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Microeconomics

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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Answer Key

Question 1¹

Because of superb graduate education at our Department, you have been handpicked by The President to serve as his special assistant to his hotel business that operates informally out of the South Wing of the White House.² Your first task is to derive the profit maximizing inputs and output of his hotel business. You quickly learn that this president strictly prefers simple answers. So you consider a Cobb-Douglas production function with labor and capital as inputs. In terms of notation, output q is given by

$$q = Az_1^{\alpha_1} z_2^{\alpha_2}$$

where $A, \alpha_1, \alpha_2, z_1, z_2 \geq 0$. We denote by z_1 and z_2 the amount of labor and capital, respectively.

- a.) Unfortunately, the parameters A , α_1 , and α_2 are only known to the IRS and the Russians. Lacking the right connections to either of them, you set out to estimate the parameters of the production function. How would you estimate it using OLS? Set up the “econometric model” and state the estimator.

Use a logarithmic transformation

$$\ln(q) = \ln(A) + \alpha_1 \ln(z_1) + \alpha_2 \ln(z_2).$$

Now we can consider a regression model

$$\ln(q_i) = \alpha_0 + \alpha_1 \ln(z_{i1}) + \alpha_2 \ln(z_{i2}) + \varepsilon_i,$$

where $\alpha_0 = \ln(A)$ and ε_i is an unobserved random variable with $\mathbb{E}[\varepsilon_i] = 0$. Written in matrix notation, we have

$$Q = Z\alpha + \varepsilon$$

where

$$Q = \begin{bmatrix} \ln(q_1) \\ \ln(q_2) \\ \vdots \\ \ln(q_n) \end{bmatrix}, Z = \begin{bmatrix} 1 & \ln(z_{11}) & \ln(z_{12}) \\ \vdots & \vdots & \vdots \\ 1 & \ln(z_{n1}) & \ln(z_{n2}) \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}, \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}.$$

Having Q and Z as data, we get

$$\hat{\alpha} = (Z^T Z)^{-1} Z^T Q,$$

and thus,

$$\hat{A} = e^{\hat{\alpha}_0}.$$

¹You may find it easier to start with parts g.) and h.).

²Names, characters, businesses, places, events and incidents are either the products of the author’s imagination or used in a fictitious manner. Any resemblance to actual persons, living or dead, or actual events is purely coincidental.

- b.) After firing up your (illegal) copy of Stata, you estimate parameters A , α_1 , and α_2 and find that $\hat{A}, \hat{\alpha}_1, \hat{\alpha}_2 > 0$ with $\hat{\alpha}_1 + \hat{\alpha}_2 < 1$. Armed with these estimates, you set out to calculate profit maximizing inputs and output. The President has lamented about the competitiveness of the hotel business. Thus you feel safe to assume that his business is a price taker both in the factor and output markets. Denote by p the price of the output and by w_1 and w_2 the prices of the inputs, respectively. Set up the profit maximization problem and derive step-by-step the conditional factor demand functions (i.e., factor demand conditional on optimal output) and the supply function.

The profit maximization problem is

$$\max_{z_1, z_2 \geq \mathbb{R}_+} pAz_1^{\alpha_1}z_2^{\alpha_2} - w_1z_1 - w_2z_2$$

The first-order conditions are

$$\begin{aligned} pA\alpha_1(z_1^*)^{\alpha_1-1}(z_2^*)^{\alpha_2} - w_1 &= 0 \\ pA\alpha_2(z_1^*)^{\alpha_1}(z_2^*)^{\alpha_2-1} - w_2 &= 0 \end{aligned}$$

Multiplying the first equation with z_1^* and the second with z_2^* , we obtain

$$\begin{aligned} pA\alpha_1(z_1^*)^{\alpha_1}(z_2^*)^{\alpha_2} - w_1z_1^* &= 0 \\ pA\alpha_2(z_1^*)^{\alpha_1}(z_2^*)^{\alpha_2} - w_2z_2^* &= 0 \end{aligned}$$

Substituting q^* for $A(z_1^*)^{\alpha_1}(z_2^*)^{\alpha_2}$ we write

$$\begin{aligned} p\alpha_1q^* - w_1z_1^* &= 0 \\ p\alpha_2q^* - w_2z_2^* &= 0 \end{aligned}$$

We solve z_1^* and z_2^* , respectively, to obtain optimal factor demands conditional on optimal output q^*

$$\begin{aligned} z_1^* &= \alpha_1 \frac{p}{w_1} q^* \\ z_2^* &= \alpha_2 \frac{p}{w_2} q^* \end{aligned}$$

