

Heterogeneous Treatment Effects with Mismeasured Endogenous Treatment*

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Abstract

This paper studies the identifying power of an instrumental variable in the nonparametric heterogeneous treatment effect framework when a binary treatment variable is mismeasured and endogenous. I characterize the sharp identified set for the local average treatment effect under the following two assumptions: (1) the exclusion restriction of an instrument and (2) deterministic monotonicity of the true treatment variable in the instrument. The identification strategy allows for general measurement error. Notably, (i) the measurement error is nonclassical, (ii) it can be endogenous, and (iii) no assumptions are imposed on the marginal distribution of the measurement error, so that I do not need to assume the accuracy of the measurement. Based on the partial identification result, I provide a consistent confidence interval for the local average treatment effect with uniformly valid size control. I also show that the identification strategy can incorporate repeated measurements to narrow the identified set, even if the repeated measurements themselves are endogenous. Using the NLS-72 dataset, I demonstrate that my new methodology can produce nontrivial bounds for the return to college attendance when attendance is mismeasured and endogenous.

Keywords: Misclassification; Local average treatment effect; Endogenous measurement error; Instrumental variable; Partial identification

JEL Classification Codes: C21, C26

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1 Introduction

Treatment effect analyses often entail a measurement error problem as well as an endogeneity problem. For example, Black, Sanders, and Taylor (2003) document a substantial measurement error in educational attainments in the 1990 U.S. Census. Educational attainments are treatment variables in a return to schooling analysis, and they are endogenous because unobserved individual ability affects both schooling decisions and wages (Card, 2001). The econometric literature, however, has offered only a few solutions for addressing the two problems at the same time. Although an instrumental variable is a standard technique for correcting both endogeneity and measurement error (e.g., Angrist and Krueger, 2001), no paper has investigated the identifying power of an instrumental variable for the heterogeneous treatment effect when the treatment variable is both mismeasured and endogenous.

I consider a measurement error in the treatment variable in the framework of Imbens and Angrist (1994) and Angrist, Imbens, and Rubin (1996), and focus on identification/inference problems for the local average treatment effect (LATE). The LATE is the average treatment effect for the subpopulation (the compliers) whose true treatment status is strictly affected by an instrument. Focusing on LATE is meaningful for a few reasons.¹ First, LATE has been a widely used parameter to investigate the heterogeneous treatment effect with endogeneity. My analysis on LATE of a mismeasured treatment variable offers a tool for a robustness check to those who have already investigated LATE. Second, LATE can be used to extrapolate to the average treatment effect or other parameters of interest. Imbens (2010) emphasize the utility of reporting LATE even if the parameter of interest is obtained based on LATE, because the extrapolation often requires additional assumptions and the result of the extrapolation can be less credible than LATE.

I take a worst case scenario approach with respect to the measurement error and allow for arbitrary measurement error. The only assumption concerning the measurement error is its independence of the instrumental variable. The following types of measurement error are considered in my analysis. First, the measurement error is nonclassical; that is, it can be dependent on the true treatment variable. The measurement error for a binary variable is always nonclassical. It is because the measurement error cannot be negative (positive) when the true variable takes the lowest (highest) value. Second, I allow the measurement error to be endogenous; that is, the measured treatment variable is allowed to be dependent on the outcome variable conditional on the true treatment variable. It is also called a differential measurement error. The validation study by Black et al. (2003) finds that the measurement error is likely to be correlated with individual observed and unobserved heterogeneity. The unobserved heterogeneity causes the endogeneity of the measurement error; it affects the measurement and the outcome at the same time. For example, the measurement error for educational attainment depends on the familiarity with the educational system in the U.S., and immigrants may have a higher rate of measurement error. At the same time, the familiarity with the U.S. educational system can be related to the English language skills, which can affect the labor market outcomes. Bound, Brown, and Mathiowetz (2001) also argue that measurement error is likely to be differential in some empirical applications. Third, there is no assumption concerning the marginal distribution of the measurement error. It is not necessary to assume anything about the accuracy of the measurement.

Even if I allow for an arbitrary measurement error, this paper demonstrates that an instrumental variable can still partially identify LATE when (a) the instrument satisfies the exclusion restriction such that the instrument affects the outcome and the measured treatment only through the true treatment, and (b) the instrument weakly increases the true treatment. These assumptions are standard in the LATE framework

¹Deaton (2009) and Heckman and Urzúa (2010) are cautious about interpreting LATE as a parameter of interest. See also Imbens (2010, 2014) for a discussion.

(Imbens and Angrist, 1994 and Angrist et al., 1996). I show that the point identification for LATE is impossible unless LATE is zero, and I characterize the sharp identified set for LATE. Based on the sharp identified set, (i) the sign of LATE is identified, (ii) there are finite upper and lower bounds on LATE even for the unbounded outcome variable, and (iii) the Wald estimand is an upper bound on LATE in absolute value but sharp upper bound is in general smaller than the Wald estimand. I obtain an upper bound on LATE in absolute value by deriving a new implication of the exclusion restriction.

Inference for LATE in my framework does not fall directly into the existing moment inequality models particularly when the outcome variable is continuous. First, the upper bound for LATE in absolute value is not differentiable with respect to the data distribution. This non-differentiability problem precludes any estimator for the upper bound from having a uniformly valid asymptotic distribution, as is formulated in Hirano and Porter (2012) and Fang and Santos (2014). Second, the upper bound cannot be characterized as the infimum over differentiable functionals indexed by a compact subset in a finite dimensional space, unless the outcome variable has a finite support.² This prohibits from applying the existing methodologies in conditional moment inequalities, e.g., Andrews and Shi (2013), Kim (2009), Ponomareva (2010), Armstrong (2014, 2015), Armstrong and Chan (2014), Chetverikov (2013), and Chernozhukov, Lee, and Rosen (2013).

I construct a confidence interval for LATE which can be applied to both discrete and continuous outcome variables. To circumvent the aforementioned problems, I approximate the sharp identified set by discretizing the support of the outcome variable where the discretization becomes finer as the sample size increases. The approximation for the sharp identified set resembles many moment inequalities in Menzel (2014) and Chernozhukov, Chetverikov, and Kato (2014), who consider a finite but divergent number of moment inequalities. I adapt a bootstrap method in Chernozhukov et al. (2014) into my framework to construct a confidence interval with uniformly valid asymptotic size control. Moreover, I demonstrate that the confidence interval is consistent against the local alternatives in which a parameter value approaches to the sharp identified set at a certain rate.

As empirical illustrations, I apply the new methodology for evaluating the effect on wages of attending a college when the college attendance can be mismeasured. I use the National Longitudinal Survey of the High School Class of 1972 (NLS-72), as in Kane and Rouse (1995). Using the proximity to college as an instrumental variable (Card, 1995), the confidence interval developed in the present paper offers nontrivial bounds on LATE, even if I allow for measurement error in college attendance. Moreover, the empirical results confirm the theoretical result that the Wald estimator is an upper bound on LATE but is not the sharp upper bound.

As an extension, I demonstrate that my identification strategy offers a new use of repeated measurements as additional sources for identification. The existing practice of the repeated measurements exploits them as instrumental variables, as in Hausman, Newey, Ichimura, and Powell (1991) and Hausman, Newey, and Powell (1995). However, when the true treatment variable is endogenous, the repeated measurements are likely to be endogenous and are not good candidates for an instrumental variable. My identification strategy shows that those variables are useful for bounding LATE in the presence of measurement error, even if the repeated measurement are not valid instrumental variables. I give a necessary and sufficient condition under which the repeated measurement strictly narrows the identified set.

The remainder of the present paper is organized as follows. Subsection 1.1 explains several examples

²When the outcome variable has a finite support, the identified set is characterized by a finite number of moment inequalities. Therefore I can apply the methodologies in unconditional moment inequalities, e.g., Imbens and Manski (2004), Chernozhukov, Hong, and Tamer (2007), Romano and Shaikh (2008, 2010), Rosen (2008), Andrews and Guggenberger (2009), Stoye (2009), Andrews and Soares (2010), Bugni (2010), Canay (2010), and Andrews and Barwick (2012).

motivating mismeasured endogenous treatment variables and Subsection 1.2 reviews the related econometric literature. Section 2 introduces the LATE framework with mismeasured treatment variables. Section 3 constructs the identified set for LATE. Section 4 proposes an inference procedure for LATE. Section 5 conducts the Monte Carlo simulations. Section 6 implements the inference procedure in NLS-72 to estimate the return to schooling. Section 7 discusses how repeated measurements narrow the identified set, even if the repeated measurements themselves are not instrumental variables. Section 8 concludes. The Appendix collects proofs and remarks.

1.1 Examples for mismeasured endogenous treatment variables

I introduce several examples in which binary treatment variables can be both endogenous and mismeasured at the same time. The first example is the return to schooling, in which the outcome variable is wages and the treatment variable is educational attainment, for example, whether a person has completed college or not. It is well-known that unobserved individual ability affects both the schooling decision and wage determination, which leads to the endogeneity of educational attainment in the wage equation (see, for example, Card (2001)). Moreover, survey datasets record educational attainments based on the interviewee's answers and these self-reported educational attainments are subject to measurement error. Empirical papers by Griliches (1977), Angrist and Krueger (1999), Kane, Rouse, and Staiger (1999), Card, 2001, Black et al. (2003) have pointed out the mismeasurement. For example, Black et al. (2003) estimate that the 1990 Decennial Census has 17.7% false positive rate of reporting a doctoral degree.

The second example is labor supply response to welfare program participation, in which the outcome variable is employment status and the treatment variable is welfare program participation. Self-reported welfare program participation in survey datasets can be mismeasured (Hernandez and Pudney, 2007). The psychological cost for welfare program participation, welfare stigma, affects job search behavior and welfare program participation simultaneously; that is, welfare stigma may discourage individuals from participating in a welfare program, and, at the same time, affect an individual's effort in the labor market (see Moffitt (1983) and Besley and Coate (1992) for a discussion on the welfare stigma). Moreover, the welfare stigma gives welfare recipients some incentive not to reveal their participation status to the survey, which causes differential measurement error in that the unobserved individual heterogeneity affects both the measurement error and the outcome.

The third example is the effect of a job training program on wages (for example, Royalty, 1996). As it is similar to the return to schooling, the unobserved individual ability plays a key role in this example. Self-reported completion of job training program is also subject to measurement error (Barron, Berger, and Black, 1997). Frazis and Loewenstein (2003) develop a methodology for evaluating a homogeneous treatment effect with mismeasured endogenous treatment variable, and apply their methodology to evaluate the effect of a job training program on wages.

The last example is the effect of maternal drug use on infant birth weight. Kaestner, Joyce, and Wehbeh (1996) estimate that a mother tends to underreport her drug use, but, at the same time, she tends to report it correctly if she is a heavy user. When the degree of drug addiction is not observed, it becomes an individual unobserved heterogeneity variable which affects infant birth weight and the measurement in addition to the drug use.

1.2 Literature review

This paper is related to a few strands of the econometric literature. First, Mahajan (2006), Lewbel (2007) and Hu (2008) use an instrumental variable to correct for measurement error in a binary treatment in the heterogeneous treatment effect framework and they achieve nonparametric point identification of the average treatment effect. This result assumes the true treatment variable is exogenous, whereas I allow it to be endogenous.

Finite mixture models are related to this paper. I consider the unobserved binary treatment, whereas finite mixture models deal with unobserved type variable. Henry, Kitamura, and Salanié (2014) and Henry, Jochmans, and Salanié (2015) are the most closely related to this paper. They investigate the identification problem in finite mixture models, by using the exclusion restriction in which an instrumental variable only affects the mixing distribution of a type variable without affecting the component distribution (that is, the conditional distribution given the type variable). If I applied their approach directly to my framework, their exclusion restriction would imply conditional independence between the instrumental variable and the outcome variable given the true treatment variable. In the LATE framework, this conditional independence implies that LATE does not exhibit essential heterogeneity (Heckman, Schmierer, and Urzua, 2010) and that LATE is equal to the mean difference between the control and treatment groups.³ Instead of applying the approaches in Henry et al. (2014) and Henry et al. (2015), this paper uses a different exclusion restriction in which the instrumental variable does not affect the outcome or the measured treatment directly.

A few papers have applied an instrumental variable to a mismeasured binary regressor in the homogenous treatment effect framework. They include Aigner (1973), Kane et al. (1999), Bollinger (1996), Black, Berger, and Scott (2000) and Frazis and Loewenstein (2003). Frazis and Loewenstein (2003) is the most closely related to the present paper among them, since they consider an endogenous mismeasured regressor. In contrast, I allow for heterogeneous treatment effects. Therefore, I contribute to the heterogeneous treatment effect literature by investigating the consequences of the measurement errors in the treatment variable.

Kreider and Pepper (2007), Molinari (2008), Imai and Yamamoto (2010), and Kreider, Pepper, Gundersen, and Jolliffe (2012) apply a partial identification strategy for the average treatment effect to the mismeasured binary regressor problem by utilizing the knowledge of the marginal distribution for the true treatment. Those papers use auxiliary datasets to obtain the marginal distribution for the true treatment. Kreider et al. (2012) is the most closely related to the present paper, in that they allow for both treatment endogeneity and differential measurement error. My instrumental variable approach can be an alternative strategy to deal with mismeasured endogenous treatment. It is worthwhile because, as mentioned in Schennach (2013), the availability of an auxiliary dataset is limited in empirical research. Furthermore, it is not always the case that the results from auxiliary datasets is transported into the primary dataset (Carroll, Ruppert, Stefanski, and Crainiceanu, 2012, p.10),

³This footnote uses the notation introduced in Section 2. The conditional independence implies $E[Y | T^*, Z] = E[Y | T^*]$. Under this assumption,

$$\begin{aligned} E[Y | Z] &= P(T^* = 1 | Z)E[Y | Z, T^* = 1] + P(T^* = 0 | Z)E[Y | Z, T^* = 0] \\ &= P(T^* = 1 | Z)E[Y | T^* = 1] + P(T^* = 0 | Z)E[Y | T^* = 0] \end{aligned}$$

and therefore $\Delta E[Y | Z] = \Delta E[T^* | Z](E[Y | T^* = 1] - E[Y | T^* = 0])$. I obtain the equality

$$\frac{\Delta E[Y | Z]}{\Delta E[T^* | Z]} = E[Y | T^* = 1] - E[Y | T^* = 0]$$

This above equation implies that the LATE does not depend on the compliers of consideration, which is in contrast with the essential heterogeneity of the treatment effect (Heckman et al., 2010). Furthermore, since $E[Y | T^* = 1] - E[Y | T^* = 0]$ is equal to the LATE, I do not need to care about the endogeneity problem here.

Some papers investigate mismeasured endogenous continuous variables, instead of binary variables. Amemiya (1985); Hsiao (1989); Lewbel (1998); Song, Schennach, and White (2015) consider nonlinear models with mismeasured continuous explanatory variables. The continuity of the treatment variable is crucial for their analysis, because they assume classical measurement error. The treatment variable in the present paper is binary and therefore the measurement error is nonclassical. Hu, Shiu, and Woutersen (2015) consider mismeasured endogenous continuous variables in single index models. However, their approach depends on taking derivatives of the conditional expectations with respect to the continuous variable. It is not clear if it can be extended to binary variables. Song (2015) considers the semi-parametric model when endogenous continuous variables are subject to nonclassical measurement error. He assumes conditional independence between the instrumental variable and the outcome variable given the true treatment variable, which imposes some structure on the outcome equation (e.g., LATE does not exhibit essential heterogeneity). Instead this paper proposes an identification strategy without assuming any structure on the outcome equation.

Chalakov (2013) investigates the consequences of measurement error in the instrumental variable instead of the treatment variable. He assumes that the treatment variable is perfectly observed, whereas I allow for it to be measured with error. Since I assume that the instrumental variable is perfectly observed, my analysis is not overlapped with Chalakov (2013).

Manski (2003), Blundell, Gosling, Ichimura, and Meghir (2007), and Kitagawa (2010) have similar identification strategy to the present paper in the context of sample selection models. These papers also use the exclusion restriction of the instrumental variable for their partial identification results. Particularly, Kitagawa (2010) derives the “integrated envelope” from the exclusion restriction, which is similar to the total variation distance in the present paper because both of them are characterized as a supremum over the set of the partitions. First and the most importantly, the present paper considers mismeasurement of the treatment variable, whereas the sample selection model considers truncation of the outcome variable. It is not straightforward to apply their methodologies in sample selection models into mismeasured treatment problem. Second, the present paper offers an inference method with uniform size control, but Kitagawa (2010) derives only point-wise size control. Last, Blundell et al. (2007) and Kitagawa (2010) use their result for specification test, but I cannot use it for specification test because the sharp identified set of the present paper is always non-empty.

2 LATE model with misclassification

This section introduces measurement error in the treatment variable into the LATE framework (Imbens and Angrist, 1994, and Angrist et al., 1996). The objective is to evaluate the causal effect of a binary treatment variable $T^* \in \{0, 1\}$ on an outcome variable Y , where $T^* = 0$ represents the control group and $T^* = 1$ represents the treatment group. To control for endogeneity of T^* , the LATE framework requires a binary instrumental variable $Z \in \{z_0, z_1\}$ which shifts T^* exogenously without any direct effect on Y . The treatment variable T^* of interest is not directly observed, and instead there is a binary measurement $T \in \{0, 1\}$ for T^* . I put the $*$ symbol on T^* to emphasize that the true treatment variable T^* is unobserved. Y can be discrete, continuous or mixed; Y is only required to have some dominating finite measure μ_Y on the real line. μ_Y can be the Lebesgue measure or the counting measure.

