Micro Prelim August 2019

QUESTION 1 ANSWER KEYS

(a) First convert the table into probabilities:

$$\begin{pmatrix} (r,\kappa): & (100,8) & (120,12) & (100,16) & (100,24) & (140,24) & (120,36) & (140,36) & (120,40) & (140,72) \\ \text{residents:} & \frac{2}{24} & \frac{1}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24} & \frac{2}{24} & \frac{4}{24} & \frac{4}{24} & \frac{5}{24} \end{pmatrix}$$

First of all, it cannot be profit-maximizing to charge a price which is not in the set $\{100,120,140\}$. If you charge \$140 then the expected number of rides will be $\frac{1}{10}24+\frac{4}{10}36+\frac{5}{10}72=52.8$ and thus your expected profit per member is $\pi(140)=140-52.8c$ and your expected total profit is $50\pi(140)$.

If you charge \$120 then the expected number of rides will be $\frac{1}{17}12+\frac{1}{17}24+\frac{6}{17}36+\frac{4}{17}40+\frac{5}{17}72=45.42$ and thus your expected profit per member is $\pi(120)=120-45.42c$ and your expected total profit is $50\pi(120)$.

If you charge \$100 then the expected number of rides will be $\frac{2}{24}8+\frac{1}{24}12+\frac{3}{24}16+\frac{3}{24}24+\frac{6}{24}36+\frac{4}{24}40+\frac{5}{24}72=36.83$ and thus your expected profit per member is $\pi(100)=100-36.83c$ and your expected total profit is $50\pi(100)$.

- $\pi(140)>\pi(120)$ if and only if $c<2.71$ and $\pi(140)>\pi(100)$ if and only if $c<2.51$. Thus if $c<2.51$ you will charge \$140 (note that if $c<2.51$ then $\pi(140)>0$, since $140-52.8c>0$ if and only if $c<2.65$).
- $\pi(120)>\pi(140)$ if and only if $c>2.71$ and $\pi(120)>\pi(100)$ if and only if $c<2.33$. Thus you will never choose to charge \$120.
- $\pi(100)>\pi(120)$ if and only if $c>2.33$ and $\pi(100)>\pi(140)$ if and only if $c>2.51$, but if $c>2.51$ then you will charge \$100, as long as $\pi(100)>0$, that is, as long as $c<2.72$.

Hence the profit maximizing price is $\begin{cases} \$140 & \text{if } 0<c<2.51 \\ \$100 & \text{if } 2.51<c<2.72 \\ \infty \text{ (i.e. no operation)} & \text{otherwise} \end{cases}$

(b) **(b.1)** In this case $n_{ij}=\frac{N}{100}$, for every $i,j\in\{1,\ldots,10\}$, that is, we have the uniform distribution over $\{1,...,10\}\times\{1,,...,10\}$ [note that, since $K<\frac{N}{100}$ demand for membership will be greater than K, no matter what price you charge]. Hence the expected number of rides per customer is the same, namely $\frac{\sum_{k=1}^{10}k}{10}=\frac{\frac{10(11)}{2}}{10}=5.5$, no matter what price you charge; it follows that your expected cost per customer is 5.5c, no matter what price you charge. Thus the profit-maximizing price is \$10 and profits will be positive if and only if $c<\frac{10}{5.5}=1.82$.

(b.2) When K = 1,000, your maximum profits are $1{,}000[10-5.5c]=10{,}000-5{,}500c$. Since you want $10{,}000-5{,}500c\geq 3{,}900$ you need $c\leq 1.11$.

(c) **(c.1)** In this case, the joint distribution is:

$r\downarrow/\kappa\rightarrow$	1	2	3	4	5
16	$\frac{N}{15}$	$\frac{N}{15}$	$\frac{N}{15}$	$\frac{N}{15}$	$\frac{N}{15}$
17	0	$\frac{N}{15}$	$\frac{N}{15}$	$\frac{N}{15}$	$\frac{N}{15}$
18	0	0	$\frac{N}{15}$	$\frac{N}{15}$	$\frac{N}{15}$
19	0	0	0	$\frac{N}{15}$	$\frac{N}{15}$
20	0	0	0	0	$\frac{N}{15}$