To derive the supply function, we plug these factor demands conditional on optimal output into the production function and

solve for q^* , i.e.,

$$\begin{aligned} A \left(\alpha_1 \frac{p}{w_1} q^* \right)^{\alpha_1} \left(\alpha_2 \frac{p}{w_2} q^* \right)^{\alpha_2} &= q^* \\ A \left(\alpha_1 \frac{p}{w_1} \right)^{\alpha_1} \left(\alpha_2 \frac{p}{w_2} \right)^{\alpha_2} (q^*)^{\alpha_1 + \alpha_2} &= q^* \\ A \left(\alpha_1 \frac{p}{w_1} \right)^{\alpha_1} \left(\alpha_2 \frac{p}{w_2} \right)^{\alpha_2} &= (q^*)^{1 - \alpha_1 - \alpha_2} \\ q^* &= A^{\frac{1}{1 - \alpha_1 - \alpha_2}} \left(\alpha_1 \frac{p}{w_1} \right)^{\frac{\alpha_1}{1 - \alpha_1 - \alpha_2}} \left(\alpha_2 \frac{p}{w_2} \right)^{\frac{\alpha_2}{1 - \alpha_1 - \alpha_2}} \end{aligned}$$

- c.) Derive also the profit function (as a function of factor prices and output price). (The President is really interested in profits. Moreover, it will be relevant in the latter parts of the question.)

Plug in supply and conditional factor demands into the profit function to obtain:

$$\begin{aligned} \pi(p, w_1, w_2) &= pq^* - w_1 z_1^* - w_2 z_2^* \\ &= pq^* - w_1 \frac{\alpha_1 pq^*}{w_1} - w_2 \frac{\alpha_2 pq^*}{w_2} \\ &= pq^* - \alpha_1 pq^* - \alpha_2 pq^* \\ &= (1 - \alpha_1 - \alpha_2) pq^* \\ &= (1 - \alpha_1 - \alpha_2) p A^{\frac{1}{1 - \alpha_1 - \alpha_2}} \left(\alpha_1 \frac{p}{w_1} \right)^{\frac{\alpha_1}{1 - \alpha_1 - \alpha_2}} \left(\alpha_2 \frac{p}{w_2} \right)^{\frac{\alpha_2}{1 - \alpha_1 - \alpha_2}} \end{aligned}$$

- d.) Very proud of your achievement, you quickly run back to the oval office to cheerfully present your results. After 140 seconds, The President dismisses your results as fake news. He sternly declares that you omitted a very very important variable. He won't tell you what it is, but he knows it is there, and you better get it right otherwise you are fired. In your despair, you ask around what it could be. Not surprisingly, Jean Speiser is unwilling to tell you. In this high-stress environment, you experience nightmares the moment you try to sleep. As always, Professor Schipper appears in your dreams partying merrily between convex sets. He laughingly whispers one hint: "McKenzie (1959)". In the morning you awake with a bad headache and vaguely recall that "McKenzie (1959)" might have something to do with the entrepreneurial factor. It suddenly dawns on you that you essentially forgot The President in your calculation. You set up your new equation

$$q = Az_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}, \quad (1)$$

where z_3 denotes the entrepreneurial factor. Initially you suspect that only the FBI and the Russian have data on z_3 . Since you don't find it politically opportune to

contact either of them at this point of time, you need some other way to figure out α_3 . With the entrepreneurial factor in mind, what number should you assume for α_3 ? Explain in as much details as possible. Verify that the conditions under which one can apply McKenzie's proposition (see part g.) are satisfied in our case.

McKenzie (Econometrica 1959, Section 7) argued that any convex production set that allows for the possibility of complete shut-down can be understood as a section of a convex production set with constant returns (i.e., a convex cone) where the additional dimensions model the entrepreneurial factors of each firm. If McKenzie's argument applies here then The President was actually right in the sense that decreasing returns implied by your estimates satisfying $\alpha_1 + \alpha_2 < 1$ point to an omitted factor. This suggests to set $\alpha_3 = 1 - \alpha_1 - \alpha_2$. (See also part g.) below.)

To see whether McKenzie's argument applies, verify that the production set associated with the Cobb-Douglas function is actually convex. I.e., define the production set by

$$Y := \{(-z_1, -z_2, q) : q - Az_1^{\alpha_1} z_2^{\alpha_2} \leq 0 \text{ and } (z_1, z_2) \geq 0\}.$$

I would expect that you verify that the production function is concave. Then you could show that a production set is convex if and only if the production function is concave (solution to Exercise 5.B.3 in MWG). The observation now follows. Possibility of complete shutdown is straightforward to verify as $pA0^{\alpha_1}0^{\alpha_2} = 0$ if $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$.