To describe the data generating process, I consider the counterfactual variables. Let T_z^* denote the counterfactual true treatment variable when $Z = z$. Let Y_{t^*} denote the counterfactual outcome when $T^* = t^*$. Let T_{t^*} denote the potential measured treatment variable when $T^* = t^*$. The individual treatment

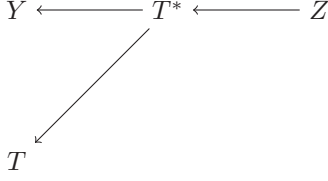


Figure 1: Three equations in the model

effect is $Y_1 - Y_0$. It is not directly observed; Y_0 and Y_1 are not observed at the same time. Only Y_{T^*} is observable. Using the notation, the observed variables (Y, T, Z) are generated by the following three equations:

$$T = T_{T^*} \tag{1}$$

$$Y = Y_{T^*} \tag{2}$$

$$T^* = T_Z^*. \tag{3}$$

Figure 2 describes the above three equations graphically. (1) is the measurement equation, which is the arrow from Z to T^* in Figure 2. $T - T^*$ is the measurement error; $T - T^* = 1$ (or $T_0 = 1$) represents a false positive and $T - T^* = -1$ (or $T_1 = 0$) represents a false negative. The next two equations (2) and (3) are standard in the LATE framework. (2) is the outcome equation, which is the arrow from T^* to Y in Figure 2. (3) is the treatment assignment equation, which is the arrow from T^* to T in Figure 2. Correlation between (Y_0, Y_1) and $(T_{z_0}^*, T_{z_1}^*)$ causes an endogeneity problem.

In a return to schooling analysis, Y is wages, T^* is the true indicator for college completion, Z is the proximity to college, and T is the measurement of T^* . The treatment effect $Y_1 - Y_0$ in the return to schooling is the effect of college attendance T^* on wages Y . The college attendance is not correctly measured in a dataset, such that only the proxy T is observed.

The only assumption for my identification analysis is as follows.

Assumption 1. (i) Z is independent of $(T_{t^*}, Y_{t^*}, T_{z_0}^*, T_{z_1}^*)$ for each $t^* = 0, 1$. (ii) $T_{z_1}^* \geq T_{z_0}^*$ almost surely.

Part (i) is the exclusion restriction and I consider stochastic independence instead of mean independence. Although it is stronger than the minimal conditions for the identification for LATE without measurement error, a large part of the existing applied papers assume stochastic independence (Huber and Mellace, 2015, p.405). Z is also independent of T_{t^*} conditional on $(Y_{t^*}, T_{z_0}^*, T_{z_1}^*)$, which is the only assumption on the measurement error for the identified set in Section 3.

Part (ii) is the monotonicity condition for the instrument, in which the instrument Z increases the value of T^* for all the individuals. de Chaisemartin (2015) relaxes the monotonicity condition, and the following analysis of the present paper only requires the complier-defiers-for-marginals condition in de Chaisemartin (2015) instead of the monotonicity condition. Moreover, Part (ii) implies that the sign of the first stage regression, which is the effect of the instrumental variable on the true treatment variable, is known. It is a reasonable assumption because most empirical applications of the LATE framework assume the sign is known. For example, Card (1995) claims that the proximity-to-college instrument weakly increases the likelihood of going to a college. Last, I do not assume a relevance condition for the instrumental variable, such as $T_{z_1}^* \neq T_{z_0}^*$. The relevance condition is a testable assumption when $T^* = T$, but it is not testable in my analysis. I will discuss the relevance condition in my framework after Theorem 1.

As I emphasized in the introduction, the framework here does not assume anything on measurement error except for the independence from Z . I do not impose any restriction on the marginal distribution of the measurement error or on the relationship between the measurement error and $(Y_{t^*}, T_{z_0}^*, T_{z_1}^*)$. Particularly, the measurement error can be differential, that is, T_{t^*} can depend on Y_{t^*} .

In this paper, I focus on the local average treatment effect (LATE), which is defined by

$$\theta \equiv E[Y_1 - Y_0 \mid T_{z_0}^* < T_{z_1}^*].$$

LATE is the average of the treatment effect $Y_1 - Y_0$ over the subpopulation (the compliers) whose treatment status depend on the instrument. Imbens and Angrist (1994, Theorem 1) show that LATE equals

$$\frac{\Delta E[Y \mid Z]}{\Delta E[T^* \mid Z]},$$

where I define $\Delta E[X \mid Z] = E[X \mid Z = z_1] - E[X \mid Z = z_0]$ for a random variable X . The present paper introduces measurement error in the treatment variable, and therefore the fraction $\Delta E[Y \mid Z]/\Delta E[T^* \mid Z]$ is not equal to the Wald estimand

$$\frac{\Delta E[Y \mid Z]}{\Delta E[T \mid Z]}.$$

Since $\Delta E[T^* \mid Z]$ is not point identified, I cannot point identify LATE. The failure for the point identification comes purely from the measurement error, because LATE would be point identified under $T = T^*$.

3 Sharp identified set for LATE

This section considers the partial identification problem for LATE. Before defining the sharp identified set, I express LATE as a function of the underlying distribution P^* of $(Y_0, Y_1, T_0, T_1, T_{z_0}^*, T_{z_1}^*, Z)$. I use the $*$ symbol on P^* to clarify that P^* is the distribution of the unobserved variables. In the following arguments, I denote the expectation operator E by E_{P^*} when I need to clarify the underlying distribution. The local average treatment effect is a function of the unobserved distribution P^* :

$$\theta(P^*) \equiv E_{P^*}[Y_1 - Y_0 \mid T_{z_0}^* < T_{z_1}^*].$$

The sharp identified set is the set of parameter values for LATE which is consistent with the distribution of the observed variables. I use P for the distribution of the observed variables (Y, T, Z) . The equations (1), (2), and (3) induce the distribution of the observables (Y, T, Z) from the unobserved distribution P^* , and I denote by $P_{(Y,T,Z)}^*$ the induced distribution. When the distribution of (Y, T, Z) is P , the set of P^* which induces $P_{(Y,T,Z)}^* = P$ is

$$\{P^* \in \mathcal{P}^* : P = P_{(Y,T,Z)}^*\},$$

where \mathcal{P}^* is the set of P^* 's satisfying Assumptions 1. For every distribution P of (Y, T, Z) , the sharp identified set for LATE is defined as

$$\Theta_I(P) \equiv \{\theta(P^*) : P^* \in \mathcal{P}^* \text{ and } P = P_{(Y,T,Z)}^*\}.$$

The proof of Theorem 1 in Imbens and Angrist (1994) provides a relationship between $\Delta E[Y \mid Z]$ and

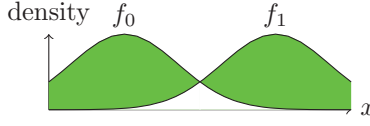


Figure 2: Definition of the total variation distance

LATE:

$$\theta(P^*)P^*(T_{z_0}^* < T_{z_1}^*) = \Delta E_{P^*}[Y | Z], \quad (4)$$

This equation gives the two pieces of information of $\theta(P^*)$. First, the sign of $\theta(P^*)$ is the same as $\Delta E_{P^*}[Y | Z]$. Second, the absolute value of $\theta(P^*)$ is at least the absolute value of $\Delta E_{P^*}[Y | Z]$. The following lemma summarizes these two pieces.

Lemma 1.

$$\theta(P^*)\Delta E_{P^*}[Y | Z] \geq 0$$

$$|\theta(P^*)| \geq |\Delta E_{P^*}[Y | Z]|.$$

I derive a new implication from the exclusion restriction for the instrumental variable in order to obtain an upper bound on $\theta(P^*)$ in absolute value. To explain the new implication, I introduce the total variation distance. The total variation distance

$$TV(f_1, f_0) = \frac{1}{2} \int |f_1(x) - f_0(x)| d\mu_X(x)$$

is the distance between the distribution f_1 and f_0 . In Figure 3, the total variation distance is the half of the area for the shaded region. I use the total variation distance to evaluate the distributional effect of a binary variable, particularly the distributional effect of Z on (Y, T) . The distributional effect of Z on (Y, T) reflects the dependency of $f_{(Y,T)|Z=z}(y, t)$ on z , and I interpret the total variation distance $TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0})$ as the magnitude of the distributional effect. Even when the variable X is discrete, I use the density f for X to represent the probability function for X .

The new implication is based on the exclusion restriction imposes that the instrumental variable has direct effect on the true treatment variable T^* and has indirect effect on the outcome variable Y and on the measured treatment variable T . The new implication formalizes the idea that the magnitude of the direct effect of Z on T^* is no smaller than the magnitude of the indirect effect of Z on (Y, T) .

Lemma 2. *Under Assumption 1, then*

$$TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) = TV(f_{(Y_1, T_1)|T_{z_0}^* < T_{z_1}^*}, f_{(Y_0, T_0)|T_{z_0}^* < T_{z_1}^*})TV(f_{T^*|Z=z_1}, f_{T^*|Z=z_0})$$

and therefore

$$TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) \leq TV(f_{T^*|Z=z_1}, f_{T^*|Z=z_0}) = P^*(T_{z_0}^* < T_{z_1}^*).$$

The new implication in Lemma 2 gives a lower bound on $P^*(T_{z_0}^* < T_{z_1}^*)$ and therefore yields an upper bound on LATE in absolute value, combined with Eq. (4). Therefore, I use these relationships to derive an

upper bound on LATE in absolute value, that is,

$$|\theta(P^*)| = \frac{|\Delta E_{P^*}[Y | Z]|}{P^*(T_{z_0}^* < T_{z_1}^*)} \leq \frac{|\Delta E_{P^*}[Y | Z]|}{TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0})}$$

as long as $TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) > 0$.

The next theorem shows that the above observations characterize the sharp identified set for LATE.

Theorem 1. *Suppose that Assumption 1 holds, and consider an arbitrary data distribution P of (Y, T, Z) .*

(i) *The sharp identified set $\Theta_I(P)$ for LATE is included in $\Theta_0(P)$, where $\Theta_0(P)$ is the set of θ 's which satisfies the following three inequalities.*

$$\begin{aligned} \theta \Delta E_P[Y | Z] &\geq 0 \\ |\theta| &\geq |\Delta E_P[Y | Z]| \\ |\theta| TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) &\leq |\Delta E_P[Y | Z]|. \end{aligned}$$

(ii) *If Y is unbounded, then $\Theta_I(P)$ is equal to $\Theta_0(P)$.*

Corollary 1. *Consider an arbitrary data distribution P of (Y, T, Z) . If $TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) > 0$,*

$$\Theta_0(P) = \begin{cases} \left[\Delta E_P[Y | Z], \frac{\Delta E_P[Y | Z]}{TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0})} \right] & \text{if } \Delta E_P[Y | Z] > 0 \\ \{0\} & \text{if } \Delta E_P[Y | Z] = 0 \\ \left[\frac{\Delta E_P[Y | Z]}{TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0})}, \Delta E_P[Y | Z] \right] & \text{if } \Delta E_P[Y | Z] < 0. \end{cases}$$

If $TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) = 0$, then $\Theta_0(P) = \mathbb{R}$.

The total variation distance $TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0})$ measures the strength for the instrumental variable in my analysis, that is, $TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) > 0$ is the relevance condition in my identification analysis. $TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) = 0$ means that the instrumental variable Z does not affect Y and T , in which case Z has no identifying power for the local average treatment effect. When $TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) > 0$, the interval in the above theorem is always nonempty and bounded, which implies that Z has some identifying power for the local average treatment effect.

The Wald estimand $\Delta E_P[Y | Z] / \Delta E_P[T | Z]$ can be outside the identified set. The inequality

$$\left| \frac{\Delta E_P[Y | Z]}{TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0})} \right| \leq \left| \frac{\Delta E_P[Y | Z]}{\Delta E_P[T | Z]} \right|$$

holds and a strict inequality holds unless the sign of $f_{(Y,T)|Z=z_1}(y, t) - f_{(Y,T)|Z=z_0}(y, t)$ is constant in y for every t . It might seem counter-intuitive that the Wald estimand equals to LATE without measurement error but that it is not necessarily in the identified set $\Theta_0(P)$ when measurement error is allowed. Recall that my framework includes no measurement error as a special case. As in Balke and Pearl (1997) and Heckman and Vytlacil (2005), the the LATE framework has the testable implications:

$$f_{(Y,T)|Z=z_1}(y, 1) \geq f_{(Y,T)|Z=z_0}(y, 1) \text{ and } f_{(Y,T)|Z=z_1}(y, 0) \leq f_{(Y,T)|Z=z_0}(y, 0).$$

When the data distribution does not satisfy the testable implications and there is no measurement error on

the treatment variable, the identified set for LATE becomes empty and, therefore, the Wald estimand is no longer equal to LATE anymore. My framework has no testable implications, because the identified set is always non-empty. The recent papers by Huber and Mellace (2015), Kitagawa (2014) and Mourifié and Wan (2014) propose the testing procedures for the testable implications.

4 Inference

Having derived what can be identified about LATE under Assumption 1, this section considers statistical inference about LATE. I construct a confidence interval for LATE based on $\Theta_0(P)$ in Theorem 1. There are two difficulties with directly using Theorem 1 for statistical inference. First, as is often the case in the partially identified models, the length of the interval is unknown ex ante, which causes uncertainty of how many moment inequalities are binding for a given value of the parameter. Second, the identified set depends on the total variation distance which involves absolute values of the data distribution. I cannot apply the delta method to derive the asymptotic distribution for the total variation distance, because of the failure of differentiability (Hirano and Porter, 2012, Fang and Santos, 2014). This non-differentiability problem remains even if the support of Y is finite.

I will take three steps to construct a confidence interval. In the first step, I use the supremum representation of the total variation distance to characterize the identified set $\Theta_0(P)$ by the moment inequalities with differentiable moment functions. When the outcome variable Y has a finite support, I can apply methodologies developed for a finite number of moment inequalities (e.g., Andrews and Soares, 2010) to construct a confidence interval for LATE. When the outcome variable Y has an infinite support, however, none of the existing methods can be directly applied because the moment inequalities are not continuously indexed by a compact subset of the finite dimensional space. In the second step, therefore, I discretize the support of Y to make the number of the moment inequalities to be finite in the finite sample. I let the discretization finer as the sample size goes to the infinity, such that eventually the approximation error from the discretization vanishes. The number of the moment inequalities become finite but growing, particularly diverging to the infinity when Y has an infinite support. This structure resembles many moment inequalities in Chernozhukov et al. (2014). The third step is to adapt a bootstrapped critical value construction in Chernozhukov et al. (2014) to my framework.

4.1 Supremum representation of the total variation distance

In order to avoid the non-differentiability problem, I characterize the identified set by the moment inequalities which are differentiable with respect to the data distribution.

Lemma 3. *Let P be an arbitrary data distribution of (Y, T, Z) .*

1. *Let \mathbf{Y} be the support for the random variable Y and $\mathbf{T} \equiv \{0, 1\}$ be the support for T . Denote by \mathbf{H} the set of measurable functions on $\mathbf{Y} \times \mathbf{T}$ taking a value in $\{-1, 1\}$. Then*

$$TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) = \sup_{h \in \mathbf{H}} \Delta E_P[h(Y, T) | Z]/2$$

2. The identified set $\Theta_0(P)$ for LATE is the set of θ 's which satisfy the following conditions

$$\begin{aligned} & -(1\{\theta \geq 0\} - 1\{\theta < 0\})\Delta E_P[Y | Z] \leq 0 \\ & (1\{\theta \geq 0\} - 1\{\theta < 0\})\Delta E_P[Y | Z] - |\theta| \leq 0 \\ & |\theta|\Delta E_P[h(Y, T) | Z]/2 - (1\{\theta \geq 0\} - 1\{\theta < 0\})\Delta E_P[Y | Z] \leq 0 \text{ for every } h \in \mathbf{H}. \end{aligned}$$

The number of elements \mathbf{H} can be large; \mathbf{H} is an infinite set when Y is continuous, and \mathbf{H} has the same elements of the power set of $\mathbf{Y} \times \mathbf{T}$ when Y takes only finite values.