Hence

- if you charge 20 then the expected number of rides per customer is $5=\frac{15}{3}$ and the expected cost per customer is $\frac{15}{3}c$,
- if you charge 19 then the expected number of rides per customer is $\frac{1}{3}4+\frac{2}{3}5=\frac{14}{3}$ and the expected cost per customer is $\frac{14}{3}c$,
- if you charge 18 then the expected number of rides per customer is $\frac{1}{6}3+\frac{2}{6}4+\frac{3}{6}5=\frac{13}{3}$ and the expected cost per customer is $\frac{13}{3}c$,
- if you charge 17 then the expected number of rides per customer is $\frac{1}{10}2+\frac{2}{10}3+\frac{3}{10}4+\frac{4}{10}5=\frac{12}{3}$ and the expected cost per customer is $\frac{12}{3}c$,
- if you charge 16 then the expected number of rides per customer is $\frac{1}{15}1+\frac{2}{15}2+\frac{3}{15}3+\frac{4}{15}4+\frac{5}{15}5=\frac{11}{3}$ and the expected cost per customer is $\frac{11}{3}c$.

Thus, if you increase the price by 1, your expected cost per customer increases by $\frac{c}{3}$. Hence if $c<3$ then the profit-maximizing price is 20 and if $c>3$ then the profit maximizing price is 16, provided that $c<\frac{48}{11}=4.36$ (which is necessary and sufficient for profits to be positive when you charge 16).

(c.2) If $c<3$ then you charge 20 and your profits are $1,000(20-5c)>5,000$ and thus you are willing to operate. If $c>3$ then you charge 16 and your profits are $1,000(16-\frac{11}{3}c)$. Since you want $1,000(16-\frac{11}{3}c)\geq 3,900$ it must be $c\leq 3.3$. Thus you are willing to run the business if and only if $c\leq 3.3$.

Question 2

When we studied consumer theory in ECN200A, we introduced various functions like utility functions, indirect utility functions, expenditure functions, Walrasian demand functions, Hicksian demand functions etc. There are other functions that are sometimes useful in the context of consumer theory. Let us use our knowledge of consumer theory and the techniques we learned to study one such function that is not discussed in Mas-Colell, Whinston, and Green (1995).

Fix a consumption bundle $g \in X \subseteq \mathbb{R}^L_+$ with $g \neq 0$. We will use this consumption bundle as a reference point. We want to define a function that measures *how many units of this reference consumption bundle g a consumer is willing to give up in order to move from some utility level $\underline{u}$ to some consumption bundle $x \in X$.* Such a function may be useful in the context of development economics of societies in which one commodity (e.g., rice) is a natural reference commodity already. It is also of conceptual significance as it helps us to understand the consumer problem as a problem of maximizing the difference between benefits and costs.

To this end, for reference consumption bundle $g \in X$, $g \neq 0$, and utility level $\underline{u}$, define the *benefit function* by

$$b(x, \underline{u}) = \begin{cases} \max\{\beta \in \mathbb{R} : u(x - \beta g) \geq \underline{u}, x - \beta g \in X\} & \text{if } x - \beta g \in X, u(x - \beta g) \geq \underline{u} \text{ for some } \beta \\ -\infty & \text{otherwise} \end{cases}$$

a. Let's try first to understand this function graphically by assuming $L = 2$. Consider first Figure 1 (a). It depicts an indifference curve representing a utility level u and a reference consumption bundle g. Further, it depicts b, the number of units of g the consumer is willing to give up to move from the indifference curve representing u to the consumption bundle x. Explain now what happens in Figure 1 (b).

   `Figure b depicts a situation in which it is not possible to get all the way back to the indifference curve (because of the need to satisfy the constraint` $x - \beta g \in X$ `but the benefit is nevertheless possible.`

b. Explain what happens in Figure 1 (c).