- e.) Having set α_3 to what McKenzie would have probably set it, you seem to struggle to figure out the supply q because you don't know the price of the entrepreneurial factor, w_3 . Argue that we can set

$$w_3 = (1 - \alpha_1 - \alpha_2)pA^{\frac{1}{1 - \alpha_1 - \alpha_2}} \left(\alpha_1 \frac{p}{w_1}\right)^{\frac{\alpha_1}{1 - \alpha_1 - \alpha_2}} \left(\alpha_2 \frac{p}{w_2}\right)^{\frac{\alpha_2}{1 - \alpha_1 - \alpha_2}}$$

(You should recognize this expression from earlier parts of the question. You are also allowed to make an assumption on z_3 but this assumption should be verbally justified.)

Armed with this insight, you rush back to the oval office and argue that both The President and you got it right. That is, it is true that you omitted the entrepreneurial factor but your earlier results on the optimal input of labor and capital and optimal supply are still correct. Verify this claim.

Note that w_3 is the profit in the two-factor case. Thus, we would fix the factor price of the entrepreneurial factor to the profit in the two-factor case. (Aside: Note that this

is a "private" price in the sense that this firm is the only firm that "purchases" this entrepreneurial factor. Other firms would "purchase" their own unique entrepreneurial factor. In a sense, it just means this president is really unique.)

We feel relieved that this president is in limited supply and certainly cannot be replicated. Moreover, it is not clear how to measure the entrepreneurial factor. Thus, we fix $z_3 = 1$.

Now conjecture that with w_3 defined above and $z_3 = 1$, the factor-demands in the two-factor case and the output in the two-factor case are still optimal in the three-factor case. Indeed, we verify this claim by plugging these expressions into the 3-factor profit function and verify that everything cancels nicely so that we are left with zero profits:

$$\begin{aligned}\pi(p, w_1, w_2, w_3) &= pq(p, w_1, w_2, w_3) - w_1 z_1(p, w_1, w_2, w_3) - w_2 z_2(p, w_1, w_2, w_3) - w_3 z_3 \\ &= pq^* - w_1 z_1^* - w_2 z_2^* - \pi(p, w_1, w_2) \cdot 1\end{aligned}$$

where q^* is the optimal supply in the two-factor case and z_1^* and z_2^* are the optimal factor demands of capital and labor, respectively, in the two-factor case. Since $\pi(p, w_1, w_2) = pq^* - w_1 z_1^* - w_2 z_2^*$, we have that $\pi(p, w_1, w_2, \pi(p, w_1, w_2)) = 0$. This is consistent with profit maximization in the case of constant returns to scale. (See also answer to the next part.)

- f.) The President turns to his trusted trade advisor, Piotr Novato, to verify your analysis. Being educated both in Cobb-Douglas production functions and constant returns to scale, he rejects your analysis claiming that it implies that The President's hotel business makes zero profits. This flies in the face of empirical evidence of strictly positive profits made by The President. As often, he is just confused but The President is already furious over alleged zero profits. How can you clear up the confusion? Explain.

Indeed, he is confused. First, it is true that the three-factor production function has constant returns to scale. (You may want to verify this explicitly.) Second, it is true that with constant returns to scale, profits are zero or if positive they are infinite. (Again, you may want to verify this explicitly.) Since you actually used the property of zero profits to claim optimality of the two-factor factor-demands in the three-factor case, profits are indeed zero in the three-factor case. But it does not mean that The President earns zero profit. Rather, his profit is the price earned by him for his entrepreneurial factor, also a condition we used in answering earlier parts of the question. So The President's profit is really the profit in the two-factor model.

- g.) The President seems pleased to hear that after all he makes positive profits. He invites you to teach his staff about your analysis. You start by stating McKenzie's insight in general terms:

"For any convex production set $Y \subseteq \mathbb{R}^L$ with $\mathbf{0} \in Y$, there is a constant returns, convex production set $Y' \subseteq \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$."

Provide a proof of this proposition. (There is a Twitter-like proof in less than 140 characters.) (Hint: Think about the defining property of a production set with constant returns.)

I would expect something like MWG, p. 135.

- h.) You continue with the following proposition: "If $y \in Y \subseteq \mathbb{R}^L$ is profit maximizing at $p \in \mathbb{R}_{++}^L$ then $(y, -1) \in Y' \subseteq \mathbb{R}^{L+1}$ is profit maximizing at $(p, \pi(p))$. Conversely, if $(y, -1) \in Y'$ is profit maximizing at (p, p_{L+1}) , then $y \in Y$ is profit maximizing at p and the profit is p_{L+1} ." Prove each direction of this proposition.

I would expect something like the answer to MWG Exercise 5.C.12.

ECN200, General Equilibrium Theory
 Sketched answers to prelim exam, June 2017

Question 2

Consider a smooth exchange economy with I consumers and $L \geq 2$ goods, $\{\mathcal{J}, (u^i, w^i)_{i \in \mathcal{J}}\}$, but suppose that, unlike in class, for some reason the agents in this economy cannot exchange the commodities directly. Instead, some institutional arrangement forces them to trade *in bundles of commodities*.