4.2 Discretizing the outcome variable

To make the inference problem statistically and computationally feasible, I discretize the support for Y and make the number of the moment inequalities finite. Consider a partition $\mathbf{I}_n = \{I_{n,1}, \dots, I_{n,K_n}\}$ over \mathbf{Y} , in which $I_{n,k}$ depends on n , and K_n can grow with sample size. Let $h_{n,j}$ be a generic function of $\mathbf{Y} \times \mathbf{T}$ into $\{-1, 1\}$ that is constant over $I_{n,k} \times \{t\}$ for every $k = 1, \dots, K_n$ and every $t = 0, 1$. Let $\{h_{n,1}, \dots, h_{n,4K_n}\}$ be the set of all such functions. Note that $\{h_{n,1}, \dots, h_{n,4K_n}\}$ is a subset of \mathbf{H} . Using these $h_{n,j}$'s, I consider the following set $\Theta_n(P)$ characterized by the moment inequalities

$$\Theta_n(P) = \{\theta \in \Theta : \Delta E_P[g_{z,j}((Y, T), \theta) | Z] \leq 0 \text{ for every } 1 \leq j \leq p_n\}$$

where $p_n = 4^{K_n} + 2$ is the number of the moment inequalities, and

$$\begin{aligned} g_{z,j}((y, t), \theta) &= |\theta|h_{n,j}(y, t)/2 - (1\{\theta \geq 0\} - 1\{\theta < 0\})y \text{ for every } j = 1, \dots, 4^{K_n} \\ g_{z,4^{K_n}+1}((y, t), \theta) &= (1\{\theta \geq 0\} - 1\{\theta < 0\})y - 1\{z = z_1\}|\theta| \\ g_{z,4^{K_n}+2}((y, t), \theta) &= -(1\{\theta \geq 0\} - 1\{\theta < 0\})y \end{aligned}$$

for every $z = z_0, z_1$. That the set $\Theta_n(P)$ is an outer identified set. That is, it is a superset of the identified set $\Theta_0(P)$. The next subsections consider consistency with respect to the identified set $\Theta_0(P)$ by letting $\Theta_n(P)$ converge to $\Theta_0(P)$. This point is different than the usual use of the outer identified set and is similar to sieve estimation. To clarify the convergence of $\Theta_n(P)$ to $\Theta_0(P)$, I call $\Theta_n(P)$ the *approximated identified set*.

4.3 Confidence interval for LATE

The approximated identified set $\Theta_n(P)$ consists of a finite number of moments inequalities, but the number of moment inequalities depends on the sample size. As Section 4.4 requires, the number of moment inequalities $p_n = 4^{K_n} + 2$ needs to diverge in order to obtain the consistency for the confidence interval when Y is continuous. The approximated identified set is defined to converge to the sharp identified set, so that the confidence interval in the present paper exhausts all the information in the large sample. As in Subsection 4.4, the confidence interval has asymptotic power 1 against all the fixed alternatives outside the sharp identified set.

This divergent number of the moment inequalities in the approximated identified set resembles the identified set in Chernozhukov et al. (2014), who considers testing a growing number of moment inequalities in which each moment inequalities are based on different random variables. I modify their methodology into

the two sample framework where one sample is the group with $Z = z_1$ and the other sample is the group with $Z = z_0$. For simplicity, I assume that Z is deterministic, which makes the notation in the following analysis simpler. The assumption of deterministic Z yields two independent samples: $(Y_{z_0,1}, T_{z_0,1}), \dots, (Y_{z_0,n_0}, T_{z_0,n_0})$ are the observations with $Z = z_0$ and $(Y_{z_1,1}, T_{z_1,1}), \dots, (Y_{z_1,n_1}, T_{z_1,n_1})$ are the observations with $Z = z_1$. n_0 is the sample size for the observations with $Z = z_0$ and n_1 is for $Z = z_1$. The total sample size is $n = n_0 + n_1$. I assume $n_1/n_0, n_0/n_1 = o(1)$.

In order to discuss a test statistic and a critical value, I introduce estimators for the moment functions and the estimated standard deviations for the moment functions. For an estimator for the moment function,

$$\hat{m}_j(\theta) = \hat{m}_{1,j}(\theta) - \hat{m}_{0,j}(\theta)$$

estimates the j th moment function $m_j(\theta) = \Delta E_P[g_{Z,j}((Y, T), \theta) | Z]$, where

$$\hat{m}_{z,j}(\theta) = n_z^{-1} \sum_{i=1}^{n_z} g_{z,j}((Y_{z,i}, T_{z,i}), \theta).$$

Denote by $\sigma_{z,j}(\theta)$ the standard deviation of $n_z^{-1/2} \sum_{i=1}^{n_z} g_{z,j}((Y_{z,i}, T_{z,i}), \theta)$. The standard deviation $\sigma_j(\theta)$ for $\sqrt{n}\hat{m}_j(\theta)$ is $\sigma_j^2(\theta) = n^{-1}(n_1\sigma_{1,j}^2(\theta) + n_0\sigma_{0,j}^2(\theta))$. Denote by $\hat{\sigma}_{z,j}(\theta)$ the estimated standard deviation of $n_z^{-1/2} \sum_{i=1}^{n_z} g_{z,j}((Y_{z,i}, T_{z,i}), \theta)$, that is,

$$\hat{\sigma}_{z,j}(\theta) = n_z^{-1} \sum_{i=1}^{n_z} (g_{z,j}((Y_{z,i}, T_{z,i}), \theta) - \hat{m}_{z,j}(\theta))^2.$$

$\hat{\sigma}_j(\theta)$ estimates the standard deviation $\sigma_j(\theta)$, that is,

$$\hat{\sigma}_j^2(\theta) = n^{-1}(n_1\hat{\sigma}_{1,j}^2(\theta) + n_0\hat{\sigma}_{0,j}^2(\theta)).$$

The test statistics for $\theta_0 = \theta$ is

$$T(\theta) = \max_{1 \leq j \leq p_n} \frac{\sqrt{n}\hat{m}_j(\theta)}{\max\{\hat{\sigma}_j(\theta), \xi\}}$$

where ξ is a small positive number which prohibits the fraction from becoming too large when the estimated standard deviation is near zero. The truncation via ξ controls the effect of the approximation error from the approximated identified set on the power against local alternatives, as in Subsection 4.4. The size is $\alpha \in (0, 1/2)$ and the pretest size for the moment inequality selection is $\beta \in (0, \alpha/2)$. The critical value $c^{2S}(\alpha, \theta)$ for $T(\theta)$ is based on the two-step multiplier bootstrap (Chernozhukov et al., 2014), described in Algorithm 1. The $(1 - \alpha)$ -confidence interval for LATE is

$$\{\theta \in \Theta : T(\theta) \leq c^{2S}(\alpha, \theta)\}.$$

Under the following three assumptions, I show that this confidence uniformly valid asymptotic size control for the confidence interval, by adapting Theorem 4.4 in Chernozhukov et al. (2014) into the two independent samples.

Assumption 2. $\Theta \subset \mathbb{R}$ is bounded.

Assumption 3. There are constants $c_1 \in (0, 1/2)$ and $C_1 > 0$ such that $\log^{7/2}(p_n n) \leq C_1 n^{1/2-c_1}$ with $p_n = 4^{K_n} + 2$.

Algorithm 1 Two-step multiplier bootstrap (Chernozhukov et al., 2014)

- 1: For each $z = 0, 1$, generate independent random variables $\epsilon_{z,1}, \dots, \epsilon_{z,n_z}$ from $N(0, 1)$.
- 2: Construct the bootstrap test statistics for the moment inequality selection by

$$W(\theta) = \max_{1 \leq j \leq p_n} \frac{\sqrt{n} \hat{m}_j^B(\theta)}{\max\{\hat{\sigma}_j(\theta), \xi\}},$$

where $\hat{m}_{z,j}^B(\theta) = n_z^{-1} \sum_{i=1}^{n_z} \epsilon_{z,i} (g_{z,j}((Y_{z,i}, T_{z,i}), \theta) - \hat{m}_{z,j}(\theta))$ and $\hat{m}_j^B(\theta) = \hat{m}_{1,j}^B(\theta) - \hat{m}_{0,j}^B(\theta)$.

- 3: Construct the bootstrap critical value $c(\beta, \theta)$ for the moment inequality selection as the conditional $(1 - \beta)$ -quantile of $W(\theta)$ given $\{(Y_{z,i}, T_{z,i})\}$.
- 4: Select the moment inequalities and save

$$\hat{J} = \left\{ j = 1, \dots, p_n : \frac{\sqrt{n} \hat{m}_j(\theta)}{\max\{\hat{\sigma}_j(\theta), \xi\}} > -2c(\beta, \theta) \right\}.$$

- 5: Construct the bootstrap test statistics by

$$W_j(\theta) = \max_{j \in \hat{J}} \frac{\sqrt{n} \hat{m}_j^B(\theta)}{\max\{\hat{\sigma}_j(\theta), \xi\}}$$

where $W_j(\theta) = 0$ if \hat{J} is empty.

- 6: Construct the bootstrap critical value $c^{2S}(\alpha, \theta)$ as the conditional $(1 - \alpha + 2\beta)$ -quantile of $W_j(\theta)$ given $\{(Y_{z,i}, T_{z,i})\}$.
-

Assumption 4. (i) There is a constant $C_0 > 0$ such that $\max\{E[Y^3]^{2/3}, E[Y^4]^{1/2}\} < C_0$. (ii) $0 < \sigma_{z,j}(\theta) < \infty$.

Theorem 2. Under Assumptions 2 and 3,

$$\liminf_{n \rightarrow \infty} \inf_{(\theta, P) \in \mathcal{H}_0} P(T(\theta) \leq c^{2S}(\alpha, \theta)) \geq 1 - \alpha,$$

where \mathcal{H}_0 is the set of (θ, P) such that P satisfies Assumption 4 and $\theta \in \Theta_0(P)$.

4.4 Power against fixed and local alternatives

This section discusses the power properties of the confidence interval. First, I assume that the density function $f_{(Y,T)|Z=z}$ satisfies the Hölder continuity. This assumption justifies the approximation of the total variation distance via step functions, which is similar to the sieve estimation.

Assumption 5. The density function $f_{(Y,T)|Z=z}$ is Hölder continuous in (y, t) with the Hölder constant D_0 and exponent d .

I restrict the number of the moment inequalities and, in turn, restrict the magnitude of the critical value. Note that the number of the moment inequalities is the tuning parameter in this framework. The tradeoff is as follows: the approximation error is large if $p_n \rightarrow \infty$ slow, and the sampling error is large if $p_n \rightarrow \infty$ fast.

Assumption 6. $\log^{1/2}(p_n) \leq C_1 n^{1/2 - c_1}$.

The last condition is that the grids in \mathbf{I}_n becomes finer as the sample size goes to infinity.

Assumption 7. There is a positive constant D_1 such that $I_{n,k}$ is a subset of some open ball with radius D_1/K_n in $\mathbf{Y} \times \mathbf{T}$.

I obtain the following power property against local alternatives, based on Corollary 5.1 in Chernozhukov et al. (2014).

Theorem 3. Fix $\delta, \epsilon > 0$ and $\tau_n \rightarrow \infty$ with $\tau_n = o(n)$. Denote by $\mathcal{H}_{1,n}$ the set of local alternatives (θ, P) 's satisfying Assumptions 4, 5 and at least one of the following inequalities:

$$-\theta \Delta E_P[Y | Z] \geq \kappa_n \quad (5)$$

$$|\Delta E_P[Y | Z]| - |\theta| \geq \kappa_n \quad (6)$$

$$|\theta| TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) - |\Delta E_P[Y | Z]| \geq \kappa_n + \sup_{\theta \in \Theta} |\theta| 2^{d+2} D_0 D_1^d K_n^{-d} \mu_Y(\mathbf{Y}), \quad (7)$$

where $\kappa_n = \xi(1 + \delta)(1 + \epsilon) \sqrt{2 \log(\max\{p_n, \tau_n\} / (\alpha - 2\beta)) / n}$. Under Assumptions 2 and 3, 6 and 7,

$$\lim_{n \rightarrow \infty} \inf_{(\theta, P) \in \mathcal{H}_{1,n}} P(T(\theta) > c^{2S}(\alpha, \theta)) = 1.$$

The violation of the moment inequalities includes local alternatives in the sense that K_n^{-d} and κ_n go to zero in the large sample.

5 Monte Carlo simulations

This section illustrates the theoretical properties for the confidence interval in Section 7, using simulated datasets. Consider four independent random variables U_1, U_2, U_3, U_4 from $U(0, 1)$. Using the $N(0, 1)$ cumulative distribution function Φ , I generate (Y, T, Z) in the following way:

$$\begin{aligned} Z &= 1\{U_1 \leq 0.5\} \\ T^* &= 1\{U_2 \leq 0.5 + \gamma_1(Z - 0.5)\} \\ Y &= \Phi\left(\gamma_2 T^* + \frac{\Phi^{-1}(U_4) + 0.5\Phi^{-1}(U_2)}{1 + 0.5^2}\right) \\ T &= \begin{cases} T^* & \text{if } U_3 \leq \gamma_3 \\ 1 - T^* & \text{otherwise.} \end{cases} \end{aligned}$$

I have the three parameters in the model: γ_1 represents the strength of the instrumental variable, γ_2 represents the magnitude of treatment effect, and γ_3 represents the degree of the measurement error. This is the heterogeneous treatment effect model, because Φ is nonlinear. I select several values for $(\gamma_1, \gamma_2, \gamma_3)$ as in Table 1. The treatment effect is small ($\gamma_2 = 1$) in Designs 1-4 and large ($\gamma_1 = 3$) in Designs 5-8. The measurement error is small ($1 - \gamma_2 = 20\%$) in Designs 3,4,7,8 and large ($1 - \gamma_2 = 40\%$) in Designs 1,2,5,6. The instrumental variable is strong ($\gamma_1 = 0.5$) in Designs 2,4,6,8 and weak ($\gamma_1 = 0.1$) in Design 1,3,5,7.

In Table 1, I compute the three population objects: LATE, the Wald estimand, and the sharp identified set for LATE. As expected, LATE is included in the sharp identified set in all the designs. The comparison between the Wald estimand and the sharp identified set hints that the Wald estimand is relatively large compared to the upper bound of the identified set $\Theta_0(P)$ when the measurement error has a large degree in Design 1,2,5,6. For those designs, the Wald estimand is too large to be interpreted as an upper bound on LATE, because the upper bound of the identified set $\Theta_0(P)$ is much smaller.

I choose the sample size $n = 500, 1000, 5000$ for the Monte Carlo simulations. Note that the numbers

covers the sample size (2,909) in NLS-72. I simulate 2,000 datasets of three sample sizes. For each dataset, I construct the different confidence set with confidence size $1 - \alpha = 95\%$, as in Section 4. I use the partition of equally spaced grids over \mathbf{Y} with the number the partitions $K_n = 1, 3, 5$. In all the confidence intervals, I use 5,000 bootstraps repetitions. Figures 3-10 describe the coverage probabilities of the confidence intervals for each parameter value.

For each design, two figures are displayed. First, the left figures demonstrate the coverage probabilities according to $n = 500, 100, 5000$ given $K_n = 3$. The left figures of all the designs support the consistency results in the previous section; as the sample size increases, the coverage probabilities of the confidence intervals accumulate over $\Theta_0(P)$. Second, the right figures demonstrate the coverage probabilities according to $K_n = 1, 3, 5$ given $n = 1000$. When the Wald estimand is close to the upper bound of $\Theta_0(P)$ (Design 3,4,7,8), it seems advantageous to use $K_n = 1$. It is presumably because $K_n = 3, 5$ uses more inequalities for inference compared to $K_n = 1$ but these inequalities are not informative for LATE. When the Wald estimand is significantly larger than the upper bound of $\Theta_0(P)$ (Design 1,2,5,6), it seems advantageous to use $K_n = 3, 5$, particularly for the coverage probabilities near the upper bound of $\Theta_0(P)$. In these designs, the coverage probabilities are not sensitive the choice of $K_n = 3$ or $K_n = 5$.

γ_1	γ_2	γ_3	LATE	Identified Set	Wald Estimand
0.1	1	0.6	0.28	[0.03, 0.58]	1.43
0.5	1	0.6	0.28	[0.14, 0.58]	1.40
0.1	1	0.8	0.28	[0.03, 0.43]	0.46
0.5	1	0.8	0.28	[0.14, 0.43]	0.47
0.1	3	0.6	0.49	[0.05, 0.52]	2.52
0.5	3	0.6	0.49	[0.24, 0.52]	2.42
0.1	3	0.8	0.49	[0.05, 0.52]	0.84
0.5	3	0.8	0.49	[0.24, 0.52]	0.81

Table 1: Parameter values for Monte Carlo simulations. I numerically calculate LATE $\theta(P^*)$, the identified set $\Theta_0(P)$ and the Wald estimand $\Delta E_P[Y | Z] / \Delta E_P[T | Z]$ for each parameter value.

