Meaning of oligopoly: an economic state of limited competition, in which a market is shared by a small number of sellers. (Greek: oligo = a few, small number; poly = from poló = to sell)

There are several models of oligopoly:

- Cooperative and non-cooperative
- Cournot-Nash (firms choose quantity of commodity)
- Stackelberg (leading agent)
- Bertrand (firms choose price)

We will deal with two models: A) non-cooperative Cournot-Nash oligopoly. B) Bertrand-Nash equilibrium with differentiated products.

Cournot-Nash-quantity equilibrium
Antoine Augustine Cournot was a French mathematician who, in 1838, wrote a book entitled Researches on the Mathematical Principles of the Theory of Wealth in which he introduced, for the first time, the idea of duopoly, the concept of a (pure strategy) Nash equilibrium, the Reaction function and best-response dynamics. John Forbes Nash Jr. was an American mathematician who, in 1951, made fundamental contributions to game theory, differential geometry and the study of partial differential equations.

Assumptions of non-cooperative Cournot-Nash oligopoly:
1. There are $K$ firms in the market.
2. All $K$ firms face the same inverse demand function $p = c - dx$ (only one commodity, $c > 0$, $d > 0$).
3. The commodity is perceived as homogeneous by the $K$ firms each of which produces it in the amount $x_k$ using a linear technology $A_k, k = 1,...,K$ and a vector of resources (inputs) $b_k$.
4. Sellers (producers, oligopoly firms) are basically unaware of the competitive strategies of each other. In other words, each firm maximizes profit with respect to its choice variable assuming that the choice variables of the other firms are unaffected (via output price) by the given firm's choice. Off the equilibrium path, this belief is not true, but it is true at the equilibrium point, which is all that matters because this model is static and we are looking for equilibrium.
5. Entry (in the industry, market) of other firms is blocked (because this model is static).
6. Sellers do not decide the price of the commodity. Each oligopolist makes his quantity decision on the basis of the market demand price, $p$, at which the total commodity from all firms, $x$, can be sold.
Toward the specification of a general model for solving the non-cooperative Cournot-Nash equilibrium of \(K\) firms, we study first the behavioral objective of the \(k\)th oligopoly firm which maximizes profit (decomposed into a primal and a dual problem, as usual).

Primal

\[
\begin{align*}
\text{max} & \quad TR_k = px_k = (c - dx)x_k = (c - d \sum_{h=1}^{K} x_h)x_k \\
\text{subject to} & \quad A_k x_k \leq b_k, \quad x_k \geq 0
\end{align*}
\]

Note that \(\sum_{h} x_h = x\) is total industry output from all the oligopoly firms. The Lagrange function is

\[
L_k = (c - d \sum_{h=1}^{K} x_h)x_k + y'_k (b_k - A_k x_k)
\]

and the relevant KKT conditions

\[
\begin{align*}
\frac{\partial L_k}{\partial x_k} & = (c - d \sum_{h=1}^{K} x_h) - dx_k - A'_k y_k \leq 0 \quad \rightarrow \quad p - dx_k \leq A'_k y_k \quad (2) \\
x_k \frac{\partial L_k}{\partial x_k} & = x_k p - x_k dx_k - x_k A'_k y_k = 0 \quad \rightarrow \quad px_k = x_k dx_k + x_k A'_k y_k \quad (3)
\end{align*}
\]

Note that relation (2) is the familiar \(MR \leq MC\). Recall that profit is always defined as: profit = total revenue minus total cost of physical plant (physical inputs). From equation (3), therefore, profit of the \(k\)th oligopoly firm is \(x_k dx_k\) or \(dx_k^2\) since we are dealing with only one commodity. Marginal revenue is divided into “price”, \(p\), and \(dx_k\) that corresponds to a measure of market power of the \(k\)th oligopoly firm:

\[
\text{price} - \left(\text{market power}\right) \leq \left(\text{marginal cost}\right)
\]

Using CSC (3) to simplify the Lagrange function we obtain

\[
L_k = (c - d \sum_{h=1}^{K} x_h)x_k + y'_k (b_k - A_k x_k)
\]

\[
= x_k dx_k + x_k A'_k y_k + y'_k b_k - y'_k A_k x_k
\]

\[
= y'_k b_k + x_k dx_k
\]