There are N bundles of commodities that they can trade, $n = 1, \dots, N$. Bundle n is a vector $b^n = (b_1^n, \dots, b_L^n) \in \mathbb{R}^L$; it contains $b_\ell^n > 0$ units of commodity ℓ . Denote by $q_n > 0$ the price of bundle n , and let $q = (q_1, \dots, q_N)$ be the vector of bundle prices.

Each individual can buy and sell these bundles at the given prices. Let y_n^i denote the number of units of bundle n bought by individual i , with the convention that this number is negative if the individual is actually selling the bundle. Denote by $y^i = (y_1^i, \dots, y_N^i)$ the individual's bundle demand. This demand results in consumption of commodities

$$x^i = w^i + \sum_n y_n^i b^n,$$

where $w^i \in \mathbb{R}_{++}^L$ denotes the individual's endowment. Her budget constraint is that

$$q \cdot y^i = \sum_n y_n^i q_n \leq 0,$$

which means that she can only afford positive expenditure in some bundles if she raises enough liquidity from the sales of other bundles.

A *competitive equilibrium in bundles* is a pair (q, y) , where q is a vector of bundle prices and $y = (y^1, \dots, y^I)$ a profile of bundle demands such that

- i. each individual is individually rational: for each i , bundle y^i solves

$$\max_{\hat{y}} \{u^i(w^i + \sum_n \hat{y}_n b^n) : q \cdot \hat{y} \leq 0\};$$

- ii. all markets clear: $\sum_i y^i = 0$.

An *allocation of commodities* is (still) a profile $x = (x^1, \dots, x^I)$ of consumption bundles, such that $\sum_i x^i = \sum_i w^i$. Allocation of commodities x is said to be *first best* if there does not exist an alternative allocation \hat{x} such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all individuals, with strict inequality for some. It is said to be *second best* if there does not exist a profile of bundle demands $\hat{y} = (\hat{y}^1, \dots, \hat{y}^I)$ such that $\sum_i \hat{y}^i = 0$ and

$$u^i(w^i + \sum_n \hat{y}_n^i b^n) \geq u^i(x^i)$$

for all individuals, with strict inequality for some.

With respect to this model:

1. Argue that if allocation $x \gg 0$ is first best, then

$$\frac{\partial_{x_\ell} u^i(x^i)}{\partial_{x_{\ell'}} u^i(x^i)} = \frac{\partial_{x_\ell} u^{i'}(x^{i'})}{\partial_{x_{\ell'}} u^{i'}(x^{i'})}$$

for all pairs of individuals and commodities.

Answer: This is routine: If $x = (x^i)_{i=1}^I$ is first best, then it must solve

$$\max_{(\hat{x}^i)_{i=1}^I} \{u^1(\hat{x}^1) \mid u^i(\hat{x}^i) \geq u^i(x^i), i = 2, \dots, I, \text{ and } \sum_i \hat{x}^i = \sum_i w^i\}.$$

The Lagrangean of this problem is

$$\mathcal{L}(\hat{x}, \lambda, \mu) := u^1(\hat{x}^1) + \sum_{i \geq 2} \lambda^i [u^i(\hat{x}^i) - u^i(x^i)] + \mu \sum_i (w^i - \hat{x}^i).$$

Let $\lambda^1 := 1$. Since $x \gg 0$ the first-order conditions of this problem are that $\lambda^i \partial_{x_\ell} u^i(x^i) = \mu_\ell$, for every i and ℓ . Taking ratios, we have that

$$\frac{\partial_{x_\ell} u^i(x^i)}{\partial_{x_{\ell'}} u^i(x^i)} = \frac{\mu_\ell}{\mu_{\ell'}}$$

must be true for all i . This implies the result.

2. Suppose that $b_\ell^1 \geq 0$ for all ℓ , with strict inequality for some. Argue that, given that all preferences are strictly increasing, if (q, y) is a competitive equilibrium in bundles, then the resulting allocation of commodities, with

$$x^i = w^i + \sum_n y_n^i b_n,$$

is second best.

Answer: Suppose not: let (q, y) be a competitive equilibrium in bundles, and assume that $x = (w^i + \sum_n y_n^i b_n)_{i=1}^I$ is *not* second best. Then, there must exist $\hat{y} = (\hat{y}^i)_{i=1}^I$ such that $\sum_i \hat{y}^i = 0$ and

$$u^i(w^i + \sum_n \hat{y}_n^i b_n) \geq u^i(x^i) \quad (*)$$

for all i , with strict inequality for some. For those individuals for whom the inequality is strict, it is immediate from the definition of equilibrium that $q \cdot \hat{y}^i > 0$. On the other hand, if for any i we had $q \cdot \hat{y}^i < 0$, then, for some small enough $\varepsilon > 0$ we would have that $q \cdot \hat{y}^i + q_1 \varepsilon \leq 0$ while, by Eq. (*),

$$u^i(w^i + \sum_n \hat{y}_n^i b_n + \varepsilon b_1) > u^i(w^i + \sum_n \hat{y}_n^i b_n) \geq u^i(x^i),$$

since all individuals have strictly monotonic preferences and $b_1 > 0$. Now, this would contradict the fact that (q, y) is a competitive equilibrium in bundles, so it must be that $q \cdot \hat{y}^i \geq 0$ for all i . Aggregating, then, $\sum_i q \cdot \hat{y}^i = q \cdot \sum_i \hat{y}^i > 0$, which is impossible since $\sum_i \hat{y}^i = 0$.