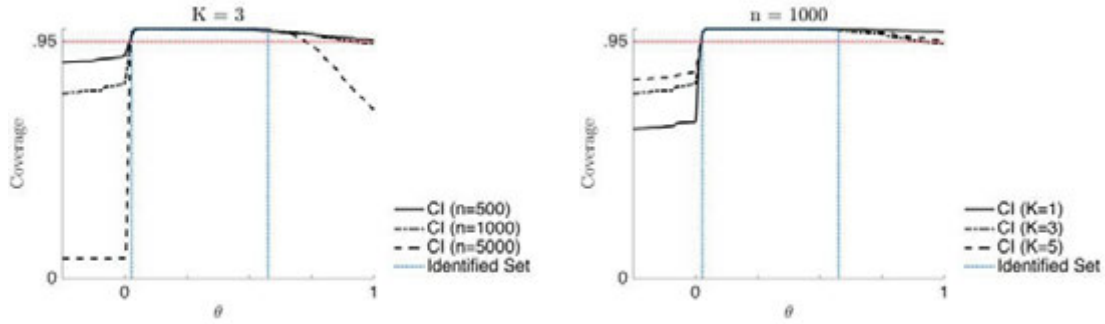


Figure 3: Coverage of the confidence interval for $(\gamma_1, \gamma_2, \gamma_3) = (0.1, 1, 0.6)$

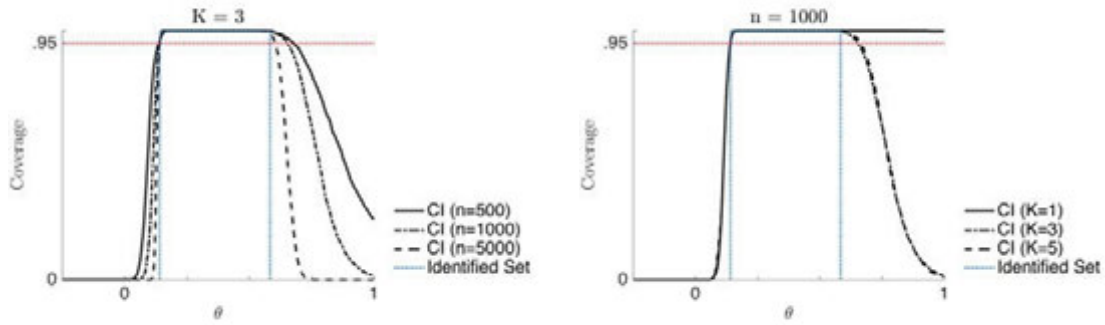


Figure 4: Coverage of the confidence interval for $(\gamma_1, \gamma_2, \gamma_3) = (0.5, 1, 0.6)$

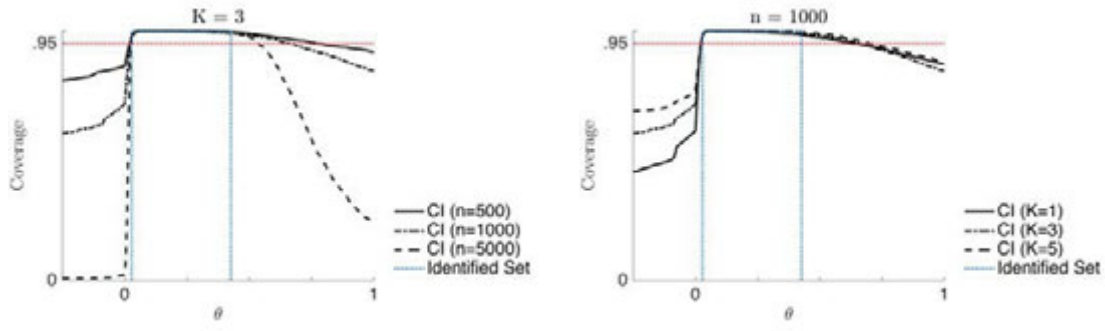


Figure 5: Coverage of the confidence interval for $(\gamma_1, \gamma_2, \gamma_3) = (0.1, 1, 0.8)$

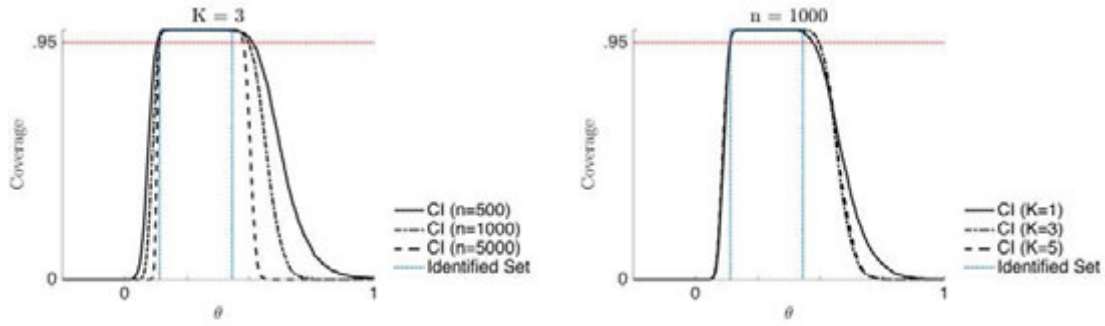


Figure 6: Coverage of the confidence interval for $(\gamma_1, \gamma_2, \gamma_3) = (0.5, 1, 0.8)$

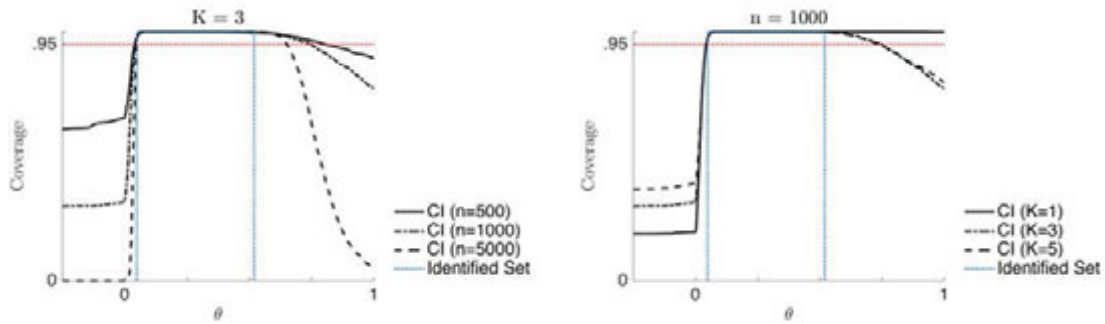


Figure 7: Coverage of the confidence interval for $(\gamma_1, \gamma_2, \gamma_3) = (0.1, 3, 0.6)$

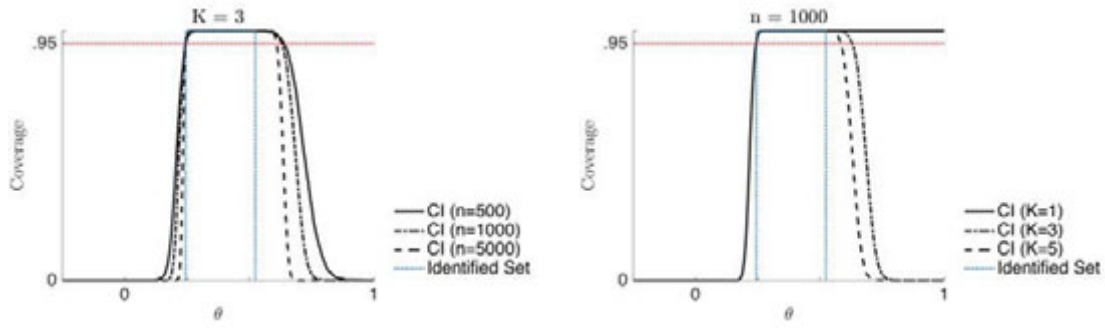


Figure 8: Coverage of the confidence interval for $(\gamma_1, \gamma_2, \gamma_3) = (0.5, 3, 0.6)$

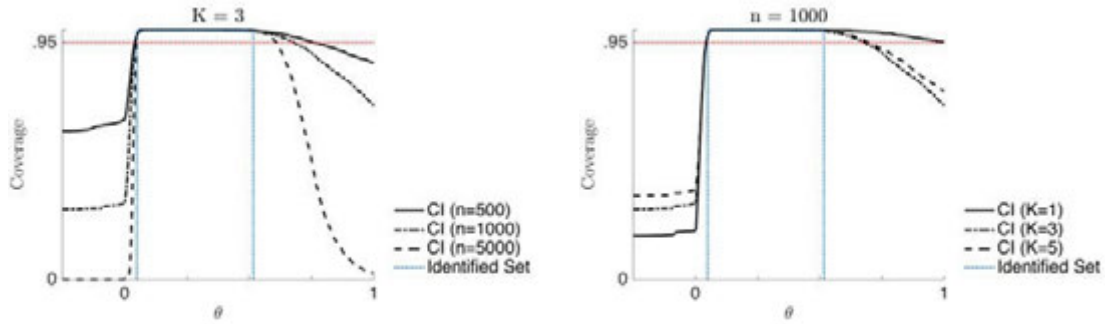


Figure 9: Coverage of the confidence interval for $(\gamma_1, \gamma_2, \gamma_3) = (0.1, 3, 0.8)$

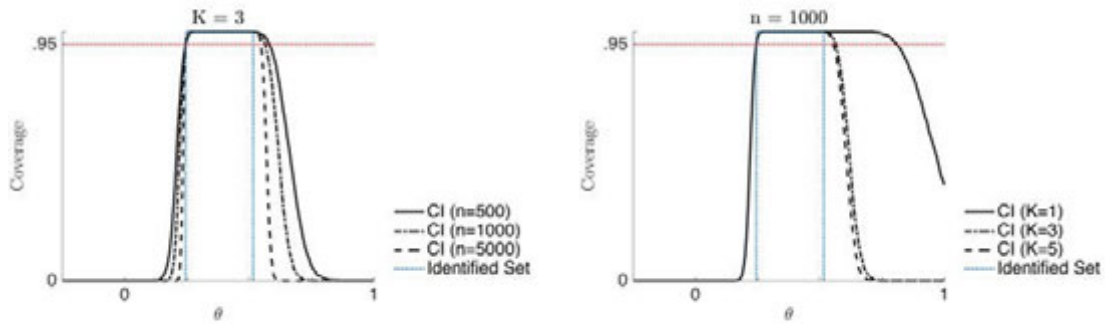


Figure 10: Coverage of the confidence interval for $(\gamma_1, \gamma_2, \gamma_3) = (0.5, 3, 0.8)$

6 Empirical illustrations

To illustrate the theoretical results on identification and inference, this section uses the National Longitudinal Survey of the High School Class of 1972 (NLS-72) to investigate the effect on wages of attending a college when the college attendance can be mismeasured. Kane and Rouse (1995) and Kane et al. (1999) use the same dataset to investigate the educational effect on wages in the presence of the endogeneity and the measurement error in the educational attainments. However, they do not consider the two problems and their results are dependent on the constant return to schooling. For an instrument, I follow the strategy in Card (1995) and Kane and Rouse (1995) closely and use the proximity to college as an instrumental variable for the college attendance.

NLS-72 was conducted by the National Center for Education Statistics with the U.S. Department of Education, and it contains 22,652 seniors (as of 1972) from 1,200 schools across the U.S. The sampled individuals were asked to participate in multiple surveys from 1972 through 1986. The survey collects labor market experiences, schooling information and demographic characteristics. I drop the individuals with college degree or more, to focus on the comparison between high school graduates and the individuals with some college education. I also drop those who have missing values for wages in 1986 or educational attainments. The resulting size is 2,909.

I consider the effect of the college attendance T^* on Y (the log of wages in 1986). The treatment group with $T^* = 1$ is the individuals who have attended a college without a degree, and the control group is the individuals who have never been to a college. Some summary statistics are on Tables 2 and 3. I allow for the possibility that T^* is mismeasured, that is, the college attendance T in the dataset can be different from the truth T^* . I define the instrumental variable Z as an indicator for whether an individual grew up near 4 year college. I use 10 miles as a threshold for the proximity-to-college to similar to the strategy in Card (1995).

I present inference results in Table 4. The first row is the Wald estimate and the 95% confidence interval for the Wald estimand. The second row is the 95% confidence interval for LATE based on the identified set $\Theta_0(P)$ in Theorem 1. For the calculation of this confidence interval, I use the partition of equally spaced grids over \mathbf{Y} with the number of the partitions equal to $K_n = 3$. In all the confidence intervals, I use 5000 bootstrap repetitions. The results are consistent with my identification analysis in the following two points. First, the Wald estimate is too large for the effect of attending a college. For example, Card (1999) documents the existing estimates for the return to schooling and most of them fall in the range of 5-15% as the percentage increases for one additional year of education. According to my analysis, the large value of the Wald estimate can result from the mismeasurement of the college attendance. Second, when I compare the upper bounds of the confidence intervals for the Wald estimand and LATE, the upper bound (1.04) based on $\Theta_0(P)$ is strictly lower than that (1.61) of the Wald estimand. This implies that the Wald estimator is an upper bound for LATE but it does not offer the sharp upper bound for LATE. These two findings are still valid when I consider six subgroups (Table 5).

All sample	Mean	Std. Dev.
Y (log hourly wage)	2.08	0.47
T (college attendance)	0.30	0.46
Z (lived near college)	0.51	0.50

Table 2: Summary Statistics for (Y, T, Z)

	white	black	male	female	non-south	south
observations	2,249	329	1,267	1,642	1,884	1,025
mean of Y	2.09	1.97	2.28	1.91	2.10	2.03
mean of T	0.31	0.31	0.31	0.31	0.32	0.28
mean of Z	0.50	0.64	0.48	0.53	0.53	0.47

Table 3: Demographic groups

All sample	Estimates	CI
Wald estimand	0.95	[0.56, 1.61]
LATE	N.A. ⁴	[0.05, 1.04]

Table 4: 95% confidence intervals for the Wald estimand and LATE

White	Estimates	CI	Black	Estimates	CI
Wald	1.20	[0.68, 2.25]	Wald	0.95	[0.54, 1.66]
LATE	N.A.	[0.06, 1.16]	LATE	N.A.	[0.05, 1.06]

Male	Estimates	CI	Female	Estimates	CI
Wald	0.93	[0.40, 2.07]	Wald	1.31	[0.76, 2.72]
LATE	N.A.	[0.04, 1.44]	LATE	N.A.	[0.07, 1.28]

Non-south	Estimates	CI	South	Estimates	CI
Wald	1.15	[0.60, 2.30]	Wald	0.58	[0.00, 1.63]
LATE	N.A.	[0.06, 1.47]	LATE	N.A.	[-0.02, 1.04]

Table 5: 95% confidence intervals for various subpopulations

⁴Estimates for θ are not available. In this paper, I focus on an inference about the true parameter value θ , instead of the identified set $\Theta_0(P)$. I do not have a consistent set estimator for $\Theta_0(P)$.

7 Identifying power of repeated measurements

This section explores the identifying power of repeated measurements. Repeated measurements (for example, Hausman et al., 1991) is a popular approach in the literature on measurement error, but they cannot be instrumental variables in this framework. This is because the true treatment variable T^* is endogenous and it is natural to suspect that a measurement of T^* is also endogenous. The more accurate the measurement is, the more likely it is to be endogenous. Nevertheless, the identification strategy of the present paper incorporates repeated measurements as an additional information to narrow the identified set for LATE, when they are coupled with the instrumental variable Z . Unlike the other paper on repeated measurements, I do not need to assume the independence of measurement errors among multiple measurements. The strategy of the present paper also benefits from having more than two measurements unlike Hausman et al. (1991) who achieve the point identification with two measurements.

Consider a second measurement R for T^* . I do not require that R is binary, so R can be discrete or continuous. Like $T = T_{T^*}$, I model R using the counterfactual outcome notations. R_1 is a counterfactual second measurement when the true variable T^* is 1, and R_0 is a counterfactual second measurement when the true variable T^* is 0. Then the data generation of R is

$$R = R_{T^*}.$$

I assume that the instrumental variable Z is independent of R_{t^*} conditional on $(Y_{t^*}, T_{t^*}, T_{z_0}^*, T_{z_1}^*)$.

Assumption 8. (i) Z is independent of $(R_{t^*}, T_{t^*}, Y_{t^*}, T_{z_0}^*, T_{z_1}^*)$ for each $t^* = 0, 1$. (ii) $T_{z_1}^* \geq T_{z_0}^*$ almost surely.

Note that I do not assume the independence between R_{t^*} and T_{t^*} , where the independence between the measurement errors is a key assumption when the repeated measurement is an instrumental variable.

Under this assumption, I refine the identified set for LATE as follows.

Theorem 4. Suppose that Assumption 8 holds, and consider an arbitrary data distribution P of (R, Y, T, Z) . (i) The sharp identified set $\Theta_I(P)$ for LATE is included in $\Theta_0(P)$, where $\Theta_0(P)$ is the set of θ 's which satisfies the following three inequalities.

$$\begin{aligned} \theta \Delta E_P[Y | Z] &\geq 0 \\ |\theta| &\geq |\Delta E_P[Y | Z]| \\ |\theta| TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0}) &\leq |\Delta E_P[Y | Z]|. \end{aligned}$$

(ii) If Y is unbounded, then $\Theta_I(P)$ is equal to $\Theta_0(P)$.