Therefore, the dual specification of the \(k\)th oligopoly firm is as follows

Dual

\[
\begin{align*}
\text{min} & \quad TC'_k = TP'_{pp,k} + TCMO_k \\
\text{subject to} & \quad MC \geq MR \\
A'_k y_k & \geq (c - d \sum_{h=1}^{K} x_h) - dx_k
\end{align*}
\]
The primal (or the dual) of the $k$th oligopoly firm cannot be solved because the industry overall quantity $x = \sum_{h=1}^{K} x_h$ is unknown and not recoverable from model (1) (or its dual). We must specify a comprehensive model that accounts for all oligopoly firms simultaneously. Such a model is the following specification:

Primal

$$\begin{align*}
\text{max } AUX &= (c - \frac{1}{2} dx)x - \frac{1}{2} \sum_{k=1}^{K} dx_k^2 \\
\text{subject to } A_k x_k &\leq b_k \\
\sum_{k=1}^{K} x_k &= x 
\end{align*}$$

dual variables

$$\begin{align*}
A_k x_k &= b_k \\
y_k &= p \\
p x_k &= x_k dx_k + x_k A_k' y_k \\
p &= \sum_{k=1}^{K} x_k - x
\end{align*}$$

with $k = 1, \ldots, K$. The objective function is an auxiliary function intended as such for generating the proper KKT conditions. The rationale for this type of auxiliary objective function will become clear after the dual model is developed. Recall that any mathematical programming problem is solved via its corresponding KKT conditions. Therefore, we must check that the KKT conditions derived from the above auxiliary model include the KKT conditions developed for the $k$th oligopoly firm. In the overall oligopoly model it is necessary to have this list of KKT conditions:

1. The technology relations for every oligopoly firm (first primal constraint).
2. The overall quantity of commodity produced by these firms (second primal constraint).
3. The marginal cost/marginal revenue relation for every oligopoly firms (KKT (2)).
4. The price formation relation, that is, the inverse demand function.

Let us check whether these required KKT relations are generated by the $AUX$ model formulated above. Lagrange function:

$$L = (c - \frac{1}{2} dx)x - \frac{1}{2} \sum_{k=1}^{K} dx_k^2 + \sum_{k=1}^{K} y_k' (b_k - A_k x_k) + p \left( \sum_{k=1}^{K} x_k - x \right)$$

Relevant KKT conditions:

$$\begin{align*}
\frac{\partial L}{\partial x_k} &= -dx_k - A_k' y_k + p \leq 0 \quad \text{dual constraints} \\
x_k \frac{\partial L}{\partial x_k} &= x_k (-dx_k - A_k' y_k + p) = 0 \quad \rightarrow \quad px_k = x_k dx_k + x_k A_k' y_k \\
\frac{\partial L}{\partial x} &= c - dx - p \leq 0 \quad \text{dual constraint} \\
x \frac{\partial L}{\partial x} &= x(c - dx - p) = 0 \quad \rightarrow \quad cx = dx^2 + px
\end{align*}$$

KKT conditions (4) and (5) are identical to KKT conditions (2) and (3) of the $k$th oligopoly firm. Furthermore, relation (6) is precisely the demand function (price formation) of the given commodity. Therefore, this correspondence guarantees that the overall (comprehensive, auxiliary) oligopoly model is correct. To develop the comprehensive dual model we use, as usual, the complementary slackness conditions (5) and (7) to simplify the Lagrange function that will become the objective function of the dual model. First of all we add up, over the index $k$, the complementary slackness condition (5):
\[ p \sum_{k=1}^{K} x_k = \sum_{k=1}^{K} dx_k^2 + \sum_{k=1}^{K} x_k A_k' y_k \]  

(8)

We now simplify the Lagrange function using equations (7) and (8):

\[
L = (c - \frac{1}{2} dx) x - \frac{1}{2} \sum_{k=1}^{K} dx_k^2 + \sum_{k=1}^{K} y_k' (b_k - A_k x_k) + p \left( \sum_{k=1}^{K} x_k - x \right)
\]

\[
= cx - \frac{1}{2} dx^2 - \frac{1}{2} \sum_{k=1}^{K} dx_k^2 + \sum_{k=1}^{K} y_k' b_k - \sum_{k=1}^{K} y_k' A_k x_k + p \sum_{k=1}^{K} x_k - px
\]

\[
= (dx^2 + px) - \frac{1}{2} dx^2 - \frac{1}{2} \sum_{k=1}^{K} dx_k^2 + \sum_{k=1}^{K} y_k' b_k - \sum_{k=1}^{K} y_k' A_k x_k + \left( \sum_{k=1}^{K} dx_k^2 + \sum_{k=1}^{K} x_k A_k' y_k \right) - px
\]

\[
= \frac{1}{2} dx^2 + \sum_{k=1}^{K} y_k' b_k + \frac{1}{2} \sum_{k=1}^{K} dx_k^2
\]