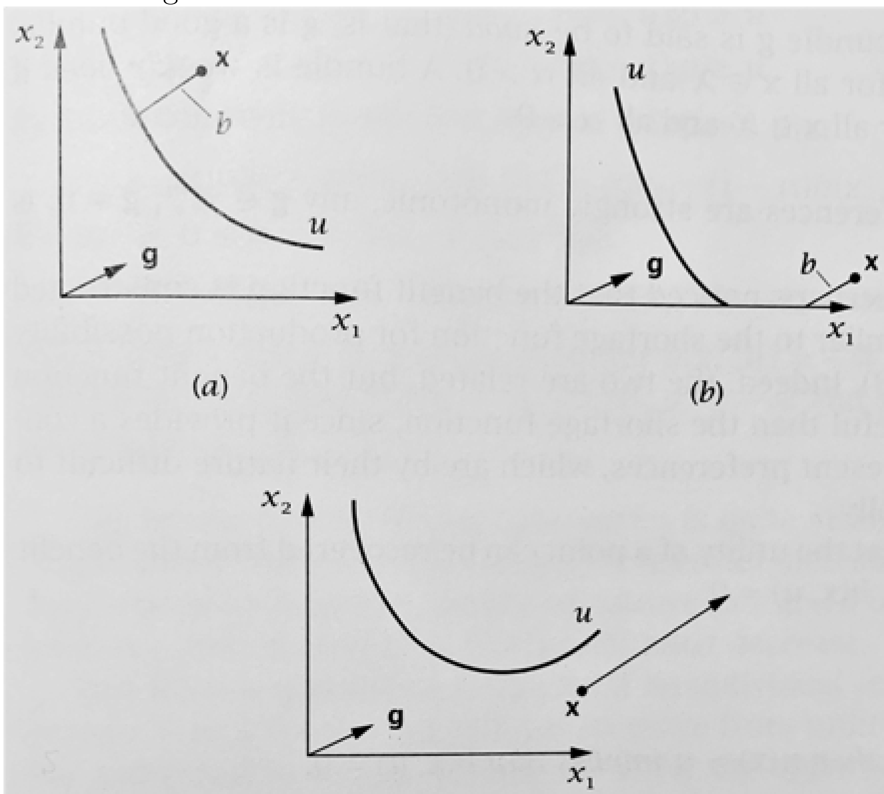
   `Figure (c) shows a case where it is impossible to get a utility level of at least` u `by moving in the direction of` g`. Thus, the corresponding value of the benefit function is` $-\infty$`.`

c. Let's derive the benefit function for the case of a Cobb-Douglas utility function $u(x) = \prod_{\ell=1}^L x_\ell^{\alpha_\ell}$ for $\alpha_\ell > 0$, $\ell = 1, ..., L$, $x \in \mathbb{R}^L_{++}$. Set $g = (1, 0, ..., 0)$. Then

$$b(x, \underline{u}) = \max \beta \text{ s.t. } (x_1 - \beta)^{\alpha_1} \prod_{\ell \geq 2} x_\ell^{\alpha_\ell} \geq \underline{u}.$$

   Derive $b(x, \underline{u})$ (I.e., solve for the β that corresponds to $b(x, \underline{u})$.)

Figure 1: Three Cases of the Benefit Function



We solve the equation

$$(x_1 - \beta)^{\alpha_1} \prod_{\ell \geq 2} x_\ell^{\alpha_\ell} = \underline{u}.$$

The solution for β is $b(x, \underline{u}) = x_1 - \frac{\underline{u}^{\frac{1}{\alpha_1}}}{\prod_{\ell \geq 2} x_\ell^{\frac{\alpha_\ell}{\alpha_1}}}$.

d. Consider now again the general definition of the benefit function defined above. Argue that $b(x, \underline{u})$ is nonincreasing in $\underline{u}$.

Argue with the definition.

e. Argue that if $x \in \mathbb{R}^L_+$ and $x + \alpha g \in \mathbb{R}^L_+$, then $b(x + \alpha g, \underline{u}) = \alpha + b(x, \underline{u})$.

Argue with the definition.

f. Show that if the utility function u is quasiconcave with respect to x, then $b(x, \underline{u})$ is concave with respect to the x.

Fix $x, y \in \mathbb{R}^L_+$ and $\underline{u}$.

Case $b(x, \underline{u})$, $b(y, \underline{u})$ are both finite: By definition

$$\begin{aligned} u(x - b(x, \underline{u})g) &\geq \underline{u} \\ u(y - b(y, \underline{u})g) &\geq \underline{u} \end{aligned}$$

By quasiconcavity of u,

$$u(\alpha x - \alpha b(x, \underline{u})g + (1-\alpha)y - (1-\alpha)b(y, \underline{u})g) \geq \underline{u}$$

for all $\alpha \in [0, 1]$. This implies

$$b(\alpha x + (1-\alpha)y, \underline{u}) \geq \alpha b(x, \underline{u}) + (1-\alpha)b(y, \underline{u}),$$

which demonstrates that b is concave.

