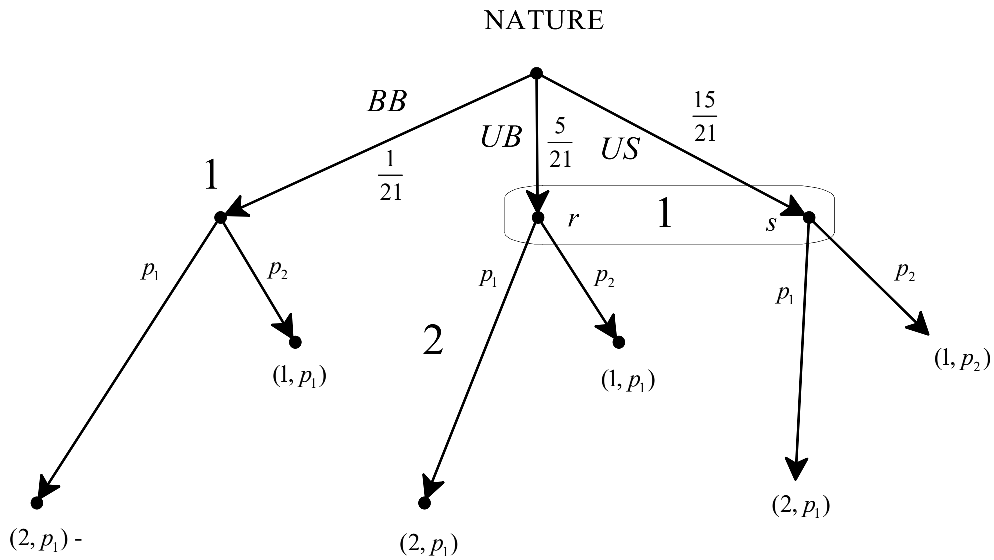


(h.1) The extensive form is as follows (note that there is a common prior):



(h.2) It is possible to find a weak sequential equilibrium. The reasoning is as follows.

At the right-most singleton node of Player 2, Player 2 is of type *S* and thus prefers $(1, p_2)$ to $(1, p_1)$, so that p_2 is the only sequentially rational choice. At the other singleton node of Player 2, Player 2 is, again, of type *S* and prefers $(2, p_1)$ to $(1, p_1)$, so that the only sequentially rational choice is p_2. At both nodes *t* and *u* Player 2 is of type *B* and thus prefers $(1, p_1)$ to $(1, p_2)$, so that p_1 strictly dominates p_2 at information set {*t,u*}; thus p_1 is the only sequentially rational choice there. At both nodes *x* and *y,* Player 2 is of type *B* and prefers $(2, p_1)$ to $(1, p_1)$, so that p_2 strictly dominates p_1 at information set {*x,y*}; thus p_2 is the only sequentially rational choice there. Hence the game can be reduced to the following:



At the singleton Player 1 prefers $(1, p_1)$ to $(2, p_1)$ and thus p_2 is the only sequentially rational choice. At nodes *r* and *s* Player 1 is of type *U* and thus (1) prefers $(1, p_1)$ to $(1, p_2)$ and (2) is indifferent between $(1, p_2)$ to $(2, p_1)$ so that, at information set {*r,s*}, p_2 weakly dominates p_1 and hence p_2 must be the only sequentially rational choice for Player 1 no matter what her von Neumann-Morgenstern function is, because by Bayesian updating she must have beliefs $\begin{pmatrix} r & s \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$ (thus attaching positive probability to node *r*). Thus the following is a weak sequential equilibrium of the original game:

- Player 1 chooses p_2 at her singleton information set and also at her information set {*r,s*}.
- Player 2 chooses p_2 at information set {*x,y*}, p_1 at information set {*t,u*}, and p_2 at both singleton nodes. -
- The beliefs are $\begin{pmatrix} r & s \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$, $\begin{pmatrix} t & u \\ \frac{1}{6} & \frac{5}{6} \end{pmatrix}$ and $\begin{pmatrix} x & y \\ p & 1-p \end{pmatrix}$ for any $p \in [0,1]$.