The comprehensive dual model, therefore, is

**Dual**

\[ \min AUX2 = \frac{1}{2} dx^2 + \sum_{k=1}^{K} y_k' b_k + \frac{1}{2} \sum_{k=1}^{K} dx_k^2 \]

subject to

\[ MC \geq MR \]

\[ A_k' y_k \geq p - dx_k \]

\[ p \geq c - dx \]

**Justification of the two auxiliary functions for the oligopoly industry.**

At industry equilibrium, the primal objective function is equal to the dual objective function. Hence

\[ (c - \frac{1}{2} dx) x - \frac{1}{2} \sum_{k=1}^{K} dx_k^2 = \frac{1}{2} dx^2 + \sum_{k=1}^{K} y_k' b_k + \frac{1}{2} \sum_{k=1}^{K} dx_k^2 \]

\[ (c - dx) x = \sum_{k=1}^{K} y_k' b_k + \sum_{k=1}^{K} dx_k^2 \]

\[ TR_{industry} = \sum_{k=1}^{K} TC_{ppk} + \sum_{k=1}^{K} TCMO_k \]

The last line of the above expression states that the total revenue of the oligopoly industry is equal to the sum of the total cost of the physical plants plus the sum of the profits of all the oligopoly firms. This conclusion justifies the specification of the two auxiliary objective functions of the primal and dual comprehensive models. Figure 1 illustrates the geometry of two oligopoly firms as developed algebraically in the above discussion.

**Alternative interpretation of the auxiliary objective functions**

A slightly different reorganization of the terms involved in the two auxiliary objective functions brings to light the idea that Congress and certain government agencies operate to keep in check the market power of an oligopoly industry. Since 1890 (Sherman Anti-Trust Act), Congress has decreed that certain tendencies toward monopolistic goals will be forbidden. This ban can be re-interpreted in economic terms as
“minimizing” the total market power of an oligopoly industry. The “minimization” is subject to a certain level of “legitimate” profit.

Let us consider again the two auxiliary functions and re-organize the terms as

\[(c - dx)x - \sum_{k=1}^{K} b_k y_k = \sum_{k=1}^{K} x_k dx_k\]

\[\max \pi = TR_{industry} - TC_{pp,industry} = \min \{Total \ market \ power\}\]

The amount \(dx_k\) was identified as a measure of market power per unit of commodity of the \(k\)th oligopoly firm. Hence the products of \(dx_k\) and the commodity quantity \(x_k\), summed over all firms, constitute the total market power of the oligopoly industry. Congress and government try to minimize this amount subject to legitimate profit levels. Or, at least, Congress and government agencies place (in principle) an upper bound on the total market power of oligopoly industries.

**Figure 1. Duopoly industry**

**Collusion of Oligopoly Firms – Cartel**

Oligopoly firms tend to collude (OPEC, for example). The colluding behavior is similar to that of a monopolist. The collusion is not successful most of the time because firms may be interested in cheating. The collusion may follow several types of agreements. One common way to collude is to create a cartel “representative council (OPEC)” that allocates a reduction on production by assigning a quota to each oligopoly firm. Let us agree that the reduction in output should be of 80 percent. This means using only 80 percent of the available (capacity) inputs. Then, the cartel-monopoly model is specified as follows

**Primal:** Cartel-monopoly \[\max TR = (c - dx)x\]

subject to
\[ A_k x_k \leq 0.8 b_k \]
\[
\sum_{k=1}^{K} x_k = x
\]

The consequence of this collusion is that the quantity sold as a cartel will be about 80 percent of the quantity sold under non-collusion oligopoly. The cartel-monopoly price will be higher as will be the profit, otherwise it would not be convenient for the oligopoly firms to behave as a cartel-monopolist.