Case $b(x, \underline{u}) = -\infty$ or $b(y, \underline{u}) = -\infty$: In this case $b(\alpha x + (1-\alpha)y, \underline{u}) \geq -\infty$. Hence, b is concave in x.

g. Assume that the utility function u is continuous, $g \geq 0$, $g \neq 0$, and $X = \mathbb{R}^L_+$. Let $p \in \mathbb{R}^L_{++}$. Assume further that $p \cdot x^* > 0$ and $b(x^*, u^*) = 0$ with $u^* = u(x^*)$. Show that if $x^* \in X$ solves the problem

$$\max_{x \in X} b(x, u^*) - p \cdot x,$$

then x^* also solves the problem

$$\max_{x \in X} u(x) \text{ s.t. } p \cdot x \leq w,$$

where as in class w represents the consumer's wealth.

Let x^* solve the benefit-minus-cost problem. Suppose by contradiction that there exists a consumption bundle $x \in X$ s.t. $u(x) > u^*$ and $p \cdot x \leq p \cdot x^*$. Since $p \cdot x^* > 0$, there exists x' nearby x and in the interior of X such that $p \cdot x' \leq p \cdot x^*$ and $b(x', u^*) > 0$. Thus, $b(x', u^*) - p \cdot x' > -p \cdot x^*$. Since by assumption $b(x^*, u^*) = 0$, we conclude

$$b(x', u^*) - p \cdot x' > b(x^*, u^*) - p \cdot x^*,$$

a contradiction.

h. Assume that the utility function u is continuous, locally nonsatiated, and that $g \geq 0$, $g \neq 0$, and $X = \mathbb{R}^L_+$. As before, let $p \in \mathbb{R}^L_{++}$ and $p \cdot g = 1$. Show that if $x^* \in X$ solves the problem

$$\max_{x \in X} u(x) \text{ s.t. } p \cdot x \leq w,$$

then x^* also solves the problem

$$\max_{x \in X} b(x, u^*) - p \cdot x,$$

with $u^* = u(x^*)$.

Let x^* solve the utility maximization problem. Suppose by contradiction that there exists $x \in X$ with

$$b(x, u^*) - p \cdot x > b(x^*, u^*) - p \cdot x^*.$$

Note that

$$p \cdot (x - b(x, u^*)g) < p \cdot x^* - b(x^*, u^*) \leq p \cdot x^*.$$

By the definition of $b(x, u^*)$,

$$u(x - b(x, u^*)g) \geq u^*.$$

Since u satisfies local nonsatiation, there exists x' near by $x - b(x, u^*)g$ with $p \cdot x' \leq p \cdot x^*$ and $u(x') > u^* = u(x^*)$, a contradiction.

For further details on benefit functions, see Luenberger, D.G. ''Microeconomic theory'', New York: McGraw-Hill, 1995.

University of California, Davis
Department of Economics
Preliminary Exam, Microeconomics
August 2019
Question 3

(a) The extended approach function is

$$F(p, x^1, x^2, \lambda^1, \lambda^2, w^1, w^2) = \begin{pmatrix} Du^1(x^1) - \lambda^1 p \\ p \cdot (w^1 - x^1) \\ Du^2(x^2) - \lambda^2 p \\ p \cdot (w^2 - x^2) \\ \tilde{x}^1 + \tilde{x}^2 - \tilde{w}^1 - \tilde{w}^2 \end{pmatrix}.$$

where $p_1 = 1$ by assumption, and the $\tilde{}$ above a bundle excludes the numéraire commodity.

(b) Since the two terms that define the function share no common argument, the Jacobean is, simply,

$$DG = \begin{pmatrix} DF(p, x^1, x^2, \lambda^1, \lambda^2, w^1, w^2) & 0 \\ 0 & DF(\hat{p}, \hat{x}^1, \hat{x}^2, \hat{\lambda}^1, \hat{\lambda}^2, \hat{w}^1, \hat{w}^2) \end{pmatrix}$$

which is block diagonal. Since we know that DF has full column rank when evaluated at any root, it follows that so does G.