The total variation distance $TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0})$ in Theorem 4 is weakly larger than that in Theorem 1, which implies that the identified set in Theorem 4 is weakly smaller than the identified set in

Theorem 1:

$$\begin{aligned}
& TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0}) \\
&= \frac{1}{2} \int_{t=0,1} |(f_{(R,Y,T)|Z=z_1} - f_{(R,Y,T)|Z=z_0})(r, y, t)| d\mu_R(r) d\mu_Y(y) \\
&\geq \frac{1}{2} \int_{t=0,1} |(f_{(R,Y,T)|Z=z_1} - f_{(R,Y,T)|Z=z_0})(r, y, t)| d\mu_R(r) | d\mu_Y(y) \\
&= \frac{1}{2} \int_{t=0,1} |(f_{(Y,T)|Z=z_1} - f_{(Y,T)|Z=z_0})(y, t)| d\mu_Y(y) \\
&= TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0})
\end{aligned}$$

and the strict inequality holds unless the sign of $(f_{(R,Y,T)|Z=z_1} - f_{(R,Y,T)|Z=z_0})(r, y, t)$ is constant in r for every (y, t) . Therefore, it is possible to test whether the repeated measurement R has additional information, by testing whether the sign of $(f_{(R,Y,T)|Z=z_1} - f_{(R,Y,T)|Z=z_0})(r, y, t)$ is constant in r .

8 Conclusion

This paper studies the identifying power of instrumental variable in the heterogeneous treatment effect framework when a binary treatment variable is mismeasured and endogenous. The assumptions in this framework are the monotonicity of the instrumental variable Z on the true treatment variable T^* and the exogeneity of Z . I use the total variation distance to characterize the identified set for LATE parameter $E[T_1 - T_0 | T_{z_0}^* < T_{z_1}^*]$. I also provide an inference procedure for LATE. Unlike the existing literature on measurement error, the identification strategy does not rely on a specific assumption on the measurement error; the only assumption on the measurement error is its independence of the instrumental variable. I apply the new methodology to study the return to schooling in the proximity-to-college instrumental variable regression using the NLS-72 dataset.

There are several directions for future research. First, the choice of the partition \mathbf{I}_n in Section 4, particularly the choice of K_n , is an interesting direction. To the best of my knowledge, the literature on many moment inequalities has not investigated how econometricians choose the numbers of the many moment inequalities. Second, it is worthwhile to investigate the other parameter for the treatment effect. This paper has focused on the local average treatment effect (LATE) for the reasons mentioned in the introduction, but the literature on heterogeneous treatment effect has emphasized the choice of treatment effect parameter as an answer to relevant policy questions.

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A Proofs of Lemmas 1, 2, and 3

Proof of Lemma 1. Eq. (4) implies

$$\begin{aligned}\theta(P^*)\Delta E_{P^*}[Y | Z] &= \theta(P^*)^2 P^*(T_{z_0}^* < T_{z_1}^*) \\ &\geq 0\end{aligned}$$

and

$$\begin{aligned}|\Delta E_{P^*}[Y | Z]| &= |\theta(P^*)P^*(T_{z_0}^* < T_{z_1}^*)| \\ &\leq |\theta(P^*)|,\end{aligned}$$

because $0 \leq P^*(T_{z_0}^* < T_{z_1}^*) \leq 1$. □

Proof of Lemma 2. I obtain

$$f_{(Y,T)|Z=z_1} - f_{(Y,T)|Z=z_0} = P^*(T_{z_0}^* < T_{z_1}^*)(f_{(Y_1,T_1)|T_{z_0}^* < T_{z_1}^*} - f_{(Y_0,T_0)|T_{z_0}^* < T_{z_1}^*})$$

by applying the same logic as Theorem 1 in Imbens and Angrist (1994):

$$\begin{aligned}f_{(Y,T)|Z=z_0} &= P^*(T_{z_0}^* = T_{z_1}^* = 1 | Z = z_0)f_{(Y,T)|Z=z_0, T_{z_0}^* = T_{z_1}^* = 1} \\ &\quad + P^*(T_{z_0}^* < T_{z_1}^* | Z = z_0)f_{(Y,T)|Z=z_0, T_{z_0}^* < T_{z_1}^*} \\ &\quad + P^*(T_{z_0}^* = T_{z_1}^* = 0 | Z = z_0)f_{(Y,T)|Z=z_0, T_{z_0}^* = T_{z_1}^* = 0} \\ &= P^*(T_{z_0}^* = T_{z_1}^* = 1)f_{(Y_1,T_1)|T_{z_0}^* = T_{z_1}^* = 1} \\ &\quad + P^*(T_{z_0}^* < T_{z_1}^*)f_{(Y_0,T_0)|T_{z_0}^* < T_{z_1}^*} \\ &\quad + P^*(T_{z_0}^* = T_{z_1}^* = 0)f_{(Y_0,T_0)|T_{z_0}^* = T_{z_1}^* = 0} \\ f_{(Y,T)|Z=z_1} &= P^*(T_{z_0}^* = T_{z_1}^* = 1)f_{(Y_1,T_1)|T_{z_0}^* = T_{z_1}^* = 1} \\ &\quad + P^*(T_{z_0}^* < T_{z_1}^*)f_{(Y_1,T_1)|T_{z_0}^* < T_{z_1}^*} \\ &\quad + P^*(T_{z_0}^* = T_{z_1}^* = 0)f_{(Y_0,T_0)|T_{z_0}^* = T_{z_1}^* = 0}.\end{aligned}$$

This implies

$$\begin{aligned}TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) &= \frac{1}{2} \int_{t=0,1} |f_{(Y,T)|Z=z_1}(y, t) - f_{(Y,T)|Z=z_0}(y, t)| d\mu_Y(y) \\ &= \frac{1}{2} \int_{t=0,1} |P^*(T_{z_0}^* < T_{z_1}^*)(f_{(Y_1,T_1)|T_{z_0}^* < T_{z_1}^*}(y, t) - f_{(Y_0,T_0)|T_{z_0}^* < T_{z_1}^*}(y, t))| d\mu_Y(y) \\ &= P^*(T_{z_0}^* < T_{z_1}^*) \frac{1}{2} \int_{t=0,1} |(f_{(Y_1,T_1)|T_{z_0}^* < T_{z_1}^*}(y, t) - f_{(Y_0,T_0)|T_{z_0}^* < T_{z_1}^*}(y, t))| d\mu_Y(y) \\ &= P^*(T_{z_0}^* < T_{z_1}^*) TV(f_{(Y_1,T_1)|T_{z_0}^* < T_{z_1}^*}, f_{(Y_0,T_0)|T_{z_0}^* < T_{z_1}^*}) \\ &\leq P^*(T_{z_0}^* < T_{z_1}^*),\end{aligned}$$

where the last inequality follows because the total variation distance is at most one. Moreover, since $T_{z_0}^* \leq T_{z_1}^*$

almost surely,

$$\begin{aligned}
P^*(T_{z_0}^* < T_{z_1}^*), &= |P^*(T_{z_0}^* = 1) - P^*(T_{z_1}^* = 1)| \\
&= \frac{1}{2}|P^*(T_{z_0}^* = 1) - P^*(T_{z_1}^* = 1)| + \frac{1}{2}|P^*(T_{z_0}^* = 0) - P^*(T_{z_1}^* = 0)| \\
&= \frac{1}{2} \int_{t^*=0,1} |f_{T^*|Z=z_1}(t^*) - f_{T^*|Z=z_0}(t^*)| \\
&= TV(f_{T^*|Z=z_1}, f_{T^*|Z=z_0}).
\end{aligned}$$

□

Proof of Lemma 3. The lemma follows from

$$\begin{aligned}
\Delta E_P[h(Y, T) | Z] &= E_P[h(Y, T) | Z = z_1] - E_P[h(Y, T) | Z = z_0] \\
&= \int_{t=0,1} h(y, t)(f_{(Y, T)|Z=z_1}(y, t) - f_{(Y, T)|Z=z_0}(y, t))d\mu_Y(y) \\
&= \int_{t=0,1} h(y, t)\Delta f_{(Y, T)|Z}(y, t)d\mu_Y(y) \\
&\leq \int_{t=0,1} |\Delta f_{(Y, T)|Z}(y, t)|d\mu_Y(y) \\
&= 2 \times TV(f_{(Y, T)|Z=z_1}, f_{(Y, T)|Z=z_0})
\end{aligned}$$

where the maximization is achieved if $h(y, t) = 1$ if $\Delta f_{(Y, T)|Z}(y, t) > 0$ and $h(y, t) = -1$ if $\Delta f_{(Y, T)|Z}(y, t) < 0$. □

B Proofs of Theorems 1 and 4

Theorem 1 is a special case of Theorem 4 with R being constant, and therefore I demonstrate the proof only for Theorem 4. Lemma 2 is modified into the following lemma in the framework of Theorem 4.

Lemma 4. *Under Assumption 8, then*

$$TV(f_{(R, Y, T)|Z=z_1}, f_{(R, Y, T)|Z=z_0}) \leq P^*(T_{z_0}^* < T_{z_1}^*).$$

Proof. The proof is the same as Lemma 2 and this lemma follows from

$$f_{(R, Y, T)|Z=z_1} - f_{(R, Y, T)|Z=z_0} = P^*(T_{z_0}^* < T_{z_1}^*)(f_{(R_1, Y_1, T_1)|T_{z_0}^* < T_{z_1}^*} - f_{(R_0, Y_0, T_0)|T_{z_0}^* < T_{z_1}^*}).$$

□

From Lemmas 1 and 4, all the three inequalities in Theorem 4 are satisfied when θ is the true value for LATE, which is the first part of Theorem 4. To prove Theorem 4, I am going to show the sharpness of the three inequalities, that is, that any point satisfying the three inequalities is generated by some data generating process P^* which is consistent with the data distribution P . I will consider two cases based on the value of $TV(f_{(R, Y, T)|Z=z_1}, f_{(R, Y, T)|Z=z_0})$.

B.1 Case 1: Zero total variation distance

Consider $TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0}) = 0$. In this case, $f_{(R,Y,T)|Z=z_1} = f_{(R,Y,T)|Z=z_0}$ almost everywhere over (r, y, t) and particularly $\Delta E_P[Y | Z] = 0$. Note that all the three inequalities in Theorem 4 have no restriction on θ in this case. For every $y \in \mathbb{R}$, consider the following two data generating processes. First, $P_{y,L}^*$ is defined by

$$\begin{aligned} Z &\sim P(Z = z) \\ (T_{z_0}^*, T_{z_1}^*) | Z &= \begin{cases} (0, 1) & \text{with probability } P(Y > y) \\ (1, 1) & \text{with probability } P(Y \leq y) \end{cases} \\ (R_1, Y_1, T_1) | (T_{z_0}^*, T_{z_1}^*, Z) &\sim f_{(R,Y,T)}(r, y, t) \\ (R_0, Y_0, T_0) | (T_{z_0}^*, T_{z_1}^*, Z) &\sim \begin{cases} f_{(R,Y,T)|Y>y}(r, y, t) & \text{if } T_{z_0}^* < T_{z_1}^* \\ f_{(R,Y,T)|Y\leq y}(r, y, t) & \text{if } T_{z_0}^* = T_{z_1}^* \end{cases} \end{aligned}$$

where (R_1, Y_1, T_1) and (R_0, Y_0, T_0) are conditionally independent of $(T_{z_0}^*, T_{z_1}^*, Z)$. Second, $P_{y,U}^*$ is defined by

$$\begin{aligned} Z &\sim P(Z = z) \\ (T_{z_0}^*, T_{z_1}^*) | Z &= \begin{cases} (0, 1) & \text{with probability } P(Y < y) \\ (1, 1) & \text{with probability } P(Y \geq y) \end{cases} \\ (R_1, Y_1, T_1) | (T_{z_0}^*, T_{z_1}^*, Z) &\sim f_{(R,Y,T)}(r, y, t) \\ (R_0, Y_0, T_0) | (T_{z_0}^*, T_{z_1}^*, Z) &\sim \begin{cases} f_{(R,Y,T)|Y<y}(r, y, t) & \text{if } T_{z_0}^* < T_{z_1}^* \\ f_{(R,Y,T)|Y\geq y}(r, y, t) & \text{if } T_{z_0}^* = T_{z_1}^* \end{cases} \end{aligned}$$

where (R_1, Y_1, T_1) and (R_0, Y_0, T_0) are conditionally independent of $(T_{z_0}^*, T_{z_1}^*, Z)$.

Lemma 5. *Consider the assumptions in Theorem 4. If $TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0}) = 0$, then, for every $y \in \mathbb{R}$ and every $\pi \in [0, 1]$,*

1. *the mixture distribution $\pi P_{y,L}^* + (1 - \pi)P_{y,U}^*$ satisfies Assumption 8;*
2. *the mixture distribution $\pi P_{y,L}^* + (1 - \pi)P_{y,U}^*$ generates the data distribution P ;*
3. *under the mixture distribution $\pi P_{y,L}^* + (1 - \pi)P_{y,U}^*$, LATE is equal to $E_P[Y] - \pi E_P[Y | Y > y] - (1 - \pi)E_P[Y | Y < y]$.*

Proof. (1) Both $P_{y,L}^*$ and $P_{y,U}^*$ satisfy the independence between Z and $(R_{t^*}, T_{t^*}, Y_{t^*}, T_{z_0}^*, T_{z_1}^*)$ for each $t^* = 0, 1$. Furthermore, $P_{y,L}^*$ and $P_{y,U}^*$ have the same marginal distribution for Z : $P(Z = z)$. Therefore, the mixture of $P_{y,L}^*$ and $P_{y,U}^*$ also satisfies the independence. Since the mixture of $P_{y,L}^*$ and $P_{y,U}^*$ satisfies $T_{z_1}^* \geq T_{z_0}^*$ almost surely, the first part of this lemma is established.

(2) The second part follows from the fact that both $P_{y,L}^*$ and $P_{y,U}^*$ generate the data distribution P . Since the proof is essentially the same for $P_{y,L}^*$ and $P_{y,U}^*$, I demonstrate it only for $P_{y,L}^*$. Denote by f^* the

density function of $P_{y,L}^*$. Then

$$\begin{aligned}
f_{(R,Y,T)|Z=z_0}^*(r, y, t) &= P_{y,L}^*(T_{z_0}^* < T_{z_1}^*) f_{(R,Y,T)|T_{z_0}^* < T_{z_1}^*, Z=z_0}^*(r, y, t) \\
&\quad + P_{y,L}^*(T_{z_0}^* = T_{z_1}^* = 1) f_{(R,Y,T)|T_{z_0}^* = T_{z_1}^* = 1, Z=z_0}^*(r, y, t) \\
&= P_{y,L}^*(T_{z_0}^* < T_{z_1}^*) f_{(R_0, Y_0, T_0)|T_{z_0}^* < T_{z_1}^*}^*(r, y, t) \\
&\quad + P_{y,L}^*(T_{z_0}^* = T_{z_1}^* = 1) f_{(R_1, Y_1, T_1)|T_{z_0}^* = T_{z_1}^* = 1}^*(r, y, t) \\
&= P(Y > y) f_{(R,Y,T)|Y > y}(r, y, t) \\
&\quad + P(Y \leq y) f_{(R,Y,T)|Y \leq y}(r, y, t) \\
&= f_{(R,Y,T)}(r, y, t) \\
f_{(R,Y,T)|Z=z_1}^*(r, y, t) &= f_{(R,Y,T)}(r, y, t),
\end{aligned}$$

where the last equality uses $Z = z_1$ implies $T^* = 1$.

(3) LATE under $P_{y,L}^*$ is

$$\begin{aligned}
E_{P_{y,L}^*}[Y_1 - Y_0 | T_{z_0}^* < T_{z_1}^*] &= E_{P_{y,L}^*}[Y_1 | T_{z_0}^* < T_{z_1}^*] - E_{P_{y,L}^*}[Y_0 | T_{z_0}^* < T_{z_1}^*] \\
&= E_P[Y] - E_P[Y | Y > y]
\end{aligned}$$

and LATE under $P_{y,U}^*$ is

$$\begin{aligned}
E_{P_{y,L}^*}[Y_1 - Y_0 | T_{z_0}^* < T_{z_1}^*] &= E_{P_{y,L}^*}[Y_1 | T_{z_0}^* < T_{z_1}^*] - E_{P_{y,L}^*}[Y_0 | T_{z_0}^* < T_{z_1}^*] \\
&= E_P[Y] - E_P[Y | Y < y].
\end{aligned}$$

They imply that LATE under the mixture distribution is equal to $E_P[Y] - \pi E_P[Y | Y > y] - (1 - \pi) E_P[Y | Y < y]$. \square

Now I will prove Theorem 4 for Case 1. Let θ be any real number. Since Y is unbounded, there are y_L and y_U with $y_L \leq E_P[Y] - \theta \leq y_U$ such that $P(Y < y_L) > 0$ and $P(Y > y_U) > 0$. Since

$$E_P[Y | Y < y_L] \leq E_P[Y] - \theta \leq E_P[Y | Y > y_U],$$

there is $\pi \in [0, 1]$ such that

$$\theta = E_P[Y] - \pi E_P[Y | Y > y_U] - (1 - \pi) E_P[Y | Y < y_L].$$

Using Lemma 5, the right hand side of the above equation is LATE under the mixture distribution $\pi P_{y_U, L}^* + (1 - \pi) P_{y_L, U}^*$. This proves that θ is LATE under some data generating process which is consistent with the observed distribution P .