QUESTION 3

3.(a) Competitive equilibrium: prices $(p, 1)$ (1 = price of labor serves as numeraire). Because of firm 1, $p = 1$, and profit of firm 1 is zero. Firm 2 maximizes $2\sqrt{L_2} - L_2 \Rightarrow L_2 = 1$. profit of firm 2 $= 1$. Demand of agent a $x^a = \frac{1}{2} \cdot 3 = \ell^a$

Demand of agent b $x^b = \frac{1}{2} \cdot 4 = \ell^b$. Equilibrium:

$(1, 1)$	$(1.5, 1.5)$	$(2, 2)$	$(1.5, 1.5)$	$(2, 1)$
prices	agent a	agent b	firm 1	firm 2

(b) If firm 2 looses ε unit of labor, its production decreases by $dy_2 = g'(1)\,\varepsilon = \varepsilon$. So the consumption of each agent decreases by $\varepsilon/2$ and the external effect decrease by ε.

change in utility for agent a: $-\frac{\partial u_a}{\partial x_a}\frac{\varepsilon}{2} + \frac{\partial u_a}{\partial \ell_a}\frac{\varepsilon}{2} + \frac{1}{8}\varepsilon$

$$= -\frac{\sqrt{1.5}}{2\sqrt{1.5}}\frac{\varepsilon}{2} + \frac{\sqrt{1.5}}{2\sqrt{1.5}}\frac{\varepsilon}{2} + \frac{1}{8}\varepsilon = \frac{1}{8}\varepsilon$$

change in utility for agent b $-\frac{\partial u_b}{\partial x_b}\frac{\varepsilon}{2} + \frac{\partial u_b}{\partial \ell_b}\frac{\varepsilon}{2} + \frac{1}{8}\varepsilon$

$$= -\frac{\sqrt{2}}{2\sqrt{2}}\frac{\varepsilon}{2} + \frac{\sqrt{2}}{2\sqrt{2}}\frac{\varepsilon}{2} + \frac{1}{8}\varepsilon = \frac{1}{8}\varepsilon$$

Thus it is possible to find a feasible allocation which improves both agent: the equilibrium is not P.O.

(c) regulated equilibrium: prices $(1, 1)$ because of firm 1 max problem of firm 2 $\max \left\{ 2\sqrt{L_2} - L_2 \mid 2\sqrt{L_2} \leq z^* \right\}$. (8)

If $z^* \leq 1$ $L_2 = \frac{(z^*)^2}{4}$ $y_2 = z^*$ profit $z^* - \frac{(z^*)^2}{4}$

demand of agent a $(1.5, 1.5)$

demand of agent b $\left(\frac{1}{2}\left(3 + z^* - \frac{(z^*)^2}{4}\right), \frac{1}{2}\left(3 + z^* - \frac{(z^*)^2}{4}\right) \right)$

equilibrium on labor market: agents want to sell

$\underbrace{3 - 1.5}_{\text{agent a}} + \underbrace{3 - 1.5 - \frac{z^*}{2} + \frac{(z^*)^2}{8}}_{\text{agent b}}$ units of labor

$= 3 - \frac{z^*}{2} + \frac{(z^*)^2}{8}$

Firm 2 uses $\frac{(z^*)^2}{4}$. Thus Firm 1 uses $3 - \frac{z^*}{2} - \frac{(z^*)^2}{8}$ units of labor to produce $3 - \frac{z^*}{2} - \frac{(z^*)^2}{8}$ units of goods.