(c) Since the second term in the definition of this function only involves prices, it follows that its partial Jacobean with respect to the other variables is the submatrix

$$\begin{pmatrix} D_{x^1, x^2, \lambda^1, \lambda^2, w^1, w^2, \hat{x}^1, \hat{x}^2, \hat{\lambda}^1, \hat{\lambda}^2, \hat{w}^1, \hat{w}^2} G \\ 0 \end{pmatrix}.$$

The submatrix

$$\begin{pmatrix} \partial_{p_2} G & \partial_{\hat{p}_2} G \\ 1 & -1 \end{pmatrix}$$

corresponds to the partial Jacobean with respect to p_2 and $\hat{p}_2$. Only the columns corresponding to derivatives with respect to other prices are needed to complete the Jacobean.

(d) The matrix obtained in (c) is block triangular and each of the block-diagonal submatrices has full column rank at the roots of the function, by part (b). It follows that the matrix has full row rank at the roots of the function, namely that the function is transverse to 0.

(e) Since the function of part (d) is transverse to 0, it follows from the transversality theorem that so does the function, generically on $(w^1, w^2, \hat{w}^1, \hat{w}^2)$, when we keep these variables fixed. Now, that function has

$$2[2(L+1) + (L-1)] + 1$$

components and only

$$2[2(L+1)+(L-1)]$$

arguments, so the only way in which it can be transverse to 0 is by *not* taking that value. Explicitly, this means that

$$\begin{array}{l} F(p, x^1, x^2, \lambda^1, \lambda^2, \omega^1, \omega^2) = 0 \\ F(\hat{p}, x^1, x^2, \hat{\lambda}^1, \hat{\lambda}^2, \hat{\omega}^1, \hat{\omega}^2) = 0 \end{array} \Rightarrow p_2 - \hat{p}_2 \neq 0,$$

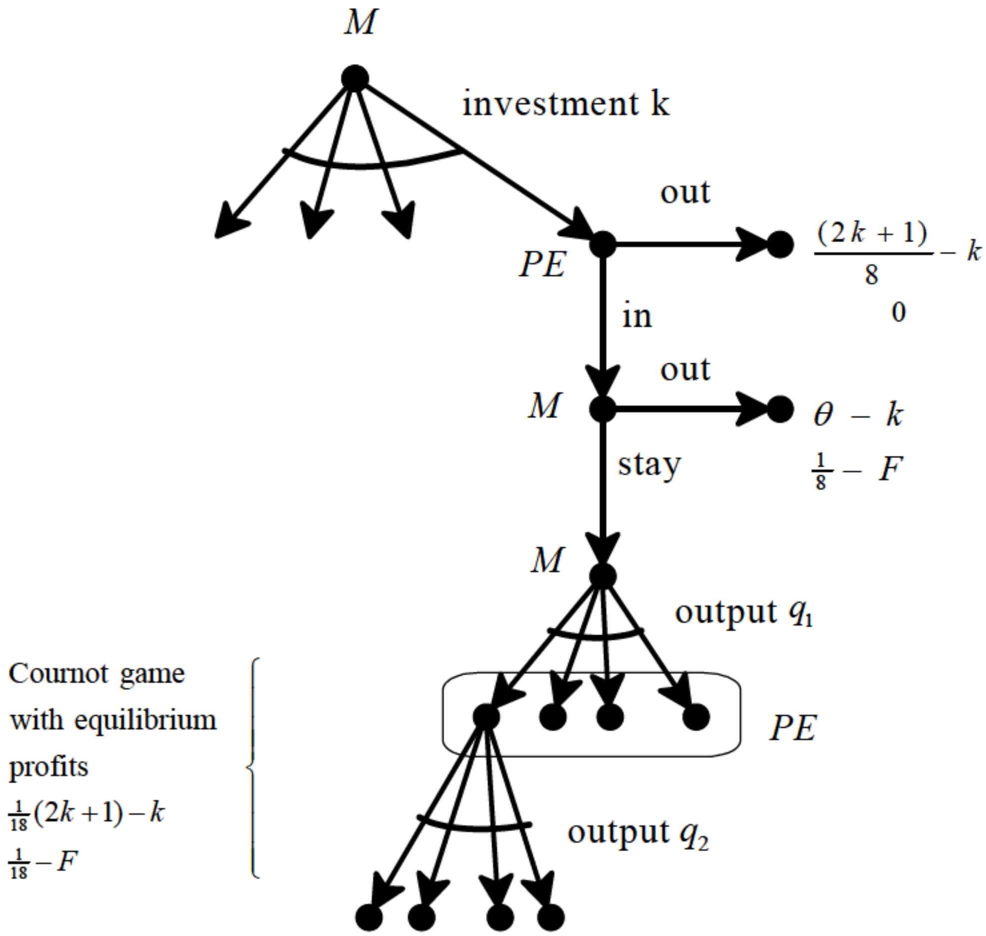
as needed.