**Examples of Cournot-Nash oligopoly**

**Example 1.** There is only one commodity to produce and sell whose inverse demand function is \( p = c - dx = 200 - 3x \). There are 3 oligopoly firms with the following technologies and input supplies:

\[
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix} = \begin{bmatrix}
3 \\
6 \\
5
\end{bmatrix}, \quad
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} = \begin{bmatrix}
18 \\
28 \\
10
\end{bmatrix}, \quad \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} = \begin{bmatrix}
3 \\
6 \\
5
\end{bmatrix}, \quad \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} = \begin{bmatrix}
15 \\
24 \\
33
\end{bmatrix}
\]

The results are:

- Cournot total quantity = 14
- Oligopoly firm 1 quantity = 6
- Oligopoly firm 2 quantity = 4.66
- Oligopoly firm 3 quantity = 3.33

- Cournot oligopoly price = 158.0
- Oligopoly firm 1 quantity = 6
- Oligopoly firm 2 quantity = 4.66
- Oligopoly firm 3 quantity = 3.33

These results are from the GAMS file: Oligopoly_MyWay.gms.

In the instructional literature (microeconomics textbooks, internet) several Cournot oligopoly examples are given where the industry inverse demand function and the marginal cost of every oligopoly firm are assumed to be known. In this case, the primal model is specified with the auxiliary objective function stated above subject to the constraints

\[ \text{given } MC_k = MR_k \quad k = 1, \ldots, K \]

To derive the marginal revenue, \( MR_k \), recall that total revenue of the \( k \)th oligopoly firm (from relation (1)) is stated as

\[ TR_k = px_k = (c - d \sum_{h=1}^{K} x_h) x_k = cx_k - d(x_1 x_k + x_2 x_k + \ldots + x_K x_k) \]

Hence,

\[ MR_k = \frac{\partial TR_k}{\partial x_k} = c - 2dx_k - d(x_1 + x_2 + \ldots + x_K) = c - 2dx_k - d \sum_{h \neq k} x_h \]

Therefore, the primal specification of the oligopoly problem **when the marginal cost is known** is stated as

**Primal**

\[
\max AUX = (c - \frac{1}{2} dx) x - \frac{1}{2} \sum_{k=1}^{K} dx_k^2
\]

subject to
given $MC_k = c - 2dx_k - d \sum_{h \neq k} x_h$

$$\sum_{k=1}^{K} x_k = x$$

If the oligopoly firms collude, their behavior is similar to that of a cartel that operates as a monopoly. In this case,

Cartel primal problem

$$\max TR = px = (c - dx)x$$

subject to

$$given \ MC = c - 2dx.$$ 

As numerical examples of this specification we study two problems appeared in the literature:

**Example 2**: American Airlines (firm 1) and United Airlines (firm 2) (Perloff, J.M. Microeconomics, 1998, pages 433-436).

The market inverse demand function is $p = 339 - x$, where $p$ is the dollar price (cost to the passenger) of a one-way flight, $x$ is the total quantity of the two airlines combined and is measured in thousand of passengers flying one way per quarter. The marginal cost is the same for the two airlines and it is equal to $147$ per passenger per flight. The solution of this duopoly problem exhibits the following results:

Cournot total quantity = 128
Oligopoly firm 1 quantity = 64
Oligopoly firm 2 quantity = 64
Cournot oligopoly price = 211
Cournot total revenue = 31,104
Cournot profit = 8,920

Cartel-monopoly quantity = 96
Cartel-Oligopoly firm 1 quantity = 48
Cartel-Oligopoly firm 2 quantity = 48
Cartel-monopoly price = 243
Cartel-monopoly total revenue = 23,328
Cartel-monopoly profit = 9,216

These results are from the GAMS file: Cartel_oligopoly_Perloff_P.435.gms.

**Example 3**: Duopoly of TutorsOnNet.com.

The market inverse demand function is $p = 200 - 4x$, where $p$ is the dollar price, $x$ is the total quantity of the commodity. The marginal cost is the same for the duopoly firms and it is equal to $8$. The solution of this duopoly problem exhibits the following results:

Cournot total quantity = 32
Oligopoly firm 1 quantity = 16
Oligopoly firm 2 quantity = 16
Cournot oligopoly price = 72
Cournot total revenue = 3,328
Cournot profit = 2,048

Cartel-monopoly quantity = 24
Cartel-Oligopoly firm 1 quantity = 12
Cartel-Oligopoly firm 2 quantity = 12
Cartel-monopoly price = 104
Cartel-monopoly total revenue = 2,496
Cartel-monopoly profit = 2,304

These results are from the GAMS file: Cartel_oligopoly_TutorsOnNet.gms.