Total amount produced: $z^* + 3 - \frac{z^*}{2} - \frac{(z^*)^2}{8}$ which serves the demand.

regulated equilibrium

$(1, 1)$ prices $(1.5, 1.5)$ agent a $\left(1.5 + \frac{z^*}{2} - \frac{(z^*)^2}{8}, 1.5 + \frac{z^*}{2} - \frac{(z^*)^2}{8}\right)$ agent b

$\left(3 - \frac{z^*}{2} - \frac{(z^*)^2}{8}, 3 - \frac{z^*}{2} - \frac{(z^*)^2}{8}\right)$ firm 1 $\left(z^*, \frac{(z^*)^2}{4}\right)$ firm 2

d) If firm 2 looses ε units of labor, the change in production is $dy_2 = -g'(L_2)\,\varepsilon = -\frac{2}{z^*}\varepsilon$ and $dz = dy_2$.

$$du_a = -\cancel{4}\,\frac{1}{2}\,\frac{\cancel{2}}{z^*}\,\frac{\varepsilon}{\cancel{2}} + \frac{1}{2}\,\frac{\varepsilon}{2} + \frac{1}{\cancel{8}_{4}}\,\frac{\cancel{2}}{z^*}\,\varepsilon = du_b$$

since $\frac{\partial u_a}{\partial x_a} = \frac{\partial u_a}{\partial \ell_a} = \frac{\partial u_b}{\partial x_b} = \frac{\partial u_b}{\partial \ell_b} = 1/2$

$$du_a = \varepsilon\left[-\frac{1}{2z^*} + \frac{1}{4} + \frac{1}{4z^*}\right] = \left(-\frac{1}{4z^*} + \frac{1}{4}\right)\varepsilon.$$

$du_a = 0 \iff z^* = 1$ which is the optimal limit on the pollution emitted by firm 2.

Question 4 - Answer Key

Ali, the crocodile, lives in the dark water of Putah Creek in the Arboretum just beside the UC Davis campus. To the delight of the ducks there, Ali is a vegetarian. Ali earns income from being employed by UC Davis as a life guard at Putah Creek. His task is to rescue students who fall into the water from time to time. He is especially busy around prelim exams when more students than usual jump into water out of despair. When being rescued by Ali, usually their mood lights up quickly in anticipation of a cool selfie with a crocodile that can be shared on Facebook. By the time they relapse upon realizing that the water damaged their cell phone beyond repair, the authorities have arrived to help (with their cell phones). Anyway, let's not digress further. While we certainly appreciate Ali's work, our academic interest is focused on his consumption behavior. Being a vegetarian, his main diet consists of almonds. They have the inconvenient feature of getting stuck between his 72 teeth. So he is also a rather heavy consumer of toothpicks. Finally there are his frequent gifts he purchases for Mathilda. Mathilda! She is the duck of the Arboretum who waddles most elegantly with her petite legs by delightfully moving well-shaped hips. He is really in love with her; at a platonic level of course so that nobody gets physically hurt.

We write $x_a, x_t, x_g \geq 0$ for the amounts of his consumption of almonds, toothpicks, and gifts, respectively. We assume for simplicity that these goods are infinitesimally divisible. Let $x = (x_a, x_t, x_g)$. We assume that his utility function is given by $u(x_a, x_t, x_g) = x_a^\alpha x_t^\theta x_g^{1-\alpha-\theta}$ with $\alpha, \theta \in (0,1)$ and $\alpha + \theta < 1$. His income or wealth is denoted by $w > 0$. Finally, we denote by $p_a, p_t, p_g > 0$ the prices of almonds, toothpicks, and gifts, respectively, and $p = (p_a, p_t, p_g)$.

a. Use the Kuhn-Tucker approach to derive step-by-step the Walrasian demand function $x(p, w)$. Verify also second-order conditions.

Our optimization problem is given by $\max_{x \in \mathbb{R}^3_+} u(x)$ s.t. $p \cdot x \leq w$. We set up the Lagrange function

$$L(x_a, x_t, x_g, \lambda) = x_a^\alpha x_t^\theta x_g^{1-\alpha-\theta} - \lambda(p_a x_a + p_t x_t + p_g x_g - w) \tag{1}$$

The first-order conditions are

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_a} = \alpha x_a^{\alpha-1} x_t^\theta x_g^{1-\alpha-\theta} - \lambda p_a \equiv 0 \tag{2}$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_t} = \theta x_a^\alpha x_t^{\theta-1} x_g^{1-\alpha-\theta} - \lambda p_t \equiv 0 \tag{3}$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial x_g} = (1-\alpha-\theta) x_a^\alpha x_t^\theta x_g^{-\alpha-\theta} - \lambda p_g \equiv 0 \tag{4}$$

$$\frac{\partial L(x_a, x_t, x_g, \lambda)}{\partial \lambda} = -p_a x_a - p_t x_t - p_g x_g + w \equiv 0 \tag{5}$$