(f) The intuition is the following. Generic determinacy of competitive equilibrium means that given a smooth economy and an equilibrium for it, we can always find an infinitesimal perturbation to its endowments such that the original prices no longer clear the markets. Using that insight, the current result is not surprising: given two profiles of endowments that share equilibrium prices, one can always find a perturbation to one of the profiles such that the original prices are not equilibrium after the perturbation. This means that it is very unlikely that two profiles of endowments share equilibrium price vectors.

Micro Prelim August 2019
QUESTION 4 ANSWER KEYS

(a) π_M is found by maximizing $\pi = Q(1-2Q)$. Thus $\pi_M = \frac{1}{8} = 0.125$. π_d is found by solving $\frac{\partial \pi_1}{\partial q_1} = 0$ and $\frac{\partial \pi_2}{\partial q_2} = 0$ where $\pi_i = q_i[1-2(q_1+q_2)]$. Thus $\pi_d = \frac{1}{18} = 0.056$.

(b) The extensive form is as follows:



(c) If *PE* enters and *M* stays, then *PE* makes negative profits $\pi_d - F < 0$. Thus *PE* will enter only if it anticipates *M* to exit. Now, *M* will stay if and only if $(2k+1)\pi_d - k \geq \theta - k$, that is (since $\pi_d = \frac{1}{18}$) $\frac{2k+1}{18} \geq \theta$ or $k \geq 9\theta - \frac{1}{2}$. If *M* chooses $k \geq 9\theta - \frac{1}{2}$ and *PE* stays out then *M*'s profit is $(2k+1)\pi_M - k = \frac{(2k+1)}{8} - k$, which is decreasing in *k* and thus maximized at the lowest admissible value of *k* subject to $k \geq 9\theta - \frac{1}{2}$, which is $\begin{cases} 9\theta - \frac{1}{2} & \text{if } \theta \geq \frac{1}{18} \\ 0 & \text{if } \theta < \frac{1}{18} \end{cases}$; at this value of *k* *M*'s profits are $\Pi(\theta) = \begin{cases} \frac{1}{2} - \frac{27}{4}\theta & \text{if } \theta \geq \frac{1}{18} \\ \frac{1}{8} & \text{if } \theta < \frac{1}{18} \end{cases}$. On the other hand, if $\theta > \frac{1}{18}$ and *M* chooses $k < 9\theta - \frac{1}{2}$ then *PE* will enter and then *M* will exit and obtain a payoff of $\theta - k$, which is maximized at $k = 0$. Thus the subgame-perfect equilibrium is as follows:

(1) If $\theta < \frac{1}{18}$ then *M* chooses k=0, and PE stays out (because *M* would stay if PE entered).