3. You are now going to argue that the previous result cannot be extended to first best. Suppose that $L = 3$, and there are only two bundles that can be traded: $b^1 = (1, 0, 0)$ and $b^2 = (0, 1, 1)$.

(a) Argue that at any (interior) competitive equilibrium allocation

$$\begin{pmatrix} \partial_{x_1} u^i(x^i) \\ \partial_{x_2} u^i(x^i) + \partial_{x_3} u^i(x^i) \end{pmatrix} = \lambda^i \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

where $\lambda^i > 0$ is a Lagrange multiplier, for every individual.

Answer: The Lagrangean of the individual rationality problem for i is

$$\mathcal{L}^i(\mathbf{y}, \lambda) := u^i(w^1 + (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_2)) - \lambda(q_1 y_1 + q_2 y_2).$$

The first-order condition for the first bundle is simply that

$$\partial_{x_1} u^i(x^i) = \lambda^i q_1. \quad (**)$$

With respect to \mathbf{y}_2 , it is that

$$\partial_{x_2} u^i(x^i) + \partial_{x_3} u^i(x^i) = \lambda^i q_2. \quad (***)$$

(b) Argue that at any (interior) competitive equilibrium allocation

$$\frac{\partial_{x_2} u^i(x^i)}{\partial_{x_1} u^i(x^i)} + \frac{\partial_{x_3} u^i(x^i)}{\partial_{x_1} u^i(x^i)} = \frac{\partial_{x_2} u^{i'}(x^{i'})}{\partial_{x_1} u^{i'}(x^{i'})} + \frac{\partial_{x_3} u^{i'}(x^{i'})}{\partial_{x_1} u^{i'}(x^{i'})}$$

for all pairs of individuals.

Answer: Taking the ratio, from Eqs. (***) and (**) we get that

$$\frac{\partial_{x_2} u^i(x^i) + \partial_{x_3} u^i(x^i)}{\partial_{x_1} u^i(x^i)} = \frac{q_2}{q_1}$$

for both individuals, which implies the result.

(c) *Intuitively*, argue that the latter condition is insufficient to guarantee that the allocation of commodities is first best.

Answer: The equilibrium condition of part (3b) will only yield the first-best condition of part (1) by accident. Intuitively, suppose that the equilibrium in bundles delivers a first-best allocation for some given endowments. We can perturb w^1 in such a way that each of the summands on the left-hand side of the equilibrium condition changes but their sum remains invariant. Then, the condition of equilibrium is preserved, but the one for first best is lost.

ECN200, General Equilibrium Theory
 Sketched answers to prelim exam, June 2017

Question 3

Consider a standard production economy

$$\{\mathcal{I}, \mathcal{J}, (u^i, w^i)_{i \in \mathcal{I}}, (Y^j)_{j \in \mathcal{J}}, (s^{i,j})_{(i,j) \in \mathcal{I} \times \mathcal{J}}\}$$

where at least one of the agents has strictly monotone preferences. Recall that an *allocation* is a pair

$$(x, y) = (x^1, \dots, x^I, y^1, \dots, y^J) \in (\mathbb{R}_+^L)^I \times (\mathbb{R}^L)^J$$

such that $y^j \in Y^j$ for each j , and $\sum_i x^i = \sum_i w^i + \sum_j y^j$; and that allocation (x, y) is *Pareto efficient* if there does not exist another allocation (\hat{x}, \hat{y}) such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all consumers, with strict inequality for some.

A profile of production plans $y = (y^1, \dots, y^J)$ is *feasible* if $y^j \in Y^j$ for each j ; a feasible profile of production plans is *technically efficient* if there does not exist an alternative feasible plan \hat{y} such that $\sum_j \hat{y}^j > \sum_j y^j$. Also, given a profile y of production plans, a profile $x = (x^1, \dots, x^I)$ of consumption bundles is *feasible* if $\sum_i x^i = \sum_i w^i + \sum_j y^j$. Finally, feasible profile x is *allocatively efficient*, given y , if there does not exist an alternative profile \hat{x} that is also feasible given y and such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all consumers, with strict inequality for some.