The second-order conditions are

$$\frac{\partial^2 L(x_a, x_t, x_g, \lambda)}{\partial x_a^2} = (\alpha - 1)\alpha x_a^{\alpha-2} x_t^{\theta} x_g^{1-\alpha-\theta} < 0 \tag{6}$$

$$\frac{\partial^2 L(x_a, x_t, x_g, \lambda)}{\partial x_t^2} = (\theta - 1)\theta x_a^{\alpha} x_t^{\theta-2} x_g^{1-\alpha-\theta} < 0 \tag{7}$$

$$\frac{\partial^2 L(x_a, x_t, x_g, \lambda)}{\partial x_g^2} = -(\alpha + \theta)(1 - \alpha - \theta) x_a^{\alpha} x_t^{\theta} x_g^{-\alpha-\theta-1} < 0 \tag{8}$$

$$\frac{\partial^2 L(x_a, x_t, x_g, \lambda)}{\partial \lambda^2} = 0 \tag{9}$$

The second-order conditions show that first-order conditions are both necessary and sufficient.

Next, we divide equation (2) by (3) as well as (2) by (4) to obtain Marginal Rates of Substitutions

$$-\frac{\alpha x_a^{\alpha-1} x_t^{\theta} x_g^{1-\alpha-\theta}}{\theta x_a^{\alpha} x_t^{\theta-1} x_g^{1-\alpha-\theta}} = -\frac{\lambda p_a}{\lambda p_t} \tag{10}$$

$$-\frac{\alpha x_a^{\alpha-1} x_t^{\theta} x_g^{1-\alpha-\theta}}{(1-\alpha-\theta) x_a^{\alpha} x_t^{\theta} x_g^{-\alpha-\theta}} = -\frac{\lambda p_a}{\lambda p_g} \tag{11}$$

which simplify to

$$-\frac{\alpha x_t}{\theta x_a} = -\frac{p_a}{p_t} \tag{12}$$

$$-\frac{\alpha x_g}{(1-\alpha-\theta) x_a} = -\frac{p_a}{p_g} \tag{13}$$

Rearranging yields

$$\frac{\alpha}{\theta} = \frac{p_a x_a}{p_t x_t} \tag{14}$$

$$\frac{\alpha}{(1-\alpha-\theta)} = \frac{p_a x_a}{p_g x_g} \tag{15}$$

These equations show already that expenditure ratios are equal to ratios of powers in the Cobb-Douglas utility function. Solving for x_t and x_g as a function of x_a, we obtain

$$x_t = \frac{\theta p_a x_a}{\alpha p_t} \tag{16}$$

$$x_g = \frac{(1-\alpha-\theta) p_a x_a}{\alpha p_g} \tag{17}$$

Substituting these two equations into (5) we get

$$p_a x_a + p_t \frac{\theta p_a x_a}{\alpha p_t} + p_g \frac{(1-\alpha-\theta) p_a x_a}{\alpha p_g} = w \tag{18}$$

and simplifying yields

$$\begin{aligned} p_a x_a \frac{\alpha}{\alpha} + \frac{\theta p_a x_a}{\alpha} + \frac{(1-\alpha-\theta) p_a x_a}{\alpha} &= w && (19) \\ \frac{p_a x_a}{\alpha} &= w && (20) \end{aligned}$$

We solve for $x_a(p, w)$

$$x_a(p, w) = \alpha \frac{w}{p_a} \tag{21}$$

Plugging this solution into (16) and (17) allows us to solve for

$$\begin{aligned} x_t(p, w) &= \theta \frac{w}{p_t} && (22) \\ x_g(p, w) &= (1-\alpha-\theta) \frac{w}{p_g} && (23) \end{aligned}$$

We note that the demand for a good does not depend on the price of other goods, which is a convenient but often unrealistic feature of Cobb-Douglas utility functions.

b. Verify that the demand function is homogenous of degree zero and satisfies Walras' Law.