From examples 1, 2 and 3, the total Cournot quantity is larger than the cartel-monopoly quantity. Correspondingly, the Cournot price is smaller than the cartel-monopoly price. The Cournot profit is smaller than the cartel-monopoly profit.
Bertrand equilibrium with differentiated products
Cournot-Nash model deals with commodity quantities. That is, oligopoly firms decide how much output to produce and sell. After forty five years from the publication of Cournot’s book (1838), Joseph Louis François Bertrand, another French mathematician, published (1883) a critique of Cournot’s model and used prices as the strategic variables of firms. In other words, he argued that, more often than not, oligopoly firms choose prices rather than quantities of commodities. Then, consumers decide on the quantity to purchase. If the commodity dealt with by oligopoly firms is homogeneous, the equilibrium price is the competitive price. But if firms deal with differentiated products, prices (and quantities) have their own equilibrium. Differentiated product markets abound: cars, tires, drinks, airlines, toothpastes, spaghetti, etc.

Coca Cola and Pepsi Cola
A coke is not the same as a pepsi, although they both can be regarded as drinks of the same category: they are differentiated products. We assume that, before setting the price of a coke, the CEO of Coca Cola observes what price the Pepsi Cola company decides for its can of soda, and vice versa. Each company employs an army of econometricians whose principal job is to estimate a demand function for its drink. It turns out that Gasmi, Laffont and Vuong did precisely that in 1992 with the following result (the estimates were rounded off somewhat):

\[
\text{Demand function for coke} \quad q_c \leq 58 - 4p_c + 2p_p
\]
\[
\text{Demand function for pepsi} \quad q_p \leq 55.3 + 1.4p_c - 3.5p_p
\]

where \(q_c, q_p\) are quantities of coke and pepsi, respectively and \(p_c, p_p\) are the corresponding prices. The subscripts refer to Coca Cola and Pepsi Cola, respectively. The meaning of these demand functions (written as weak inequalities) is that the left-hand-side quantities of either coke or pepsi, \(q_c, q_p\), are demanded by consumers while the right-hand-side quantities are supplied to the consumer markets by Coca Cola and Pepsi Cola firms, respectively. For example, for one dollar increase in the price of a pepsi, the Coca Cola company will produce and supply the market 2 more units of coke. Similarly, for one dollar unit increase in the price of a coke, the Pepsi Cola company will produce and supply the market 1.4 more units of pepsi.

Clearly, each company, in selecting the price of its product (and only implicitly the quantity of the own drink to supply the consumer market), watches what the other company does with its own price. This is the meaning of Bertrand competition.

We assume that each company has a constant marginal and average cost, \(MC_c, MC_p\) and, furthermore, both companies wish to maximize profit. Consider Coca Cola company’s decision about price

\[
\max \pi_c = p_c q_c - MC_c q_c = p_c \left[ 58 - 4p_c + 2p_p \right] - MC_c \left[ 58 - 4p_c + 2p_p \right]
\]

Profit maximization requires that

\[
\frac{\partial \pi}{\partial p_c} = 58 - 8p_c + 2p_p + 4MC_c = 0
\]

that can be restated (using the Implicit Function Theorem) as the best-response function of Coca Cola

\[
p_c = \frac{58}{8} + \frac{2}{8}p_p + \frac{4}{8}MC_c = 7.25 + 0.25p_p + 0.5MC_c
\]
Coca Cola’s price depends on Pepsi Cola’s price. Presumably, the CEO of Coca Cola forms expectations about Pepsi Cola’s price decision in order to decide at what price to offer the coke.

Similar process is assumed for Pepsi Cola company whose best-response function turns out to be

\[ p_p = \frac{55}{7} + \frac{14}{7} p_c + \frac{35}{7} \cdot MC_p = 7.9 + 0.2 p_c + 0.5 MC_p \]

A Bertrand equilibrium for the duopoly industry is achieved by solving the two best-response functions for the Coca and Pepsi prices, given (assuming) knowledge of marginal cost.

With demand functions’ information for both oligopoly firms, the comprehensive model for this Bertrand equilibrium duopoly is stated as

Primal

\[ \max TOT \pi = Coca \pi + Pepsi \pi \]

\[ = (TR_c - TC_c) + (TR_p - TC_p) \]

\[ = (p_c q_c - MC_c q_c) + (p_p q_p - MC_p q_p) \]

subject to

- Coca Cola best-response function
  \[ p_c = 7.25 + 0.25 p_p + 0.5 MC_c \]

- Coca Cola demand function
  \[ q_c = 58 - 4 p_c + 2 p_p \]

- Pepsi Cola best-response function
  \[ p_p = 7.9 + 0.2 p_c + 0.5 MC_p \]

- Pepsi Cola demand function
  \[ q_p = 55.3 + 1.4 p_c - 3.5 p_p \]

with all the prices and quantities being nonnegative.