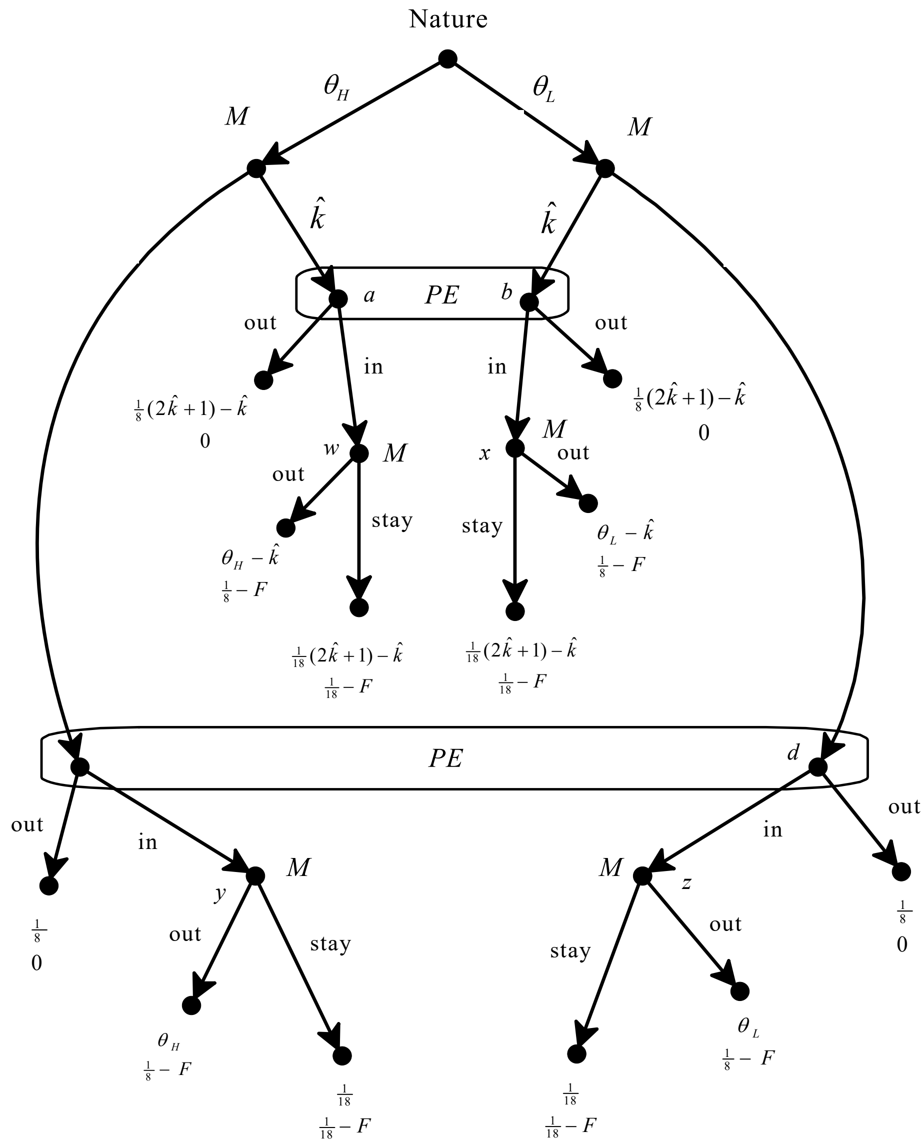
(2) If $\theta > \frac{1}{18}$ and $\frac{1}{2} - \frac{27}{4}\theta > \theta$, that is, if $\frac{1}{18} < \theta < \frac{2}{31}$ then *M* chooses $k = 9\theta - \frac{1}{2}$ and *PE* stays out (because *M* would stay if PE entered).

(3) If $\frac{1}{2} - \frac{27}{4}\theta < \theta$, that is, if $\theta > \frac{2}{31}$ then *M* chooses $k = 0$, *PE* enters and *M* exits.

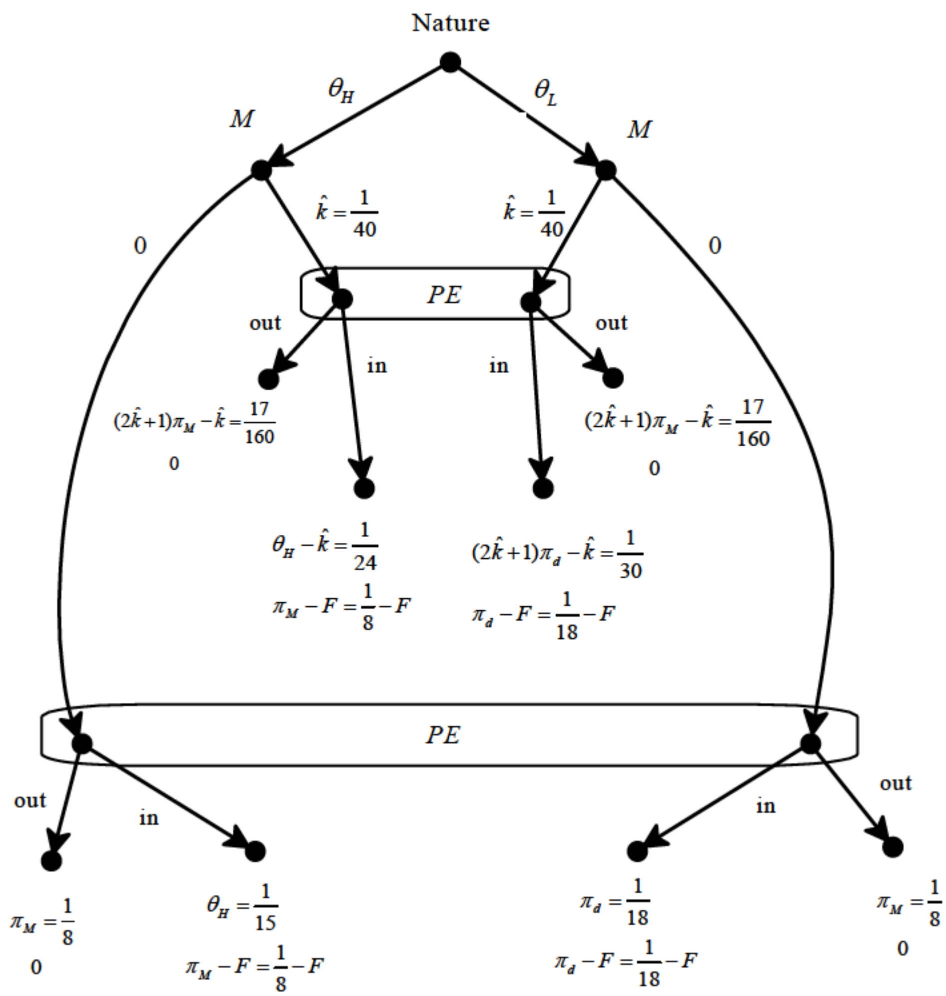
(4) If $\theta = \frac{1}{18}$ there are two subgame-perfect equilibria: (1) above (that is, *M* chooses k =0, and PE stays out) and the following: *M* chooses
k =0, PE enters and *M* exits.