For homogeneity of degree zero, we need to show $x_i(\lambda w, \lambda p) = x_i(w, p)$ for all $\lambda > 0$, $i \in \{a, t, g\}$. But this is obvious, e.g., $x_a(\lambda w, \lambda p) = \alpha \frac{\lambda w}{\lambda p_a} = \alpha \frac{w}{p_a} = x_a(p, w)$ (analogously for toothpicks and gifts).

To verify Walras' Law, we need to verify

$$p_a x_a(p, w) + p_t x_t(p, w) + p_g x_g(p, w) = w.$$

Substituting expressions for the demand functions into this equation yields

$$\begin{aligned} p_a x_a(p, w) + p_t x_t(p, w) + p_g x_g(p, w) &= w && (24) \\ p_a \alpha \frac{w}{p_a} + p_t \theta \frac{w}{p_t} + p_g (1-\alpha-\theta) \frac{w}{p_g} &= w && (25) \\ \alpha w + \theta w + (1-\alpha-\theta) w &= w && (26) \\ w &= w && (27) \end{aligned}$$

c. To be honest, we do not really know whether Ali has the Cobb-Douglas utility function stated above. Would Ali want to differently substitute a marginal amount of almonds for some toothpicks when having a differentiable utility function different from the one above and optimally demanding positive amounts of all goods? Explain.

As long as Ali consumes all goods in positive amounts, he must be at an interior solution. Thus, no matter what differentiable utility function he has, his marginal rates of substitution must equal to the price ratios. So, no, he wouldn't want to differently substitute a marginal amount of almonds for some toothpicks.

d. You would expect that the more almonds Ali eats, the more they get stuck in his teeth and the more toothpicks he purchases. In light of such considerations, does it make sense to assume Ali has the utility function above?

If the price of almonds goes up, then it follows from $x_a(p, w)$ that his demand for almonds goes down. Yet, it follows from $x_t(p, w)$ that his demand for toothpicks stays constant in p_a. So either he has also some other use for toothpicks or the Cobb Douglas utility function doesn't make much sense given the story.

e. Suppose the university would slightly raise Ali's income. (Assuming at most small changes of income is a very realistic assumption at UC Davis.) Can we learn from the Lagrange approach by how much his utility would change?

This question can be answered by observing that the Lagrange multiplier, λ, represents the marginal value of relaxing the budget constraint. We can use for instance equation (2), plug in the demands for almonds, toothpicks, and gifts, and solve for λ and simplify, i.e.,

$$\alpha\left(\alpha\frac{w}{p_a}\right)^{\alpha-1}\left(\theta\frac{w}{p_t}\right)^{\theta}\left((1-\alpha-\theta)\frac{w}{p_g}\right)^{1-\alpha-\theta} = \lambda p_a \quad (28)$$

$$\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta} = \lambda \quad (29)$$

f. Derive Ali's indirect utility function (denote it by $v(p, w)$). Simplify.

We have

$$\begin{aligned} v(p,w) &= u(x(p,w)) && (30)\\ &= (x_a(p,w))^{\alpha}(x_t(p,w))^{\theta}(x_g(p,w))^{1-\alpha-\theta} && (31)\\ &= \left(\alpha\frac{w}{p_a}\right)^{\alpha}\left(\theta\frac{w}{p_t}\right)^{\theta}\left((1-\alpha-\theta)\frac{w}{p_g}\right)^{1-\alpha-\theta} && (32)\\ &= w\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta} && (33) \end{aligned}$$

g. Can we also answer e also using Ali's indirect utility function? Briefly explain.

Yes. We note that $\frac{\partial v(p,w)}{\partial w} = \lambda$.

h. Verify that Ali satisfies Roy's identity with respect to almonds.