B.2 Case 2: Positive total variation distance

Consider $TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0}) > 0$. This means that the instrumental variable Z has non-zero indirect effect on (R, Y, T) . Consider the following two data generating processes. First, P_L^* is defined by

$$\begin{aligned} Z &\sim P(Z = z) \\ (T_{z_0}^*, T_{z_1}^*) | Z &= (0, 1) \\ (R_0, Y_0, T_0) | (T_{z_0}^*, T_{z_1}^*, Z) &\sim f_{(R,Y,T)|Z=z_0} \\ (R_1, Y_1, T_1) | (T_{z_0}^*, T_{z_1}^*, Z) &\sim f_{(R,Y,T)|Z=z_1} \end{aligned}$$

where (R_1, Y_1, T_1) and (R_0, Y_0, T_0) are conditionally independent of $(T_{z_0}^*, T_{z_1}^*, Z)$. Second, P_U^* defined as follows. Define

$$H = 1\{\Delta f_{(R,Y,T)|Z}(R, Y, T) \geq 0\} - 1\{\Delta f_{(R,Y,T)|Z}(R, Y, T) < 0\}.$$

and define P_U^* as

$$\begin{aligned} Z &\sim P(Z = z) \\ (T_{z_0}^*, T_{z_1}^*) | Z &= \begin{cases} (0, 1) & \text{with probability } \Delta E_P[H | Z]/2 \\ (0, 0) & \text{with probability } P(H = -1 | Z = z_1) \\ (1, 1) & \text{with probability } P(H = 1 | Z = z_0) \end{cases} \\ (R_1, Y_1, T_1) | (T_{z_0}^*, T_{z_1}^*, Z) &\sim \begin{cases} \frac{\Delta f_{(R,Y,T,H)|Z}(r,y,t,1)}{\Delta E_P[H|Z]/2} & \text{if } T_{z_0}^* < T_{z_1}^* \\ \text{any distribution} & \text{if } T_{z_0}^* = T_{z_1}^* = 0 \\ f_{(R,Y,T)|H=1,Z=z_0}(r,y,t) & \text{if } T_{z_0}^* = T_{z_1}^* = 1 \end{cases} \\ (R_0, Y_0, T_0) | (T_{z_0}^*, T_{z_1}^*, Z) &\sim \begin{cases} -\frac{\Delta f_{(R,Y,T,H)|Z}(r,y,t,-1)}{\Delta E_P[H|Z]/2} & \text{if } T_{z_0}^* < T_{z_1}^* \\ f_{(R,Y,T)|H=0,Z=z_1}(y,t) & \text{if } T_{z_0}^* = T_{z_1}^* = 0 \\ \text{any distribution} & \text{if } T_{z_0}^* = T_{z_1}^* = 1 \end{cases} \end{aligned}$$

where (R_1, Y_1, T_1) and (R_0, Y_0, T_0) are conditionally independent of $(T_{z_0}^*, T_{z_1}^*, Z)$.

Lemma 6. *Consider the assumptions in Theorem 4. If $TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0}) > 0$, then*

1. P_L^* generates the data distribution P and LATE under P_L^* is equal to $\Delta E_P[Y | Z]$; and
2. P_U^* generates the data distribution P and LATE under P_U^* is equal to

$$\Delta E_P[Y | Z]/TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0}).$$

Proof. (1) Denote by f^* the density function of P_L^* . P_L^* generates the data distribution P :

$$\begin{aligned}
f_{(R,Y,T)|Z=z_0}^*(r, y, t) &= f_{(R,Y,T)|T^*=0, Z=z_0}^*(r, y, t) \\
&= f_{(R_0, Y_0, T_0)|T^*=0, Z=z_0}^*(r, y, t) \\
&= f_{(R,Y,T)|Z=z_0}(r, y, t) \\
f_{(R,Y,T)|Z=z_1}^*(r, y, t) &= f_{(R,Y,T)|T^*=1, Z=z_1}^*(r, y, t) \\
&= f_{(R_1, Y_1, T_1)|T^*=1, Z=z_1}^*(r, y, t) \\
&= f_{(R,Y,T)|Z=z_1}(r, y, t)
\end{aligned}$$

where the first equality uses $T^* = 1\{Z = z_1\}$. Under P_L^* , LATE is equal to $\Delta E_P[Y | Z]$:

$$\begin{aligned}
E_{P_L^*}[Y_1 - Y_0 | T_{z_0}^* < T_{z_1}^*] &= E_{P_L^*}[Y_1] - E_{P_L^*}[Y_0] \\
&= E_P[Y | Z = z_1] - E_P[Y | Z = z_0] \\
&= \Delta E_P[Y | Z].
\end{aligned}$$

(2) Denote by f^* the density function of P_U^* . When $\Delta E_P[Y | Z] = 0$, $f_{(R_{t^*}, Y_{t^*}, T_{t^*})|(T_{z_0}^*, T_{z_1}^*)}^*$ is positive on :

$$\begin{aligned}
\Delta f_{(R,Y,T,H)|Z}(r, y, t, 1) &= \Delta f_{(R,Y,T)|Z}(r, y, t) 1\{\Delta f_{(R,Y,T)|Z}(r, y, t) \geq 0\} \\
&\geq 0 \\
\Delta f_{(R,Y,T,H)|Z}(r, y, t, -1) &= \Delta f_{(R,Y,T)|Z}(r, y, t) 1\{\Delta f_{(R,Y,T)|Z}(r, y, t) < 0\} \\
&< 0.
\end{aligned}$$

P_U^* generates the data distribution P :

$$\begin{aligned}
f_{(R,Y,T)|Z=z_0}^*(r, y, t) &= P_U^*(T_{z_0}^* < T_{z_1}^* | Z = z_0) f_{(R,Y,T)|T_{z_0}^* < T_{z_1}^*, Z=z_0}^*(r, y, t) \\
&\quad + P_U^*(T_{z_1}^* = T_{z_0}^* = 0 | Z = z_0) f_{(R,Y,T)|T_{z_1}^* = T_{z_0}^* = 0, Z=z_0}^*(r, y, t) \\
&\quad + P_U^*(T_{z_1}^* = T_{z_0}^* = 1 | Z = z_0) f_{(R,Y,T)|T_{z_1}^* = T_{z_0}^* = 1, Z=z_0}^*(r, y, t) \\
&= P_U^*(T_{z_0}^* < T_{z_1}^*) f_{(R_0, Y_0, T_0)|T_{z_0}^* < T_{z_1}^*}^*(r, y, t) \\
&\quad + P_U^*(T_{z_1}^* = T_{z_0}^* = 0) f_{(R_0, Y_0, T_0)|T_{z_1}^* = T_{z_0}^* = 0}^*(r, y, t) \\
&\quad + P_U^*(T_{z_1}^* = T_{z_0}^* = 1) f_{(R_1, Y_1, T_1)|T_{z_1}^* = T_{z_0}^* = 1}^*(r, y, t) \\
&= -\frac{\Delta E_P[H | Z]}{2} \frac{\Delta f_{(R,Y,T,H)|Z}(r, y, t, -1)}{\Delta E_P[H | Z]/2} \\
&\quad + P(H = -1 | Z = z_1) f_{(R,Y,T)|H=-1, Z=z_1}(r, y, t) \\
&\quad + P(H = 1 | Z = z_0) f_{(R,Y,T)|H=1, Z=z_0}(r, y, t) \\
&= f_{(R,Y,T)|Z=z_0}(r, y, t).
\end{aligned}$$

$$\begin{aligned}
f_{(R,Y,T)|Z=z_1}^*(r, y, t) &= P_U^*(T_{z_0}^* < T_{z_1}^* \mid Z = z_1) f_{(R,Y,T)|T_{z_0}^* < T_{z_1}^*, Z=z_1}^*(r, y, t) \\
&\quad + P_U^*(T_{z_1}^* = T_{z_0}^* = 0 \mid Z = z_1) f_{(R,Y,T)|T_{z_1}^* = T_{z_0}^* = 0, Z=z_1}^*(r, y, t) \\
&\quad + P_U^*(T_{z_1}^* = T_{z_0}^* = 1 \mid Z = z_1) f_{(R,Y,T)|T_{z_1}^* = T_{z_0}^* = 1, Z=z_1}^*(r, y, t) \\
&= P_U^*(T_{z_0}^* < T_{z_1}^*) f_{(R_1, Y_1, T_1)|T_{z_0}^* < T_{z_1}^*}^*(r, y, t) \\
&\quad + P_U^*(T_{z_1}^* = T_{z_0}^* = 0) f_{(R_0, Y_0, T_0)|T_{z_1}^* = T_{z_0}^* = 0}^*(r, y, t) \\
&\quad + P_U^*(T_{z_1}^* = T_{z_0}^* = 1) f_{(R_1, Y_1, T_1)|T_{z_1}^* = T_{z_0}^* = 1}^*(r, y, t) \\
&= \Delta E_P[H \mid Z]/2 \frac{\Delta f_{(R,Y,T,H)|Z}(r, y, t, 1)}{\Delta E_P[H \mid Z]/2} \\
&\quad + P(H = -1 \mid Z = z_1) f_{(R,Y,T)|H=-1, Z=z_1}(r, y, t) \\
&\quad + P(H = 1 \mid Z = z_0) f_{(R,Y,T)|H=1, Z=z_0}(r, y, t) \\
&= f_{(R,Y,T)|Z=z_1}(r, y, t).
\end{aligned}$$

Under P_U^* , LATE is equal to $\Delta E_P[Y \mid Z]/TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0})$:

$$\begin{aligned}
&\Delta E_{P_U^*}[H \mid Z]/2 \\
&= \frac{1}{2} \int_{t=0,1} (1\{\Delta f_{(R,Y,T)|Z}(r, y, t) \geq 0\} - 1\{\Delta f_{(R,Y,T)|Z}(r, y, t) < 0\}) \Delta f_{(R,Y,T)|Z}(r, y, t) d\mu_Y(y) d\mu_R(r) \\
&= \frac{1}{2} \int_{t=0,1} |\Delta f_{(R,Y,T)|Z}(r, y, t)| d\mu_Y(y) d\mu_R(r) \\
&= TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0})
\end{aligned}$$

$$\begin{aligned}
E_{P_U^*}[Y_1 - Y_0 \mid T_{z_0}^* < T_{z_1}^*] &= \int_{t=0,1} y \frac{\Delta f_{(R,Y,T,H)|Z}(r, y, t, 1)}{\Delta E_P[H \mid Z]/2} + \frac{\Delta f_{(R,Y,T,H)}(y, t, -1)}{\Delta E_P[H \mid Z]/2} d\mu_Y(y) d\mu_R(r) \\
&= \int_{t=0,1} y \frac{\Delta f_{(R,Y,T)|Z}(r, y, t)}{\Delta E_P[H \mid Z]/2} d\mu_Y(y) d\mu_R(r) \\
&= \frac{\Delta E_P[Y \mid Z]}{\Delta E_P[H \mid Z]/2} \\
&= \frac{\Delta E_P[Y \mid Z]}{TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0})}.
\end{aligned}$$

□

Lemma 7. Consider the assumptions in Theorem 4. If $TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0}) > 0$, then, for every $\pi \in [0, 1]$,

1. the mixture distribution $\pi P_L^* + (1 - \pi)P_U^*$ satisfies Assumption 1;
2. the mixture distribution $\pi P_L^* + (1 - \pi)P_U^*$ generates the data distribution P ;
3. under the mixture distribution $\pi P_L^* + (1 - \pi)P_U^*$, LATE is equal to

$$\pi \Delta E_P[Y \mid Z] + (1 - \pi) \frac{\Delta E_P[Y \mid Z]}{TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0})}.$$

Proof. The proof for this lemma is the same as the proof in Lemma 4. □

Now I will prove Theorem 4 for Case 2. Let θ be any real number satisfying all the three inequalities in Theorem 4. Then there is $\pi \in [0, 1]$ such that

$$\theta = \pi \Delta E_P[Y | Z] + (1 - \pi) \frac{\Delta E_P[Y | Z]}{TV(f_{(R,Y,T)|Z=z_1}, f_{(R,Y,T)|Z=z_0})}.$$

Using Lemma 7, the right hand side of the above equation is LATE under the mixture distribution $\pi P_L^* + (1 - \pi) P_U^*$. This proves that θ is LATE under some data generating process which is consistent with the observed distribution P .

C Proofs of Theorems 2 and 3

These results are obtained by applying the results in Chernozhukov et al. (2014) for two independent samples. I use the same notation as them and modify their proofs to the two samples. First, I introduce the following lemma.

Lemma 8. (1) If x_1, x_2 are p -dimensional random variables and $0 < x_{2,j} \leq 1$, then

$$\max_{1 \leq j \leq p} x_{1,j} x_{2,j} \leq \max_{1 \leq j \leq p} x_{1,j}.$$

(2) If $X_1, \bar{X}_1, X_2, \bar{X}_2$ are p -dimensional random variables and if (X_1, \bar{X}_1) and (X_2, \bar{X}_2) are independent, then

$$\begin{aligned} \sup_{\mathbf{t} \in \mathbb{R}^p} |P(X_1 + X_2 \leq \mathbf{t}) - P(\bar{X}_1 + \bar{X}_2 \leq \mathbf{t})| &\leq \sup_{\mathbf{t} \in \mathbb{R}^p} |P(X_1 \leq \mathbf{t}) - P(\bar{X}_1 \leq \mathbf{t})| \\ &\quad + \sup_{\mathbf{t} \in \mathbb{R}^p} |P(X_2 \leq \mathbf{t}) - P(\bar{X}_2 \leq \mathbf{t})|. \end{aligned}$$

(3) If x_1, x_2 are p -dimensional random variables and $x_{2,j} > 0$, then

$$\left| \max_{1 \leq j \leq p} x_{1,j} - \max_{1 \leq j \leq p} x_{1,j} x_{2,j} \right| \leq \max_{1 \leq j \leq p} x_{1,j} \max_{1 \leq j \leq p} |x_{2,j} - 1|.$$

Proof. The first statement is as follows. If $\max_{1 \leq j \leq p} x_{1,j} \geq 0$, then

$$\max_{1 \leq j \leq p} x_{1,j} x_{2,j} = \max_{1 \leq j \leq p} x_{1,j} 1\{x_{1,j} > 0\} x_{2,j} \leq \max_{1 \leq j \leq p} x_{1,j} 1\{x_{1,j} > 0\} = \max_{1 \leq j \leq p} x_{1,j}.$$

If $\max_{1 \leq j \leq p} x_{1,j} < 0$, then

$$\max_{1 \leq j \leq p} x_{1,j} x_{2,j} = \min_{1 \leq j \leq p} (-x_{1,j}) x_{2,j} \leq \min_{1 \leq j \leq p} (-x_{1,j}) = - \max_{1 \leq j \leq p} x_{1,j}.$$

The second statement is as follows.

$$\begin{aligned}
& \sup_{\mathbf{t} \in \mathbb{R}^p} |P(X_1 + X_2 \leq \mathbf{t}) - P(\bar{X}_1 + \bar{X}_2 \leq \mathbf{t})| \\
& \leq \sup_{\mathbf{t} \in \mathbb{R}^p} |P(X_1 + X_2 \leq \mathbf{t}) - P(\bar{X}_1 + X_2 \leq \mathbf{t})| \\
& \quad + \sup_{\mathbf{t} \in \mathbb{R}^p} |P(\bar{X}_1 + X_2 \leq \mathbf{t}) - P(\bar{X}_1 + \bar{X}_2 \leq \mathbf{t})| \\
& = \sup_{\mathbf{t} \in \mathbb{R}^p} |E[P(X_1 + X_2 \leq \mathbf{t} \mid X_2) - P(\bar{X}_1 + X_2 \leq \mathbf{t} \mid X_2)]| \\
& \quad + \sup_{\mathbf{t} \in \mathbb{R}^p} |E[P(\bar{X}_1 + X_2 \leq \mathbf{t} \mid \bar{X}_1) - P(\bar{X}_1 + \bar{X}_2 \leq \mathbf{t} \mid \bar{X}_1)]| \\
& \leq \sup_{\mathbf{t} \in \mathbb{R}^p} |P(X_1 \leq \mathbf{t}) - P(\bar{X}_1 \leq \mathbf{t})| \\
& \quad + \sup_{\mathbf{t} \in \mathbb{R}^p} |P(X_2 \leq \mathbf{t}) - P(\bar{X}_2 \leq \mathbf{t})|.
\end{aligned}$$

The third statement is as follows. If $\max_{1 \leq j \leq p} x_{1,j} x_{2,j} \geq \max_{1 \leq j \leq p} x_{1,j} \geq 0$, then

$$\begin{aligned}
\max_{1 \leq j \leq p} x_{1,j} - \max_{1 \leq j \leq p} x_{1,j} x_{2,j} &= \max_{1 \leq j \leq p} x_{1,j} \max_{1 \leq j \leq p} \frac{x_{1,j} 1\{x_{1,j} > 0\}}{\max_{1 \leq j \leq p} x_{1,j} 1\{x_{1,j} > 0\}} x_{2,j} - 1 \\
&\leq \max_{1 \leq j \leq p} x_{1,j} \max_{1 \leq j \leq p} x_{2,j} - 1.
\end{aligned}$$

