I. Financing a public good

The private financing of public goods is made difficult by the well-known free-rider problem. In this question, we will show that the public financing of a public good might be a good response to this problem.

We consider an economy with I agents. Agent i is endowed with the following utility function:

$$u_i(x_i, y) = x_i + \alpha_i \ln(y) \tag{1}$$

where x_i is the consumption by agent i of an aggregate private good we will refer to as the numeraire, y is the level of public good, and $\alpha_i > 0$ is an agent-specific (strictly) positive parameter reflecting the intensity of the agent's taste for the public good. We rank agents so that $\alpha_1 > \alpha_2 > \ldots > \alpha_I$, i.e., agent 1 has a stronger preference for the public good than agent 2 and so on. We assume that the public good is produced under constant returns to scale, one unit of private good giving one unit of public good. Each agent has the same endowment $\omega > 0$ of private good to start with.

The question is structured as follows: first, we will derive the socially optimal level of public good for this economy. Second, we will show that if each agent is asked to contribute voluntarily to the financing of the public good, only one agent contributes and the total amount of public good is lower than the socially optimal level. Finally, we will show that if agents are asked to vote on a tax to finance the public good, the level of public good may be improved.

1. We define the socially optimal level of public good as the value y^* that solves the following constrained optimization problem, sometimes referred to as the social planner problem:

$$\max_{\substack{x_i \geq 0 \, \forall i \\ y \geq 0}} \sum_{i=1}^{I} u_i(x_i, y) \quad \text{subject to} \quad y + \sum_{i=1}^{I} x_i \leq I\omega. \tag{2}$$

(Note: the symbol $\forall$ means "for all.")

[0.3] (a) Look at program (2) closely and interpret it in words.

Solution: Program 2 maximizes the sum of agents' utilities subject to the technical feasibility constraint that the private good left over from agents' own consumption is used to produce the public good.

[0.4] (b) Do the first-order conditions to program (2) fully characterize its solution? Briefly justify your answer.

Solution: Yes because the objective is concave and the constraint set is convex (the constraint being linear).

[0.6] (c) Ignoring non-negativity constraints, solve program (2) for the socially optimal level of public good y^*. Comment on your finding.

Solution: The first-order condition with respect to x_i gives $1 - \lambda = 0$ for all $i = 1, \dots, I$, where λ is the Lagrange multiplier on the technical feasibility constraint. The first-order condition with respect to y gives $\frac{\sum_i \alpha_i}{y} - \lambda = 0$, which yields $y^* = \sum_i \alpha_i$. The size of the public good rises with the number of individuals in the economy, or with their taste for the public good, which makes sense.

2. We now consider a mechanism by which each agent decides to contribute some of his endowment of private good ω towards the financing of the public good, taking as given the contributions of all the other agents. That is, we are considering a Nash equilibrium where the strategy of each agent is how much to contribute to the public good. We denote by z_i the contribution of agent i, and by Z_{-i} the sum of the contributions of all other agents. Agent i chooses z_i in order to solve the following optimization problem:

$$\max_{0 \leq z_i \leq \omega} u_i(\omega - z_i, z_i + Z_{-i}). \tag{3}$$

[0.3] (a) Look at program (3) closely and interpret it in words.

Solution: Agent i maximizes his own utility subject to the constraint that his private consumption is limited to his endowment minus his contribution to the public good. He correctly anticipates the amount of public good provided to be $z_i + Z_{-i}$.

[0.8] (b) Solve program (3), explicitly considering the non-negativity constraint $z_i \geq 0$. That is, find the best response of agent i, denoted $\bar{z}_i(Z_{-i})$. (Hint: Because you need to consider the non-negativity constraint, your best response should have two parts to it.) Represent the best-response function on a graph with Z_{-i} on the horizontal axis and z_i on the vertical axis and check that your function is continuous.

Solution: The first-order conditions give

$$\bar{z}_i(Z_{-i}) = \begin{cases} \alpha_i - Z_{-i} & \text{if } \alpha_i > Z_{-i} \\ 0 & \text{if } \alpha_i \leq Z_{-i} \end{cases} .$$

[0.6] (c) Show that in equilibrium, only agent 1 contributes to the public good while the other agents free ride. (Hint: First show, using your answer to part (b), that it is not possible that more than one agent contributes to the public good.)

Solution: The best response function implies that if an agent i contributes, then the resulting level of public good is $z_i + Z_{-i} = \alpha_i$. Since all the α_is are different, there can only be one agent contributing. Since the condition for agent j not to contribute is that other contributions exceed α_j, it must be that the contributing agent is the one with the highest taste for the public good, i.e., agent 1.

[0.4] (d) What is the level of public good in equilibrium? Compare it to the socially optimal level.

3. Now suppose that the government decides to implement a tax on agents' endowments to finance the public good, while letting agents vote on the tax rate to be implemented. If a tax τ is implemented, then each agent has to give $\tau\omega$ in taxes to the government, who then uses the tax proceeds $I\tau\omega$ to produce the public good. Before the vote is announced, each agent i determines their preferred tax rate $\hat{\tau}_i$ by solving the following maximization program:

$$\max_{0 \leq \hat{\tau}_i \leq 1} u_i(\omega(1-\hat{\tau}_i), I\hat{\tau}_i\omega).$$

[0.6] (a) Ignoring the inequality constraints on $\hat{\tau}_i$, derive the preferred tax rate of agent i.

> **Solution:** The first-order conditions, ignoring the inequality constraints, give $\hat{\tau}_i = \frac{\alpha_i}{\omega}$.

We now add the assumption that there is an odd number of agents in the economy. The government decides to proceed as follows: prior to the vote, each agent was asked to send through the mail their preferred tax rate $\hat{\tau}_i$. The government then conducts a series of pairwise votes between tax rates $(\hat{\tau}_i, \hat{\tau}_j)$ that were sent to him by the agents. That is, all preferred tax rates are compared to all others in pairwise votes. After all the pairwise votes have been conducted, the government selects a tax rate $\hat{\tau}_i$ if it beats all others by a simple majority, that is, if in all pairwise comparisons involving $\hat{\tau}_i$, more than half of the agents chose $\hat{\tau}_i$.

[0.6] (b) Argue that a tax rate does get selected by this (complex) procedure, and write down which one it is. (Hint: Use your intuition.)

> **Solution:** The tax rate that emerges is the one preferred by the median agent (the one with as many agents having lower α_i than agents having higher α_i than his). No tax rate can beat the one preferred by the median voter because if the competing tax rate is lower, all voters above the median voter (including the median voter) would for sure prefer the median tax rate. If the competing tax rate is higher than the median one, than for sure all voters below the median voter (including the median voter) would prefer the median tax rate. The median voter has an index equal to $\frac{I-1}{2}+1$, therefore the tax rate selected is $\hat{\tau}_{\frac{I-1}{2}+1} = \frac{\alpha_{\frac{I-1}{2}+1}}{\omega}$.

[0.4] (c) What is the level of public good financed through this procedure? Is it always higher than that obtained under the voluntary contribution equilibrium?

> **Solution:** The level of public good that emerges from the tax scheme is $I \times \omega \times \tau = \left(\alpha_{\frac{I-1}{2}+1}\right) I$. It is not necessarily higher than under the voluntary contributions equilibrium because it could be that $\alpha_1 > I \times \alpha_{\frac{I-1}{2}+1}$. This could happen if there is a lot of taste heterogeneity and agent 1 values the public good much more than other agents. But in general, we would expect that the tax scheme results in a higher level of public good because the taste parameter of the median voter is being multiplied by I, the number of agents (the tax is levied on all agents).