Roy's identity with respect to almonds reads

$$x_a(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_a}}{\frac{\partial v(p,w)}{\partial w}} \tag{34}$$

We compute the partial derivative of the indirect utility function with respect to the price of almonds,

$$\frac{\partial v(p,w)}{\partial p_a} = -\frac{\alpha}{p_a}\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta} w \tag{35}$$

Now,

$$-\frac{\frac{\partial v(p,w)}{\partial p_a}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{-\frac{\alpha}{p_a}\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta} w}{\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} = \alpha\frac{w}{p_a} = x_a(p,w) \tag{36}$$

i. When Professor Schipper interviews Ali about how exactly he arrives at his optimal consumption bundle, Ali expresses ignorance about maximizing utility subject to his budget constraint. Instead, he seems to minimize his expenditure on consumption such that he reaches a certain level of utility. A smart undergraduate student walks by and claims that this is clear evidence against the assumption of utility maximization in economics. Since Professor Schipper hates Cobb-Douglas utility functions and boring calculations, he sends the student to you so that you can show him how expenditure minimization works. Again, use the Kuhn-Tucker approach to derive the Hicksian demand function.

The expenditure minimization problem is $\min_{x\in\mathbb{R}^3_+} p\cdot x$ subject to $u(x)\geq \bar{u}$ where $\bar{u} > u(0)$. First, we set up the Lagrange function

$$L(x_a,x_t,x_g,\lambda) = p_a x_a + p_t x_t + p_g x_g - \lambda\left(x_a^{\alpha}x_t^{\theta}x_g^{1-\alpha-\theta} - \bar{u}\right) \tag{37}$$

We derive first-order conditions

$$\frac{\partial L(x_a,x_t,x_g,\lambda)}{\partial x_a} = p_a - \lambda\frac{\alpha}{x_a}x_a^{\alpha}x_t^{\theta}x_g^{1-\alpha-\theta} \equiv 0 \tag{38}$$

$$\frac{\partial L(x_a,x_t,x_g,\lambda)}{\partial x_t} = p_t - \lambda\frac{\theta}{x_t}x_a^{\alpha}x_t^{\theta}x_g^{1-\alpha-\theta} \equiv 0 \tag{39}$$

$$\frac{\partial L(x_a,x_t,x_g,\lambda)}{\partial x_g} = p_g - \lambda\frac{1-\alpha-\theta}{x_g}x_a^{\alpha}x_t^{\theta}x_g^{1-\alpha-\theta} \equiv 0 \tag{40}$$

$$\frac{\partial L(x_a,x_t,x_g,\lambda)}{\partial \lambda} = -x_a^{\alpha}x_t^{\theta}x_g^{1-\alpha-\theta} + \bar{u} \equiv 0 \tag{41}$$

Taking the ratio of (38) and (39) as well as (38) and (40), respectively, we obtain

$$\begin{aligned} \frac{p_a}{p_t} &= \frac{\alpha x_t}{\theta x_a} &(42)\\ \frac{p_a}{p_g} &= \frac{\alpha x_g}{(1-\alpha-\theta)x_a} &(43) \end{aligned}$$

We solve for x_t and x_g, respectively, as a function of x_a to obtain (16) and (17), respectively. Substituting into the utility function and factoring out any variable and parameter pertaining to almonds,

$$\begin{aligned} u(x_a, x_t, x_g) &= \left(\frac{\alpha p_a x_a}{\alpha p_a}\right)^{\alpha}\left(\frac{\theta p_a x_a}{\alpha p_t}\right)^{\theta}\left(\frac{(1-\alpha-\theta)p_a x_a}{\alpha p_g}\right)^{1-\alpha-\theta} &(44)\\ &= \frac{p_a x_a}{\alpha}\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta} &(45) \end{aligned}$$

Let $\bar{u} = u(x_a, x_t, x_g)$ and solve for x_a as a function of $\bar{u}$ and p. This is the Hicksian demand function for almonds. So may want to denote it like in class by $h_a(p, \bar{u})$, i.e.,

$$h_a(p, \bar{u}) = \frac{\alpha}{p_a}\frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} \tag{46}$$