Given these definitions:

1. Argue that if (x, y) is Pareto efficient, then profile y is technically efficient (since one agent has strictly monotone preferences).

Answer: Suppose not: (x, y) is Pareto efficient, but there exists some $\hat{y} = (\hat{y}^j)_{j=1}^J$ such that $\hat{y}^j \in Y^j$ for all j and $\sum_j \hat{y}^j > \sum_j y^j$. Then, define $\delta := \sum_j (\hat{y}^j - y^j) > 0$, let $\hat{x}^i = x^i + \delta$ for an individual whose u^i is strictly monotone, and let $\hat{x}^i = x^i$ for everybody else. Then, $u^i(\hat{x}^i) \geq u^i(x^i)$ for all i , with strict inequality for one of them; also,

$$\sum_i \hat{x}^i = \sum_i x^i + \delta = \sum_i w^i + \sum_j y^j + \sum_j \hat{y}^j - \sum_j y^j = \sum_i w^i + \sum_j \hat{y}^j.$$

But this is impossible since (x, y) is Pareto efficient.

2. Argue that if (x, y) is Pareto efficient, then profile x is feasible and allocatively efficient given y .

Answer: This is immediate: feasibility is straightforward; and if there exists \hat{x} that is allocatively superior to x given y , then (\hat{x}, y) is Pareto superior to (x, y) .

3. In what follows you will argue that, even together, technical and allocative efficiency don't suffice to guarantee Pareto efficiency. Suppose that there are two commodities, two individuals and one firm. Both individuals have smooth utility functions, while the technology of the firm is

$$Y = \{(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_- \mid y_1 \leq f(-y_2)\},$$

where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a strictly increasing and differentiable production function that transforms (input of) commodity 2 into units of (output of) commodity 1.

- (a) Argue that *any* pair $(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_-$ such that $y_1 = f(-y_2)$ is technically efficient.

Answer: Fix y such that $y_1 = f(-y_2)$. If $f(-\hat{y}_2) > y_1$, then it must be that $-\hat{y}_2 > -y_2$, since f is increasing, and hence that $\hat{y}_2 < y_2$. Alternatively, if $\hat{y}_2 > y_2$, then $-\hat{y}_2 < -y_2$ and $f(-\hat{y}_2) < f(-y_2)$, for the same reason. In any case, one cannot have $\hat{y} > y$ with $\hat{y}_1 \leq f(-\hat{y}_2)$.

- (b) Argue that, given (y_1, y_2) , allocation (x^1, x^2) is allocatively efficient only if

$$\frac{\partial_{x_1} u^1(x^1)}{\partial_{x_2} u^1(x^1)} = \frac{\partial_{x_1} u^2(x^2)}{\partial_{x_2} u^2(x^2)}.$$

Answer: Suppose that x is allocatively efficient given y . Then, it must solve the program

$$\max_{\hat{x}} \{u^1(\hat{x}^1) \mid u^2(\hat{x}^2) \geq u^2(x^2) \text{ and } \hat{x}^1 + \hat{x}^2 = w^1 + w^2 + y\}.$$

The Lagrangean for this program is

$$\mathcal{L}_A(\hat{x}, \mu, \delta) := u^1(\hat{x}^1) + \mu[u^2(\hat{x}^2) - u^2(x^2)] + \delta(w^1 + w^2 + y - \hat{x}^1 - \hat{x}^2).$$

The first-order conditions of this Lagrangean include that $\partial u^1(x^1) = \delta$ and $\mu \partial u^2(x^2) = \delta$. Taking ratios, we get the result.

- (c) Argue that Pareto efficiency of (x^1, x^2, y^1, y^2) requires that¹

$$\frac{\partial_{x_1} u^1(x^1)}{\partial_{x_2} u^1(x^1)} = \frac{\partial_{x_1} u^2(x^2)}{\partial_{x_2} u^2(x^2)} = \frac{1}{f'(-y_2)},$$

and conclude that the fact that (x^1, x^2, y^1, y^2) is both technically and allocatively efficient does not suffice for it to be Pareto efficient.

Answer: In this case, (x, y) must solve

$$\max_{\hat{x}, \hat{y}} \{u^1(\hat{x}^1) \mid u^2(\hat{x}^2) \geq u^2(x^2), \hat{y}_1 \leq f(-\hat{y}_2) \text{ and } \hat{x}^1 + \hat{x}^2 = w^1 + w^2 + \hat{y}\},$$

whose Lagrangean is

$$\mathcal{L}_P(\hat{x}, \hat{y}, \mu, \delta, \gamma) := u^1(\hat{x}^1) + \mu[u^2(\hat{x}^2) - u^2(x^2)] + \delta(w^1 + w^2 + \hat{y} - \hat{x}^1 - \hat{x}^2) + \gamma[f(-\hat{y}_2) - \hat{y}_1].$$

Besides the two previous conditions, namely that $\partial u^1(x^1) = \delta$ and $\mu \partial u^2(x^2) = \delta$, The first-order conditions now include that $\delta_1 = \gamma$ and $\delta_2 = \gamma f'(-y_2)$. These conditions imply the desired equalities.