The empirical results are as follows:

<table>
<thead>
<tr>
<th>Total duopoly profit</th>
<th>480.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coca Cola profit</td>
<td>256.00</td>
</tr>
<tr>
<td>Price</td>
<td>13.00</td>
</tr>
<tr>
<td>Quantity</td>
<td>32.00</td>
</tr>
<tr>
<td>Market power</td>
<td>8.00</td>
</tr>
<tr>
<td>Market share</td>
<td>53.3%</td>
</tr>
<tr>
<td>Pepsi Cola profit</td>
<td>224.00</td>
</tr>
<tr>
<td>Price</td>
<td>13.00</td>
</tr>
<tr>
<td>Quantity</td>
<td>28.00</td>
</tr>
<tr>
<td>Market power</td>
<td>8.00</td>
</tr>
<tr>
<td>Market share</td>
<td>46.7%</td>
</tr>
</tbody>
</table>

Let us suppose that Pepsi Cola wishes to increase its market share by investing in new machinery and equipment that will reduce its marginal cost from $5 to $4. The Bertrand equilibrium results are as follows:

<table>
<thead>
<tr>
<th>Total duopoly profit</th>
<th>498.96</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coca Cola profit</td>
<td>247.65</td>
</tr>
<tr>
<td>Price</td>
<td>12.87</td>
</tr>
<tr>
<td>Quantity</td>
<td>31.47</td>
</tr>
<tr>
<td>Market power</td>
<td>7.87</td>
</tr>
<tr>
<td>Market share</td>
<td>51.5%</td>
</tr>
<tr>
<td>Pepsi Cola profit</td>
<td>251.31</td>
</tr>
<tr>
<td>Price</td>
<td>12.47</td>
</tr>
<tr>
<td>Quantity</td>
<td>29.66</td>
</tr>
<tr>
<td>Market power</td>
<td>8.47</td>
</tr>
<tr>
<td>Market share</td>
<td>48.5%</td>
</tr>
</tbody>
</table>
Although Coca Cola lost less than 2 percent of its market share its profit fell by 3.3 percent. Coca Cola needed to lower its price as a consequence of Pepsi Cola reduction in its marginal cost which reduced Pepsi’s price by more than 50 cents. Pepsi Cola profit increased by 12.6 percent.

**More General Specification of the Bertrand-Nash Price Equilibrium**

Two firms produce and sell differentiated products. Their demand functions include an estimate of the other firm’s price:

\[ q_i = f_i(p_i, p_j), \quad i \neq j, \quad i = 1, 2, j = 1, 2 \]

\[ q_i \equiv \text{quantity of firm } i, \quad p_i \equiv \text{price of firm } i, \quad p_j \equiv \text{price of firm } j. \]

This duopoly behaves according to the Bertrand-Nash (price) equilibrium. Each firm has proprietary technology \( A_i, i = 1, 2 \) and a supply of limiting inputs \( b_i \) to produce its final output. Both firms wish to maximize profit.

Consider now the \( i \)th firm with its technological relation

\[ \max TR_i = p_i q_i \]

subject to

\[ q_i = f_i(p_i, p_j) \]

\[ A_i q_i \leq b_i \]

and all nonnegative variables. In this case, we do not know marginal cost. It will have to be found by KKT conditions:

Lagrange function

\[ L_i = p_i q_i + \lambda_i [f_i(p_i, p_j) - q_i] + \gamma_i [b_i - A_i q_i] \]

Relevant KKT conditions

\[ \frac{\partial L_i}{\partial p_i} = q_i + p_i \frac{\partial q_i}{\partial p_i} + \lambda_i \frac{\partial f_i(p_i, p_j)}{\partial p_i} - \lambda_i \frac{\partial q_i}{\partial p_i} - A_i \gamma_i \frac{\partial q_i}{\partial p_i} \leq 0 \]

\[ \frac{\partial L_i}{\partial q_i} = p_i - \lambda_i - A_i \gamma_i \leq 0 \]

From the second KKT condition it is clear that marginal cost is \( A_i \gamma_i \).