Plugging this into (16) and (17), respectively, we solve for the Hicksian demands of toothpicks and gifts, respectively,

$$\begin{aligned} h_t(p, \bar{u}) &= \frac{\theta}{p_t}\frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} &(47)\\ h_g(p, \bar{u}) &= \frac{1-\alpha-\theta}{p_g}\frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} &(48) \end{aligned}$$

j. Derive the expenditure function. Show that the expenditure function is homogeneous of degree 1 in prices, strictly increasing in $\bar{u}$ as well as nondecreasing and concave in the price of each good.

$$\begin{aligned} e(p, \bar{u}) &= p_a\frac{\alpha}{p_a}\frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} + p_t\frac{\theta}{p_t}\frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}}\\ &\quad + p_g\frac{1-\alpha-\theta}{p_g}\frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} &(49)\\ &= \frac{\bar{u}}{\left(\frac{\alpha}{p_a}\right)^{\alpha}\left(\frac{\theta}{p_t}\right)^{\theta}\left(\frac{1-\alpha-\theta}{p_g}\right)^{1-\alpha-\theta}} &(50) \end{aligned}$$

It is easy to see that it is homogenous of degree 1 in prices, strictly increasing in $\bar{u}$, and nondecreasing in prices. For concavity in prices, differentiate the expenditure function w.r.t. p_a,

$$\frac{\partial e(p,\bar{u})}{\partial p_a} = \frac{\alpha}{p_a}\frac{\bar{u}}{{\frac{\alpha}{p_a}}^{\alpha}{\frac{\theta}{p_t}}^{\theta}{\frac{1-\alpha-\theta}{p_g}}^{1-\alpha-\theta}} \tag{51}$$

which is clearly positive since $\bar{u} > u(0)$ (analogous w.r.t. to p_t and p_g).

k. Professor Schipper cannot compute the Hicksian demand using Kuhn-Tucker without a coffee. Unfortunately, Ali has no coffee to offer. Yet, he could offer Professor Schipper his expenditure function. Is there a way to quickly calculate the Hicksian demand from the expenditure function without coffee?

That's pretty obvious from the last line of j. The right hand site is the Hicksian demand for almonds. This is essentially an implication of Shephard's lemma.

ℓ. Verify the (own price) Slutsky equation for the example of almonds.

We need to verify

$$\frac{\partial h_a(p,\bar{u})}{\partial p_a} = \frac{\partial x_a(p,w)}{\partial p_a} + \frac{\partial x_a(p,w)}{\partial w}x_a(p,w) \tag{52}$$

for $\bar{u} = v(p,w)$.

Focus first on the left-hand side. Taking the partial derivative of Hicksian demand for almonds w.r.t. to p_a yields

$$\frac{\partial h_a(p,\bar{u})}{\partial p_a} = -\alpha\frac{\bar{u}}{p_a^2{\frac{\alpha}{p_a}}^{\alpha}{\frac{\theta}{p_t}}^{\theta}{\frac{1-\alpha-\theta}{p_g}}^{1-\alpha-\theta}} + \alpha^2\frac{\bar{u}}{p_a^2{\frac{\alpha}{p_a}}^{\alpha}{\frac{\theta}{p_t}}^{\theta}{\frac{1-\alpha-\theta}{p_g}}^{1-\alpha-\theta}} \tag{53}$$

Since $\bar{u} = v(p,w)$, we substitute (33) and simplify in order to obtain

$$\frac{\partial h_a(p,\bar{u})}{\partial p_a} = -\alpha\frac{w}{p_a^2} + \alpha^2\frac{w}{p_a^2} \tag{54}$$

Now turn to the right-hand side.

$$\frac{\partial x_a(p,w)}{\partial p_a} = -\alpha\frac{w}{p_a^2} \tag{55}$$

$$\frac{\partial x_a(p,w)}{\partial w} = \frac{\alpha}{p_a} \tag{56}$$

Thus,

$$\frac{\partial x_a(p,w)}{\partial w}x_a(p,w) = \frac{\alpha^2 w}{p_a^2} \tag{57}$$

Adding (55) and (57) finishes the verification.

m. Which term in the (own price) Slutsky equation refers to the substitution effect? As mentioned previously, we don't really know whether Ali has a Cobb-Douglas utility function. Suppose Ali has a continuous utility function representing locally nonsatiated preferences and that his Hicksian demand function is indeed a function (rather than a correspondence). Would a change in prices still have qualitatively the same effect on Hicksian demand as when he has the above Cobb-Douglas utility function? If yes, provide a short proof. If not, argue why not.