Importantly, technical efficiency does not suffice to imply the second equality here since, by (3a), any y such that $y_1 = f(-y_2)$ is technically efficient. Even if allocative efficiency holds, the conditions for Pareto efficiency are not guaranteed.

¹ Apologies for the typo in the last denominator of the following expression, which is now corrected.

ANSWER to QUESTION 4

- (a) (a.1) Let n be odd (thus $n \geq 3$). Let T_A be the number of people who choose route A and T_B be the number of people who choose route B. First we show that there cannot be a Nash equilibrium with $|T_A - T_B| > 1$. Suppose, w.l.o.g., that $T_A - T_B \geq 2$, that is, $T_A \geq T_B + 2$. Then a driver who takes the A route spends $\frac{T_A}{10} + 45 \geq \frac{T_B + 2}{10} + 45$ minutes on the road while a unilateral switch to the B route would involve only $\frac{T_B + 1}{10} + 45$ minutes. Next we show that any strategy profile such that $|T_A - T_B| = 1$ is a Nash equilibrium. For example, if $T_A = T_B + 1$ then a driver on the A route spends $\frac{T_A}{10} + 45 = \frac{T_B + 1}{10} + 45$ minutes on the road and if she unilaterally switched to the B route she would spend the same time, namely $\frac{T_B + 1}{10} + 45$; thus she cannot gain from deviating; on the other hand, a driver on the B route spends $\frac{T_B}{10} + 45$ minutes, while a unilateral switch to the A road would involve more time, namely $\frac{T_A + 1}{10} + 45 = \frac{T_B + 2}{10} + 45$. Thus the pure-strategy Nash equilibria are precisely those strategy profiles that involve $\frac{n-1}{2}$ drivers on one route and $\frac{n-1}{2} + 1$ on the other. (a.2) When $n = 3$ the pure-strategy Nash equilibria are: (A,A,B), (A,B,A), (A,B,B), (B,A,A), (B,A,B) and (B,B,A).
- (a) Let $n \geq 2$ be even. By a calculation similar to part (a.1), a strategy profile such that $T_A > T_B$ cannot be a Nash equilibrium because one of the drivers who chose route A would do better by switching to route B (the number of drivers is even so that if $T_A > T_B$ then $T_A \geq T_B + 2$, so that switching from A to B decreases travel time from $\frac{T_A}{10} + 45$ to $\frac{T_B + 1}{10} + 45$). Similarly there cannot be a Nash equilibrium with $T_A < T_B$. On the other hand, any strategy profile that gives $T_A = T_B = \frac{n}{2}$ is a Nash equilibrium.
- (b) When $n = 400$ the pure-strategy Nash equilibria are those strategy profiles such that $T_A = T_B = 200$. Thus each driver takes $\frac{200}{10} + 45 = 65$ minutes (if she switched route she would take 65.1 minutes).
- (c) Let A be the strategy of going to A and continuing on from there to *End* and let T_A be the number of people who choose this strategy; let AB be the strategy of going to A and then driving to B on the new road and continuing on from there and let T_{AB} be the number of people who choose this strategy; let B, T_B , BA and T_{BA} be defined similarly. Then strategy BA is strictly dominated by B, because it involves $45 + 10 + 45 = 100$ minutes of driving time, whereas B involves $45 + \frac{T_B + T_{AB}}{10} \leq 45 + \frac{400}{10} = 85$ minutes (it is also strictly dominated by A and by AB for similar reasons).
- (d) First of all, because of part (d), at any Nash equilibrium it must be that $T_{BA} = 0$. The easiest way to proceed is to look at the case where T_A, T_B and T_{AB} are such that each of the routes A, B and AB takes the same amount of time; that is, T_A, T_B and T_{AB} satisfy the following equations: (1) $\frac{T_A + T_{AB}}{10} + 45 = \frac{T_B + T_{AB}}{10} + 45$ (from which we get that $T_A = T_B$), (2) $\frac{T_A + T_{AB}}{10} + 45 = \frac{T_A + T_{AB}}{10} + 10 + \frac{T_B + T_{AB}}{10}$ (from which we get that $T_B + T_{AB} = 350$) and, of course, (3) $T_A + T_B + T_{AB} = 400$. The unique solution is $T_A = T_B = 50, T_{AB} = 300$. This is indeed a Nash equilibrium because switching from A to B increases travel time from $\frac{350}{10} + 45 = 80$ to

$45 + \frac{351}{10} = 80.1$ (and similar calculations for the other deviations). Thus we have found a set of pure-strategy Nash equilibria, namely all the strategy profiles where 50 drivers choose route A, 50 route B and 300 route AB. At each of these equilibria each driver takes $35 + 45 = 80$ minutes.