(5) If $\theta = \frac{2}{31}$ there are two subgame-perfect equilibria: (2) above (that is, *M* chooses $k = 9\theta - \frac{1}{2} = \frac{5}{62}$, and PE stays out because *M* would stay) and the following: *M* chooses k= 0, *PE* enters and *M* exits.

(d) The extensive form is as follows:



(e) Since $\pi_d = \frac{1}{18}$, when $\theta_L = \frac{1}{20}$ and $\theta_H = \frac{1}{15}$, for *M* at node *y* "out" is better than "stay" and at node *z* "stay is better than "out". Furthermore, when $\hat{k} = \frac{1}{40}$, $(2\hat{k}+1)\pi_d = \frac{7}{120}$ thus for *M* at node *w* "out" is better than "stay" and at node *x* "stay" is better than "out". Hence the game can be simplified as follows:



For a pure-strategy separating equilibrium there are only two possibilities:

(1) the θ_H type chooses $\hat{k}$ and the θ_L type chooses 0,

(2) the θ_H type chooses 0 and the θ_L type chooses $\hat{k}$.

In case (1) *PE* must assign probability 1 to the left node of its top information set and thus chooses "in" at that information set; furthermore, *PE* must assign probability 1 to the right node of its bottom information set and thus chooses "out" at that information set. But then the θ_H type of *M* gets a payoff of $\frac{1}{24}$ by choosing $\hat{k}$ and a payoff of $\frac{1}{8} > \frac{1}{24}$ by choosing 0. Hence it is not a Nash equilibrium.

In case (2) *PE* must assign probability 1 to the right node of its top information set and thus chooses "out" at that information set; furthermore, *PE* must assign probability 1 to the left node of its bottom information set and thus chooses "in" at that information set. But then the θ_H type of *M* gets a payoff of $\frac{1}{15}$ by choosing 0 and a payoff of $\frac{17}{160} > \frac{1}{15}$ by choosing $\hat{k}$. Hence it is not a Nash equilibrium.

(f) In order for the θ_H type to optimally choose $\hat{k}$, it must be that *PE*'s choice at its bottom information set is "in" and this is an optimal choice for *PE* if and only if *PE* attaches sufficiently high probability to the left node of its bottom information set. Since Bayes' rule does not apply to this information set, such beliefs are allowed by the notion of weak sequential equilibrium. Since both types of *M* choose $\hat{k}$, Bayes' rule requires *PE* to attach probability *p* to the left node of its top information set (and $1-p$ to the right node). Thus "out" is optimal if and only if $0 \geq \frac{1}{8}p + \frac{1}{18}(1-p) - F$, that is, if and only if $p \leq \frac{72F-4}{5}$.