Yes, it has the same effect. The substitution effect is always negative. This is known as the Compensated Law of Demand.

Note that we did not assume differentiability. We show that for any $p, p' > 0$, the Hickisan demand function satisfies

$$(p - p') \cdot (h(p, \bar{u}) - h(p', \bar{u})) \leq 0 \tag{58}$$

Since $h(p, \bar{u})$ minimizes expenditures at prices p over any other consumption bundle yielding a utility of at least $\bar{u}$. Thus, we must have

$$p \cdot h(p, \bar{u}) \leq p \cdot h(p', \bar{u}) \tag{59}$$

$$p' \cdot h(p', \bar{u}) \leq p' \cdot h(p, \bar{u}) \tag{60}$$

This is equivalent, respectively, to

$$p \cdot (h(p, \bar{u}) - h(p', \bar{u})) \leq 0 \tag{61}$$

$$0 \leq p' \cdot (h(p, \bar{u}) - h(p', \bar{u})) \tag{62}$$

Adding these to inequalities and bringing all terms to the left-hand side yields (58).

n. Because of the drought, the price of almonds changes from p_a^0 to p_a^1. UC Davis is committed to keep Ali as well off as before the price change. The newly hired Senior Vice Provost for Crocodile Welfare turns to Professor Schipper for advice on the exact amount to be deducted from the budget of the university and paid to Ali as compensation for the price change. Unfortunately, Professor Schipper is so immersed in exciting new research that he is extremely slow in answering his email. Luckily, the Senior Vice Provost for Crocodile Welfare spends most of his time in the hammocks on the quad, where he meets you studying for the prelims. Help him calculate the amount.

Essentially the problem asks for the Compensating Variation. It is a measure of change of welfare of a consumer due to a price change. It measures in units of wealth the amount a social planner must pay to Ali after the price change to compensate him so as to make him just as well off as before the price change. It is negative if and only if the transfer goes from the social planner to Ali.

Let $p^0 = (p_a^0, p_t, p_g)$, $p^1 = (p_a^1, p_t, p_g)$, $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$. The Compensating Variation is defined by

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) \tag{63}$$

Since Ali spends his entire wealth in the expenditure minimizing consumption bundle, i.e., $e(p^1, u^1) = w$, we have

$$CV(p^0, p^1, w) = w - e(p^1, u^0) \tag{64}$$

We compute

$$\begin{aligned}
e(p^1, u^0) &= \frac{u^0}{{\frac{\alpha}{p_a^1}}^{\alpha} {\frac{\theta}{p_t}}^{\theta} {\frac{1-\alpha-\theta}{p_g}}^{1-\alpha-\theta}} && (65)\\
&= \frac{w \; {\frac{\alpha}{p_a^0}}^{\alpha} {\frac{\theta}{p_t}}^{\theta} {\frac{1-\alpha-\theta}{p_g}}^{1-\alpha-\theta}}{{\frac{\alpha}{p_a^1}}^{\alpha} {\frac{\theta}{p_t}}^{\theta} {\frac{1-\alpha-\theta}{p_g}}^{1-\alpha-\theta}} && (66)\\
&= w \; {\frac{p_a^1}{p_a^0}}^{\alpha} && (67)
\end{aligned}$$

Thus,

$$\begin{aligned}
CV(p^0, p^1, w) &= w - w \; {\frac{p_a^1}{p_a^0}}^{\alpha} && (68)\\
&= w \;\; 1 - {\frac{p_a^1}{p_a^0}}^{\alpha} && (69)
\end{aligned}$$

Thus, the Compensating Variation is negative if and only if $p_a^1 > p_a^0$.