The **comprehensive** Bertrand duopoly model will assemble the KKT conditions of both firms as constraints into a **Nonlinear** Complementarity Problem:

\[ \min CSC = \sum_{i=1}^{2} wp_i y_i + \sum_{i=1}^{2} p_i w_{d_{ii}} + \sum_{i=1}^{2} q_i w_{d_{2i}} = 0 \]

subject to

\[ q_i = f_i(p_i, p_j) \quad \text{demand function} \]

\[ A_i q_i + w p_i = b_i \quad \text{physical plant} \]

\[ q_i + p_i \frac{\partial q_i}{\partial p_i} + \lambda_i \frac{\partial f_i(p_i, p_j)}{\partial p_i} - \lambda_i \frac{\partial q_i}{\partial p_i} - A_i \gamma_i \frac{\partial q_i}{\partial p_i} + w_{d_{ii}} = 0 \quad \text{implicit best-response function} \]

\[ p_i - \lambda_i - A_i \gamma_i + w_{d_{2i}} = 0 \quad \text{implicit market power} \]

with all nonnegative variables.

Note that I introduced slack variables in each of the constraints:
\( wp_i \equiv \text{slack primal vector variable (} p \text{ for primal)} \)

\( wd_{ij} \equiv \text{slack variable for the first dual constraint (} d \text{ for dual)} \)

\( wd_{2,i} \equiv \text{slack variable for the second dual constraint (} d \text{ for dual)} \)

The objective function is the sum of all the complementary slackness conditions that should add up to zero.

**Example 4.** The Coca Cola – Pepsi Cola example continues with the introduction of a specific technology for each firm. Suppose that Coca Cola employs \( A_1 = 0.5 \) of some input to produce one unit of coke using the available supply of that input in the amount of \( b_1 = 15 \). Pepsi Cola uses a technology \( A_2 = 0.7 \) and an input supply of \( b_2 = 20 \). With this information and the previous demand functions the empirical result of the more general specification of this duopoly is

<table>
<thead>
<tr>
<th></th>
<th>Coca Cola profit</th>
<th>Pepsi Cola profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Cost</td>
<td>112.50</td>
<td>116.62</td>
</tr>
<tr>
<td>Input Shadow Price</td>
<td>19.55</td>
<td>12.81</td>
</tr>
<tr>
<td>Price</td>
<td>13.52</td>
<td>13.05</td>
</tr>
<tr>
<td>Marginal Cost</td>
<td>9.77</td>
<td>8.96</td>
</tr>
<tr>
<td>Market power</td>
<td>3.75</td>
<td>4.09</td>
</tr>
<tr>
<td>Quantity</td>
<td>30.00</td>
<td>28.57</td>
</tr>
<tr>
<td>Market share</td>
<td>51.22%</td>
<td>48.78%</td>
</tr>
</tbody>
</table>

It is curious to note that, as of this week (February 2018), the price of a six pack of regular Coca Cola (at the local grocery store) is $6.99 while the price of a six pack of Pepsi Cola is $6.46 with a difference of 53 cents.

**Bertrand duopoly Cartel**

Let us suppose that the two oligopoly firms wish to collude and behave as a cartel (this may be unlikely between Coca Cola and Pepsi Cola, but let us imagine that this may happen to see the results between two oligopoly firms). The Primal model, then will be

Primal: Bertrand duopoly cartel

\[
\begin{align*}
\text{maxTOT } & \pi = Coca & + Pepsi \\
& \pi & = (TR_c - TC_c) (+ TR_p - TC_p) \\
& & = (p_c q_c - MC_c q_c) + (p_p q_p - MC_p q_p)
\end{align*}
\]

subject to

- Coca Cola demand function \( q_c = 58 - 4 p_c + 2 p_p \)
- Pepsi Cola demand function \( q_p = 55.3 + 1.4 p_c - 3.5 p_p \)

with all nonnegative variables. This Bertrand cartel problem differs from the oligopoly model stated above because no best-response function was stated as a constraint.
\[ \lambda_c \text{ and } \lambda_p \]

We choose \( \lambda_c \) and \( \lambda_p \) as Lagrange multipliers of the two demand constraints. As we will see in a while, these Lagrange multipliers will acquire a very interesting economic meaning.