If $0 > \max_{1 \leq j \leq p} x_{1,j} x_{2,j} \geq \max_{1 \leq j \leq p} x_{1,j}$, then

$$\begin{aligned}
\max_{1 \leq j \leq p} x_{1,j} - \max_{1 \leq j \leq p} x_{1,j} x_{2,j} &= \max_{1 \leq j \leq p} x_{1,j} \left(1 - \min_{1 \leq j \leq p} \frac{(-x_{1,j})}{\min_{1 \leq j \leq p} (-x_{1,j})} x_{2,j} \right) \\
&\leq \max_{1 \leq j \leq p} x_{1,j} \left(1 - \min_{1 \leq j \leq p} x_{2,j} \right).
\end{aligned}$$

If $\max_{1 \leq j \leq p} x_{1,j} \geq \max_{1 \leq j \leq p} x_{1,j} x_{2,j} \geq 0$, then

$$\begin{aligned}
\max_{1 \leq j \leq p} x_{1,j} - \max_{1 \leq j \leq p} x_{1,j} x_{2,j} &= \max_{1 \leq j \leq p} x_{1,j} \left(1 - \frac{\max_{1 \leq j \leq p} x_{1,j} 1\{x_{1,j} > 0\} x_{2,j}}{\max_{1 \leq j \leq p} x_{1,j} 1\{x_{1,j} > 0\}} \right) \\
&\leq \max_{1 \leq j \leq p} x_{1,j} \left(1 - \frac{\max_{1 \leq j \leq p} x_{1,j} 1\{x_{1,j} > 0\} \min_{1 \leq j \leq p} x_{2,j}}{\max_{1 \leq j \leq p} x_{1,j} 1\{x_{1,j} > 0\}} \right) \\
&= \max_{1 \leq j \leq p} x_{1,j} \left(1 - \min_{1 \leq j \leq p} x_{2,j} \right).
\end{aligned}$$

If $0 > \max_{1 \leq j \leq p} x_{1,j} \geq \max_{1 \leq j \leq p} x_{1,j} x_{2,j}$, then

$$\begin{aligned}
\max_{1 \leq j \leq p} x_{1,j} - \max_{1 \leq j \leq p} x_{1,j} x_{2,j} &= \max_{1 \leq j \leq p} x_{1,j} \left(\frac{\max_{1 \leq j \leq p} (-x_{1,j}) x_{2,j}}{\max_{1 \leq j \leq p} (-x_{1,j})} - 1 \right) \\
&\leq \max_{1 \leq j \leq p} x_{1,j} \left(\frac{\max_{1 \leq j \leq p} (-x_{1,j}) \max_{1 \leq j \leq p} x_{2,j}}{\max_{1 \leq j \leq p} (-x_{1,j})} - 1 \right) \\
&= \max_{1 \leq j \leq p} x_{1,j} \max_{1 \leq j \leq p} x_{2,j} - 1.
\end{aligned}$$

□

Now I modify Chernozhukov et al. (2014) for two independent sample. In order to simplify the notations, this section focuses on a fixed value of θ and omits θ in the following discussion. All the following results are

uniformly valid in θ . Denote $X_{z,i,j} = g_{z,j}((Y_{z,i}, T_{z,i}), \theta)$, $\hat{\mu}_{z,j} = n_z^{-1} \sum_{i=1}^{n_z} X_{z,i,j}$ and $\hat{\mu}_j = \hat{\mu}_{1,j} - \hat{\mu}_{0,j}$. For every $J \subset \{1, \dots, p_n\}$, define

$$\begin{aligned} T(J) &= \max_{j \in J} \frac{\sqrt{n} \hat{\mu}_j}{\max\{\hat{\sigma}_j, \xi\}} \\ W(J) &= \max_{j \in J} \frac{\sqrt{n} \hat{\mu}_j^B}{\max\{\hat{\sigma}_j, \xi\}} \\ \bar{T}(J) &= \max_{j \in J} \frac{\sqrt{n}(\hat{\mu}_j - E[\hat{\mu}_j])}{\max\{\hat{\sigma}_j, \xi\}} \\ T_0(J) &= \max_{j \in J} \frac{\sqrt{n}(\hat{\mu}_j - E[\hat{\mu}_j])}{\max\{\sigma_j, \xi\}} \\ \bar{W}^B(J) &= \max_{j \in J} \frac{\sqrt{n} \hat{\mu}_j^B}{\max\{\sigma_j, \xi\}}. \end{aligned}$$

Denote by $c(\gamma, J)$ the conditional $(1 - \gamma)$ -quantile of $W(J)$ given $\{(Y_{z,i}, T_{z,i})\}$.

Denote by $\Sigma_{z,J}$ the variance-covariance matrix of $\{X_{z,j}\}_{j \in J}$. $\Sigma_J = n^{-1}(n_1 \Sigma_{1,J} + n_0 \Sigma_{0,J})$ is the variance-covariance matrix of $\{\sqrt{n} \hat{\mu}_j\}_{j \in J}$. Denote by

$$\Omega_{z,J} = \frac{n_z}{n} \max\{\text{diag}(\Sigma_J), \xi^2 I_{|J|}\}^{-1} \Sigma_{z,J}$$

where \max is the element-wise maximum. Let U_1 and U_0 be $|J|$ -dimensional independent normal random variable with

$$U_1 \sim N(0, \Omega_{1,J}) \text{ and } U_0 \sim N(0, \Omega_{0,J}).$$

Denote by $c_0(\gamma, J)$ the $(1 - \gamma)$ quantile of $\max_{j \in J}(U_{1,j} - U_{0,j})$. Define

$$\begin{aligned} \rho_{n,J} &= \sup_{t \in \mathbb{R}} P(T_0(J) \leq t) - P(\max_{j \in J}(U_{1,j} - U_{0,j}) \leq t) \\ \rho_{n,J}^B &= \sup_{t \in \mathbb{R}} P(\bar{W}^B(J) \leq t \mid \text{Data}) - P(\max_{j \in J}(U_{1,j} - U_{0,j}) \leq t) \\ \rho_{z,n,J} &= \sup_{\mathbf{t} \in \mathbb{R}^{|J|}} P \left(\frac{\sqrt{n_z}(\hat{\mu}_{z,j} - E[\hat{\mu}_{z,j}])}{\max\{\sigma_j, \xi\}} \leq \mathbf{t}_j, \forall j \in J \right) - P(U_{z,j} \leq \mathbf{t}_j, \forall j \in J) \\ \rho_{z,n,J}^B &= \sup_{\mathbf{t} \in \mathbb{R}^{|J|}} P \left(\frac{\sqrt{n_z} \hat{\mu}_{z,j}^B}{\max\{\sigma_j, \xi\}} \leq \mathbf{t}_j, \forall j \in J \mid \{(Y_{z,i}, T_{z,i})\} \right) - P(U_{z,j} \leq \mathbf{t}_j, \forall j \in J). \end{aligned}$$

Note that \mathbf{t} is a $|J|$ -dimensional vector in the definitions of $\rho_{z,n,J}$, and therefore I need to use the central limit and bootstrap theorems for hyper-rectangles, which is slightly different from Chernozhukov et al. (2014).

Note the following statements are taken from Chernozhukov et al. (2014) and Chernozhukov, Chetverikov, and Kato (2015).

Lemma 9. (1) There are positive numbers $c \in (0, 1/2)$ and $C > 0$ such that

$$\rho_{z,n,J} \leq Cn_z^{-c} \quad (8)$$

$$P(\rho_{z,n,J}^B < C\nu_{z,n}) \geq 1 - Cn_z^{-c} \text{ with some } \nu_{z,n} = Cn_z^{-c} \quad (9)$$

$$P \max_{j \in J} \frac{\sigma_{z,j}^2}{\hat{\sigma}_{z,j}^2} - 1 > n_z^{-1/2+c_1/4} B_{z,n_z}^2 \log(|J|) \leq Cn_z^{-c} \quad (10)$$

$$P \max_{j \in J} \frac{\sqrt{n_z}(\hat{\mu}_{z,j} - E[\hat{\mu}_{z,j}])}{\sigma_{z,j}} > n_z^{c_1/4} \sqrt{\log(|J|)} \leq Cn_z^{-c}. \quad (11)$$

(2) The following inequalities hold:

$$E_P \left[\max_{j \in J} \frac{\sqrt{n_z} \hat{\mu}_{z,j}^B}{\hat{\sigma}_{z,j}} \mid \text{Data} \right] \leq \sqrt{2 \log(2|J|)} \quad (12)$$

$$c_0(\gamma, J) \leq \sqrt{2 \log(|J|)} + \sqrt{2 \log(1/\gamma)} \quad (13)$$

$$P \max_{j \in J} (U_{1,j} - U_{0,j}) - c_0(\gamma, J) \leq \epsilon \leq 4\epsilon(\sqrt{\log(|J|)} + 1) \quad (14)$$

$$c(\gamma, J) \leq \sqrt{2 \log(|J|)} + \sqrt{2 \log(1/\gamma)}. \quad (15)$$

Proof. Note that

$$|g_{z,j}((Y_{z,i}, T_{z,i}), \theta) - E_P[g_{z,j}((Y_{z,i}, T_{z,i}), \theta)]| \leq |Y_{z,i} - E_P[Y_{z,i}]| + \max_{\theta \in \Theta} |\theta|.$$

By Assumptions 2, 3, and 4,

$$(M_{z,n_z,3}(\theta, P))^3 \vee M_{z,n_z,4}(\theta, P)^2 \vee B_{z,n_z}(\theta, P)^2 \log^{7/2}(p_n/n_z) \leq \frac{C_0 C_1}{\xi^2} n_z^{1/2-c_1},$$

where

$$M_{z,n_z,k}(\theta, P) = \max_{1 \leq j \leq p_n} E_P \left[\frac{g_{z,j}((Y_{z,i}, T_{z,i}), \theta) - E_P[g_{z,j}((Y_{z,i}, T_{z,i}), \theta)]}{\max\{\sigma_{z,j}(\theta), \xi\}} \right]^k \Bigg]^{1/k}$$

$$B_{z,n_z}(\theta, P) = E_P \left[\max_{1 \leq j \leq p_n} \frac{g_{z,j}((Y_{z,i}, T_{z,i}), \theta) - E_P[g_{z,j}((Y_{z,i}, T_{z,i}), \theta)]}{\max\{\sigma_{z,j}(\theta), \xi\}} \right]^4 \Bigg]^{1/4}.$$

This is the key assumption in Chernozhukov et al. (2014, Eq.(49)) and therefore we can borrow their results. The first statement is Theorem 2.1 in Chernozhukov et al. (2015). The second is Corollary 4.3 in Chernozhukov et al. (2015). The third and fourth are Step 3 of Theorem 4.3. The last two statements are Lemma A.4 in Chernozhukov et al. (2014). \square

Lemma 9 yields the following lemma in the two sample setting.

Lemma 10.

$$\rho_{n,J} \leq C(n_1^{-c} + n_0^{-c}) \quad (16)$$

$$P(\rho_{n,J}^B < \nu_{1,n} + \nu_{0,n}) \geq 1 - C(n_1^{-c} + n_0^{-c}) \quad (17)$$

$$P\left(\max_{j \in J} \frac{\max\{\sigma_j, \xi\}}{\max\{\hat{\sigma}_j, \xi\}} - 1 > (n_1^{-1/2+c_1/4} + n_0^{-1/2+c_1/4})B^2 \log(p_n)\right) \leq C(n_1^{-c} + n_0^{-c}) \quad (18)$$

$$P\left(\max_{j \in J} \frac{\sqrt{n}(\hat{\mu}_j - E[\hat{\mu}_j])}{\hat{\sigma}_j} > \left(\frac{n}{n_1}n_1^{c_1/4} + \frac{n}{n_0}n_0^{c_1/4}\right)\sqrt{\log(|J|)}\right) \leq C(n_1^{-c} + n_0^{-c}) \quad (19)$$

$$E_P[|W(J)| \mid \text{Data}] \leq 2\left(\frac{n}{n_1} + \frac{n}{n_0}\right)\sqrt{2\log(2|J|)}. \quad (20)$$

where $B^2 = C_0/\xi^2$.

Proof. Using the inequality in Lemma 8 (2), $\rho_{n,J} \leq \rho_{1,n} + \rho_{0,n}$ and $\rho_{n,J}^B \leq \rho_{1,n}^B + \rho_{0,n}^B$, which implies $\rho_{n,J} \leq Cn^{-c}$ and $P(\rho_{n,J}^B < Cn^{-c}) \geq 1 - Cn^{-c}$.

Using the inequality $|\sqrt{a} - 1| \leq |a - 1|$ for $a > 0$,

$$\begin{aligned} & \max_{j \in J} \frac{\max\{\sigma_j, \xi\}}{\max\{\hat{\sigma}_j, \xi\}} - 1 \\ &= \max_{j \in J} \frac{\hat{\sigma}_j}{\max\{\hat{\sigma}_j, \xi\}} \max\left\{\frac{\sigma_j}{\hat{\sigma}_j}, \frac{\xi}{\hat{\sigma}_j}\right\} - \max\left\{1, \frac{\xi}{\hat{\sigma}_j}\right\} \\ &\leq \max_{j \in J} \frac{\sigma_j}{\hat{\sigma}_j} - 1 \\ &\leq \max_{j \in J} \frac{\sigma_j^2}{\hat{\sigma}_j^2} - 1 \\ &= \max_{j \in J} \frac{\frac{n_1}{n}\sigma_{1,j}^2 + \frac{n_0}{n}\sigma_{0,j}^2}{\frac{n_1}{n}\hat{\sigma}_{1,j}^2 + \frac{n_0}{n}\hat{\sigma}_{0,j}^2} - 1 \\ &\leq \max_{j \in J} \frac{\frac{n_1}{n}\hat{\sigma}_{1,j}^2}{\frac{n_1}{n}\hat{\sigma}_{1,j}^2 + \frac{n_0}{n}\hat{\sigma}_{0,j}^2} \frac{\sigma_{1,j}^2}{\hat{\sigma}_{1,j}^2} - 1 + \max_{j \in J} \frac{\frac{n_0}{n}\hat{\sigma}_{0,j}^2}{\frac{n_1}{n}\hat{\sigma}_{1,j}^2 + \frac{n_0}{n}\hat{\sigma}_{0,j}^2} \frac{\sigma_{0,j}^2}{\hat{\sigma}_{0,j}^2} - 1 \\ &\leq \max_{j \in J} \frac{\sigma_{1,j}^2}{\hat{\sigma}_{1,j}^2} - 1 + \max_{j \in J} \frac{\sigma_{0,j}^2}{\hat{\sigma}_{0,j}^2} - 1. \end{aligned}$$

Applying the triangle inequality and $\sigma_j \geq \frac{\sqrt{n_z}}{\sqrt{n}}\sigma_{z,j}$

$$\begin{aligned} \max_{j \in J} \frac{\sqrt{n}(\hat{\mu}_j - E[\hat{\mu}_j])}{\max\{\sigma_j, \xi\}} &\leq \max_{j \in J} \frac{\frac{\sqrt{n}}{\sqrt{n_1}}\sqrt{n_1}(\hat{\mu}_{1,j} - E[\hat{\mu}_{1,j}])}{\max\{\sigma_j, \xi\}} + \max_{j \in J} \frac{\frac{\sqrt{n}}{\sqrt{n_0}}\sqrt{n_0}(\hat{\mu}_{0,j} - E[\hat{\mu}_{0,j}])}{\max\{\sigma_j, \xi\}} \\ &\leq \max_{j \in J} \frac{\frac{\sqrt{n}}{\sqrt{n_1}}\sqrt{n_1}(\hat{\mu}_{1,j} - E[\hat{\mu}_{1,j}])}{\frac{\sqrt{n_1}}{\sqrt{n}}\sigma_{1,j}} + \max_{j \in J} \frac{\frac{\sqrt{n}}{\sqrt{n_0}}\sqrt{n_0}(\hat{\mu}_{0,j} - E[\hat{\mu}_{0,j}])}{\frac{\sqrt{n_0}}{\sqrt{n}}\sigma_{0,j}} \\ &= \frac{n}{n_1} \max_{j \in J} \frac{\sqrt{n_1}(\hat{\mu}_{1,j} - E[\hat{\mu}_{1,j}])}{\sigma_{1,j}} + \frac{n}{n_0} \max_{j \in J} \frac{\sqrt{n_0}(\hat{\mu}_{0,j} - E[\hat{\mu}_{0,j}])}{\sigma_{0,j}}. \end{aligned}$$