- (e) Opening the new road has made everybody worse off: before every driver was taking 65 minutes and now she is taking 80 minutes! This is known as Braess's paradox. Braess's paradox, credited to the German mathematician Dietrich Braess, states that adding extra capacity to a network when the moving entities choose their route selfishly, can in some cases make every entity worse off and thus reduce overall performance.
- (f) We already know that at a Nash equilibrium nobody will choose strategy BA. Denote by $A \not\rightarrow B$ the fact that somebody who has chosen route A cannot gain (i.e. reduce her travel time) by unilaterally switching to route B and similarly for the other cases. Then the inequalities are as follows:

$$(1) \text{ (reflecting } A \not\rightarrow B) \quad \frac{T_A + T_{AB}}{10} + 45 \leq 45 + \frac{T_B + 1 + T_{AB}}{10}, \text{ i.e. } \boxed{T_A \leq T_B + 1}$$

$$(2) \text{ (reflecting } B \not\rightarrow A) \quad 45 + \frac{T_B + T_{AB}}{10} \leq \frac{T_A + 1 + T_{AB}}{10} + 45, \text{ i.e. } \boxed{T_B \leq T_A + 1}$$

$$(3) \text{ (reflecting } A \not\rightarrow AB) \quad \frac{T_A + T_{AB}}{10} + 45 \leq \frac{T_A - 1 + T_{AB} + 1}{10} + 10 + \frac{T_B + T_{AB} + 1}{10}, \text{ i.e. } \boxed{349 \leq T_B + T_{AB}}$$

$$(4) \text{ (reflecting } B \not\rightarrow AB) \quad 45 + \frac{T_B + T_{AB}}{10} \leq \frac{T_A + T_{AB} + 1}{10} + 10 + \frac{T_B - 1 + T_{AB} + 1}{10}, \text{ i.e. } \boxed{349 \leq T_A + T_{AB}}$$

$$(5) \text{ (reflecting } AB \not\rightarrow A) \quad \frac{T_A + T_{AB}}{10} + 10 + \frac{T_B + T_{AB}}{10} \leq \frac{T_A + 1 + T_{AB} - 1}{10} + 45, \text{ i.e. } \boxed{T_B + T_{AB} \leq 350}$$

$$(6) \text{ (reflecting } AB \not\rightarrow B) \quad \frac{T_A + T_{AB}}{10} + 10 + \frac{T_B + T_{AB}}{10} \leq 45 + \frac{T_B + 1 + T_{AB} - 1}{10}, \text{ i.e. } \boxed{T_A + T_{AB} \leq 350}$$

Note: the above inequalities assume that all of T_A, T_B and T_{AB} are ≥ 1 . For completeness (and to display brilliance!) one would have to prove this fact. Here is the proof. First we show that there is no Nash equilibrium where $T_{AB} = 0$. If there were a Nash equilibrium where only strategies A and B were chosen, then by the argument of part (b) it would have to be that $T_A = T_B = 200$, implying that each driver takes 65 minutes, but then a driver who is currently choosing A would gain by switching to AB because she would take only $20 + 10 + 20.1 = 50.1$ minutes. Next we show that there is no Nash equilibrium where $T_{AB} = 400$, because the AB route would take $40 + 10 + 40 = 90$ minutes while switching to A (or B) would take only $40 + 45 = 85$ minutes. Thus at a Nash equilibrium it must be that $0 < T_{AB} < 400$ and hence $T_A + T_B > 0$. Next we prove that it cannot be that

$T_A = 0$. If $T_A = 0$ then $T_B + T_{AB} = 400$ so that taking route AB requires

$$\frac{T_{AB}}{10} + 10 + \frac{T_{AB} + T_B}{10} \stackrel{\text{since } T_B + T_{AB} = 400}{=} \frac{T_{AB}}{10} + 50 \text{ minutes, but taking route A would take only } \frac{T_{AB}}{10} + 45 \text{ and hence}$$

it would pay for an AB driver to switch to A. The proof it cannot be that $T_B = 0$ is similar.

- (g) To the above inequalities we have to add also $T_A + T_B + T_{AB} = 400$. This equation, together with inequalities (3)-(6), yields $\boxed{50 \leq T_A \leq 51 \text{ and } 50 \leq T_B \leq 51}$. Thus the pure-strategy Nash equilibria are all the strategy profiles inducing one of the following:

1. $T_A = T_B = 50, T_{AB} = 300$
2. $T_A = 50, T_B = 51, T_{AB} = 299$
3. $T_A = 51, T_B = 50, T_{AB} = 299$
4. $T_A = 51, T_B = 51, T_{AB} = 298$