As usual, we are interested in deriving the dual problem and interpret all of its components in economic terms. Be begin with formulating the Lagrange function:

\[ L = p_c q_c - MC_c q_c + p_p q_p - MC_p q_p + \lambda_c (58 - 4 p_c + 2 p_p - q_c) + \lambda_p (55.3 + 1.4 p_c - 3.5 p_p - q_p). \]

Relevant KKT conditions are

\[ \frac{\partial L}{\partial p_c} = q_c + p_c \frac{\partial q_c}{\partial p_c} - MC_c \frac{\partial q_c}{\partial p_c} - 4 \lambda_c + 1.4 \lambda_p \leq 0 \]
\[ = (q_c - 4 p_c) + 4 MC_c - 4 \lambda_c + 1.4 \lambda_p \leq 0 \]
\[ = MR_c - TotMC_c \leq 0 \]

\[ p_c \frac{\partial L}{\partial p_c} = p_c q_c - 4 p_c^2 + 4 MC_c p_c - 4 \lambda_c p_c + 1.4 \lambda_p p_c = 0 \]

\[ \frac{\partial L}{\partial p_p} = q_p - 3.5 p_p + 3.5 MC_p - 3.5 \lambda_p + 2 \lambda_c \leq 0 \]

\[ p_p \frac{\partial L}{\partial p_p} = p_p q_p - 3.5 p_p^2 + 3.5 MC_p p_p - 3.5 \lambda_p p_p + 2 \lambda_c p_p = 0 \]

\[ \frac{\partial L}{\partial q_c} = p_c - MC_c - \lambda_c \leq 0 \]

\[ q_c \frac{\partial L}{\partial q_c} = p_c q_c - MC_c q_c - \lambda_c q_c = 0 \]

\[ \frac{\partial L}{\partial q_p} = p_p - MC_p - \lambda_p \leq 0 \]

\[ q_p \frac{\partial L}{\partial q_p} = p_p q_p - MC_p q_p - \lambda_p q_p = 0 \]

Note that KKT condition (13) and (15) can be interpreted as a measure of market power (difference between price and marginal cost per unit of output) of the two soft drink companies. Therefore, the Lagrange multipliers \( \lambda_c \) and \( \lambda_p \) acquire the meaning of market power indicators of the two companies. Then, the complementary slackness conditions (14) and (16) are a measure of total market power for each company as indicated by the expressions \( \lambda_c q_c \) and \( \lambda_p q_p \). It turns out that the Lagrange function can be simplified (with some boring algebra) by using the complementary slackness conditions (10), (12), (14) and (16) to obtain precisely

\[ L = \lambda_c q_c + \lambda_p q_p \equiv \text{Total market power of the duopoly}. \]

The dual model of the Bertrand cartel equilibrium can thus be stated as \((\text{TMP} \equiv \text{Total Market Power of Duopoly})\)

Dual of Bertrand duopoly cartel

\[ \min \text{TMP} = \lambda_c q_c + \lambda_p q_p \]

subject to
\[ MC \geq MR \]
\[-4MC_c + 4\lambda_c - 1.4\lambda_p \geq q_c - 4p_c \quad \text{profit equilibrium conditions} \]
\[-3.5MC_p - 2\lambda_c + 3.5\lambda_p \geq q_p - 3.5p_p \]
\[ \lambda_c \geq p_c - MC_c \quad \text{market power conditions} \]
\[ \lambda_p \geq p_p - MC_p \]
\[ q_c = 58 - 4p_c + 2p_p \quad \text{demand functions} \]
\[ q_p = 55.3 + 1.4p_c - 3.5p_p \]

with all nonnegative variables.

The economic meaning of the dual specification is of interest. We said that the Lagrange multipliers \( \lambda_c \) and \( \lambda_p \) measure the per-unit of product market power of the respective duopoly firms. Congress, on its wisdom, has approved anti-trust laws to regulate and control the concentration of market power. There are government agencies that administer those laws and attempt to minimize the total market power of cartels. Often, the antitrust agency mandates a divestment of some oligopoly firm with the explicit objective to reduce (minimize) the market power of that firm. Therefore, one can interpret the dual objective function as the attempt to minimize the total market power of the Bertrand cartel. The dual constraints specify the economic equilibrium conditions for profit maximizing.

**Example 5.** The solution of the Coca Cola and Pepsi Cola cartel problem results in the following equilibrium: Marginal cost was fixed at $5 for both firms

| Total cartel profit 526.6318 = Total market power of the cartel |
| Coca Cola profit 302.4106 |
| Pepsi Cola profit 224.2212 |
| Price 15.99 |
| Price 16.74 |
| Quantity 27.5112 |
| Quantity 19.1035 |
| Market power 10.9883 |
| Market power 11.7372 |
| Market share 59.02% |
| Market share 40.98% |