Applying the triangle inequality

$$\begin{aligned}
|W(J)| &= \max_{j \in J} \frac{\frac{\sqrt{n}}{\sqrt{n_1}} \sqrt{n_1} \hat{\mu}_{1,j}^B - \frac{\sqrt{n}}{\sqrt{n_0}} \sqrt{n_0} \hat{\mu}_{0,j}^B}{\max\{\hat{\sigma}_j, \xi\}} \\
&\leq \max_{j \in J} \frac{\frac{\sqrt{n}}{\sqrt{n_1}} \sqrt{n_1} \hat{\mu}_{1,j}^B}{\max\{\hat{\sigma}_j, \xi\}} + \max_{j \in J} \frac{\frac{\sqrt{n}}{\sqrt{n_0}} \sqrt{n_0} \hat{\mu}_{0,j}^B}{\max\{\hat{\sigma}_j, \xi\}} \\
&= \frac{n}{n_1} \max_{j \in J} \frac{\sqrt{n_1} \hat{\mu}_{1,j}^B}{\hat{\sigma}_{1,n_1,j}} \frac{\sqrt{n_1} \hat{\sigma}_{1,n_1,j}}{\max\{\hat{\sigma}_j, \xi\}} + \frac{n}{n_0} \max_{j \in J} \frac{\sqrt{n_0} \hat{\mu}_{0,j}^B}{\hat{\sigma}_{0,n_0,j}} \frac{\sqrt{n_0} \hat{\sigma}_{0,n_0,j}}{\max\{\hat{\sigma}_j, \xi\}} \\
&\leq 2 \frac{n}{n_1} \max_{j \in J} \frac{\sqrt{n_1} \hat{\mu}_{1,j}^B}{\hat{\sigma}_{1,n_1,j}} + 2 \frac{n}{n_0} \max_{j \in J} \frac{\sqrt{n_0} \hat{\mu}_{0,j}^B}{\hat{\sigma}_{0,n_0,j}}.
\end{aligned}$$

□

Lemma 11. *If (θ, P) satisfies at least one of the inequalities in Theorem 3, then*

$$\max_{1 \leq j \leq p_n} \frac{\sqrt{n} \mu_j}{\max\{\sigma_j, \xi\}} \geq (1 + \delta)(1 + \epsilon) \sqrt{2 \log(\max\{p_n, \tau_n\} / (\alpha - 2\beta))}.$$

Proof. When one of the first two inequalities in Theorem 3 holds, this lemma is immediate. I will show the case when the last inequality holds. By Assumptions 5 and 7,

$$\max_{(y,t) \in I_{n,k}} \Delta f_{(Y,T)|Z}(y,t) - \min_{(y,t) \in I_{n,k}} \Delta f_{(Y,T)|Z}(y,t) \leq 2D_0 \left(2 \frac{D_1}{K_n}\right)^d = 2^{d+1} D_0 D_1^d K_n^{-d}.$$

Define $D = 2^{d+1} D_0 D_1^d$ and then

$$\max_{(y,t) \in I_{n,k}} \Delta f_{(Y,T)|Z}(y,t) - \min_{(y,t) \in I_{n,k}} \Delta f_{(Y,T)|Z}(y,t) \leq DK_n^{-d}.$$

for every $P \in \mathcal{P}$ and $k = 1, \dots, K_n$. Define

$$\begin{aligned}
h^* &= \arg \max_{h \in \mathbf{H}} \Delta E[h(Y, T) | Z] \\
h_{n,j^*} &= \arg \max_{h_{n,j}: 1 \leq j \leq 4^{K_n}} \Delta E[h(Y, T) | Z].
\end{aligned}$$

If $|\Delta f_{(Y,T)|Z}(y,t)| > DK_n^{-d}$ for some (y,t) with $y \in I_{n,k}$, then $h^*(y,t)$ is constant on $I_{n,k}$, and therefore $h^* = h_{n,j^*}$ is constant on $I_{n,k}$. Then, on every $I_{n,k}$, either $h^* = h_{n,j^*}$ or $|\Delta f_{(Y,T)|Z}(y,t)| \leq DK_n^{-d}$. It implies

$$\begin{aligned}
\Delta E_P[h^*(Y, T) - h_{n,j}(Y, T) | Z] &= \int_{t=0,1}^{K_n} (h^*(y,t) - h_{n,j}(y,t)) \Delta f_{(Y,T)|Z}(y,t) d\mu_Y(t) \\
&= \int_{k=1}^{K_n} \int_{t=0,1}^{I_{n,k}} (h^*(y,t) - h_{n,j}(y,t)) \Delta f_{(Y,T)|Z}(y,t) d\mu_Y(t) \\
&\leq \int_{k=1}^{K_n} \int_{t=0,1}^{I_{n,k}} DK_n^{-d} d\mu_Y(t) \\
&= 2DK_n^{-d} \mu_Y(\mathbf{Y})
\end{aligned}$$

and

$$\begin{aligned}
TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) &= \Delta E_P[h^*(Y, T) | Z] \\
&= \Delta E_P[h_{n,j^*}(Y, T) | Z] + \Delta E_P[h^*(Y, T) - h_{n,j^*}(Y, T) | Z] \\
&= \Delta E_P[h_{n,j^*}(Y, T) | Z] + 2DK_n^{-d}\mu_Y(\mathbf{Y}).
\end{aligned}$$

Then

$$\begin{aligned}
\sqrt{n}\mu_{j^*} &= \sqrt{n}(|\theta|\Delta E_P[h_{n,j^*}(Y, T) | Z]/2 - |\Delta E_P[Y | Z]|) \\
&= \sqrt{n}(|\theta|TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) - |\Delta E_P[Y | Z]|) \\
&\quad + |\theta|\sqrt{n}(\Delta E_P[h_{n,j^*}(Y, T) | Z]/2 - TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0})) \\
&\geq \sqrt{n}(|\theta|TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) - |\Delta E_P[Y | Z]|) - |\theta|\sqrt{n}2DK_n^{-d}\mu_Y(\mathbf{Y}) \\
&\geq \sqrt{n}(|\theta|TV(f_{(Y,T)|Z=z_1}, f_{(Y,T)|Z=z_0}) - |\Delta E_P[Y | Z]|) - \sqrt{n}\sup_{\theta \in \Theta} |\theta|2^{d+2}D_0D_1^dK_n^{-d}\mu_Y(\mathbf{Y}).
\end{aligned}$$

By the last inequality (7) in Theorem 3,

$$\frac{\sqrt{n}\mu_{j^*}}{\max\{\sigma_{j^*}, \xi\}} \geq (1 + \delta)(1 + \epsilon)\sqrt{2 \log(\max\{p_n, \tau_n\}/(\alpha - 2\beta))}.$$

□

Define

$$\begin{aligned}
\zeta_{n1} &= \frac{n}{n_1}n_1^{c_1/4} + \frac{n}{n_0}n_0^{c_1/4} (n_1^{-1/2+c_1/4} + n_0^{-1/2+c_1/4})B^2 \log^{3/2}(p_n) \\
\zeta_{n2} &= 8 \frac{\left(\frac{n}{n_1} + \frac{n}{n_0}\right)}{\left(\frac{n}{n_1}n_1^{c_1/4} + \frac{n}{n_0}n_0^{c_1/4}\right)}.
\end{aligned}$$

Lemma 12.

$$\begin{aligned}
P(|\bar{T}(J) - T_0(J)| > \zeta_{n1}) &\leq Cn^{-c} \\
P(P(|W(J) - \bar{W}^B(J)| > \zeta_{n1} | Data) > \zeta_{n2}) &\leq 2C(n_1^{-c} + n_0^{-c}).
\end{aligned}$$

Proof. Since

$$\begin{aligned}
\bar{T}(J) &= \max_{j \in J} \frac{\sqrt{n}(\hat{\mu}_j - E[\hat{\mu}_j])}{\max\{\hat{\sigma}_j, \xi\}} \\
&= \max_{j \in J} \frac{\sqrt{n}(\hat{\mu}_j - E[\hat{\mu}_j])}{\max\{\sigma_j, \xi\}} \frac{\max\{\sigma_j, \xi\}}{\max\{\hat{\sigma}_j, \xi\}} \\
T_0(J) &= \max_{j \in J} \frac{\sqrt{n}(\hat{\mu}_j - E[\hat{\mu}_j])}{\max\{\sigma_j, \xi\}},
\end{aligned}$$

Lemma 8 implies

$$|\bar{T}(J) - T_0(J)| \leq \max_{j \in J} \frac{\sqrt{n}(\hat{\mu}_j - E[\hat{\mu}_j])}{\max\{\sigma_j, \xi\}} \max_{j \in J} \frac{\max\{\sigma_j, \xi\}}{\max\{\hat{\sigma}_j, \xi\}} - 1$$

By Lemma 10, I have

$$P(|\bar{T}(J) - T_0(J)| > \zeta_{n1}) \leq 2C(n_1^{-c} + n_0^{-c}).$$

Since Lemma 8 implies

$$\begin{aligned} |W(J) - \bar{W}^B(J)| &\leq \max_{j \in J} \frac{\sqrt{n} \hat{\mu}_j^B}{\max\{\hat{\sigma}_j, \xi\}} \max_{j \in J} 1 - \frac{\max\{\hat{\sigma}_j, \xi\}}{\max\{\sigma_j, \xi\}} \\ &= |W(J)| \max_{j \in J} 1 - \frac{\max\{\hat{\sigma}_j, \xi\}}{\max\{\sigma_j, \xi\}}, \end{aligned}$$

Markov inequality implies

$$\begin{aligned} P(|W(J) - \bar{W}^B(J)| > \zeta_{n1} \mid Data) &\leq \frac{1}{\zeta_{n1}} E[|W(J) - \bar{W}^B(J)| \mid Data] \\ &\leq \frac{1}{\zeta_{n1}} E[|W(J)| \mid Data] \max_{j \in J} 1 - \frac{\max\{\hat{\sigma}_j, \xi\}}{\max\{\sigma_j, \xi\}} \\ &\leq \frac{1}{\zeta_{n1}} 2 \frac{n}{n_1} + \frac{n}{n_0} \sqrt{2 \log(2|J|)} \max_{j \in J} 1 - \frac{\max\{\hat{\sigma}_j, \xi\}}{\max\{\sigma_j, \xi\}} \\ &\leq \zeta_{n2} \frac{\max_{j \in J} 1 - \frac{\max\{\hat{\sigma}_j, \xi\}}{\max\{\sigma_j, \xi\}}}{(n_1^{-1/2+c_1/4} + n_0^{-1/2+c_1/4}) B^2 \log(|J|)}. \end{aligned}$$

Lemma 10 implies

$$P(P(|W(J) - \bar{W}^B(J)| > \zeta_{n1} \mid Data) > \zeta_{n2}) \leq 2C(n_1^{-c} + n_0^{-c}).$$

□

Define

$$\varphi_n = \zeta_{n2} + \nu_{1,n} + \nu_{0,n} + 4\zeta_{n1}(\sqrt{\log(p_n)} + 1).$$

Lemma 13.

$$c_0(\gamma + 4\zeta_{n1}(\sqrt{\log(|J|)} + 1), J) + \zeta_{n1} \leq c_0(\gamma, J)$$

$$P(c(\gamma, J) \geq c_0(\gamma + \varphi_n, J)) \geq 1 - 3C(n_1^{-c} + n_0^{-c}).$$

Proof. By Lemma 9, I have

$$P \max_{j \in J} (U_{1,j} - U_{0,j}) \leq t + \zeta_{n1} \leq P \max_{j \in J} (U_{1,j} - U_{0,j}) \leq t + 4\zeta_{n1}(\sqrt{\log(|J|)} + 1)$$

and therefore

$$c_0(\gamma + 4\zeta_{n1}(\sqrt{\log(|J|)} + 1), J) + \zeta_{n1} \leq c_0(\gamma, J).$$

If $\rho_{n,J}^B < \nu_{1,n} + \nu_{0,n}$ and $P(|W - \bar{W}| > \zeta_{n1} \mid Data) < \zeta_{n2}$, then

$$\begin{aligned}
P(W(J) \leq t \mid Data) &\leq P(\bar{W} \leq t + \zeta_{n1} \mid Data) + P(|W - \bar{W}| > \zeta_{n1} \mid Data) \\
&\leq P \max_{j \in J} (U_{1,j} - U_{0,j}) \leq t + \zeta_{n1} \mid Data \\
&\quad + \rho_{n,J}^B + P(|W - \bar{W}| > \zeta_{n1} \mid Data) \\
&\leq P \max_{j \in J} (U_{1,j} - U_{0,j}) \leq t + 4\zeta_{n1}(\sqrt{\log(|J|)} + 1) \\
&\quad + \rho_{n,J}^B + P(|W - \bar{W}| > \zeta_{n1} \mid Data) \\
&\leq P \max_{j \in J} (U_{1,j} - U_{0,j}) \leq t \\
&\quad + 4\zeta_{n1}(\sqrt{\log(|J|)} + 1) + \nu_{1,n} + \nu_{0,n} + \zeta_{n2} \\
&\leq P \max_{j \in J} (U_{1,j} - U_{0,j}) \leq t + \varphi_n
\end{aligned}$$

and, at $t = c_0(\gamma + \varphi_n, J)$,

$$P(W(J) \leq c_0(\gamma + \varphi_n, J) \mid Data) \leq 1 - \gamma.$$

Therefore

$$c(\gamma, J) \geq c_0(\gamma + \varphi_n, J)$$

if $\rho_{n,J}^B < \nu_{1,n} + \nu_{0,n}$ and $P(|W - \bar{W}| > \zeta_{n1} \mid Data) < \zeta_{n2}$. By Lemmas 10 and 12, this lemma is established. \square

Define $J_2 = \{j \in \{1, \dots, p_n\} : \sqrt{n}\mu_j / \max\{\sigma_j, \xi\} > -c_0(\beta + \varphi_n, \{1, \dots, p_n\})\}$ and I obtain the following lemma.

Lemma 14.

$$\begin{aligned}
P(J_2 \subset \hat{J}) &< \beta + \varphi_n + 4r_n \left(\sqrt{2 \log(p_n)} + \sqrt{-2 \log(\beta + \varphi_n)} \right) \sqrt{\log(p_n) + 1} \\
&\quad + \rho_{n,J} + 4C(n_1^{-c} + n_0^{-c}).
\end{aligned}$$

Proof. Define $r_n = 2(n_1^{-1/2+c_1/4} + n_0^{-1/2+c_1/4})B^2 \log(p_n)$ and consider $J = \{1, \dots, p_n\}$. Note that $J_2 \subset \hat{J}$ implies

$$\sqrt{n} \frac{\mu_j}{\max\{\sigma_j, \xi\}} > -c_0(\beta + \varphi_n, J) \text{ and } \frac{\sqrt{n}\mu_j}{\max\{\hat{\sigma}_j(\theta), \xi\}} > -2c(\beta, J) \text{ for some } j \in J.$$

If $c(\beta, J) \geq c_0(\beta + \varphi_n, J)$ and $\max_{j \in J} |(\max\{\hat{\sigma}_j, \xi\} / \max\{\sigma_j, \xi\}) - 1| \leq r_n/2$, then $J_2 \subset \hat{J}$ implies that

$$\sqrt{n}(\hat{\mu}_j - \mu_j) - (1 - r_n) \max\{\sigma_j, \xi\} c_0(\beta + \varphi_n, J) > 0 \text{ for some } j \in J.$$

Therefore

$$\begin{aligned}
& P(J_2 \subset \hat{J}) \\
& \leq P \max_{1 \leq j \leq p_n} \sqrt{n}(\hat{\mu}_j - \mu_j) / \max\{\sigma_j, \xi\} > (1 - r_n)c_0(\beta + \varphi_n, J) \\
& \quad + P(c(\beta, J) < c_0(\beta + \varphi_n, J)) \\
& \quad + P \max_{1 \leq j \leq p_n} |(\max\{\hat{\sigma}_j, \xi\} / \max\{\sigma_j, \xi\}) - 1| > r_n/2 \\
& = P \max_{j \in J} (U_{1,j} - U_{0,j}) > (1 - r_n)c_0(\beta + \varphi_n, J) \quad + \rho_{n,J} \\
& \quad + P(c(\beta, J) < c_0(\beta + \varphi_n, J)) \\
& \quad + P \max_{1 \leq j \leq p_n} |(\max\{\hat{\sigma}_j, \xi\} / \max\{\sigma_j, \xi\}) - 1| > r_n/2 \\
& \leq P \max_{j \in J} (U_{1,j} - U_{0,j}) > c_0(\beta + \varphi_n, J) \\
& \quad + P \max_{j \in J} (U_{1,j} - U_{0,j}) - c_0(\beta + \varphi_n, J) < r_n c_0(\beta + \varphi_n, J) \quad + \rho_{n,J} \\
& \quad + P(c(\beta, J) < c_0(\beta + \varphi_n, J)) \\
& \quad + P \max_{1 \leq j \leq p_n} |(\max\{\hat{\sigma}_j, \xi\} / \max\{\sigma_j, \xi\}) - 1| > r_n/2 \\
& = \beta + \varphi_n + P \max_{j \in J} (U_{1,j} - U_{0,j}) - c_0(\beta + \varphi_n, J) < r_n c_0(\beta + \varphi_n, J) \quad + \rho_{n,J} \\
& \quad + P(c(\beta, J) < c_0(\beta + \varphi_n, J)) \\
& \quad + P \max_{1 \leq j \leq p_n} |(\max\{\hat{\sigma}_j, \xi\} / \max\{\sigma_j, \xi\}) - 1| > r_n/2 \quad .
\end{aligned}$$

By Lemma 9,

$$\begin{aligned}
P \max_{j \in J} (U_{1,j} - U_{0,j}) - c_0(\beta + \varphi_n, J) < r_n c_0(\beta + \varphi_n, J) \\
\leq 4r_n c_0(\beta + \varphi_n, J) \sqrt{\log(p_n) + 1} \\
\leq 4r_n \left(\sqrt{2 \log(p_n)} + \sqrt{-2 \log(\beta + \varphi_n)} \right) \sqrt{\log(p_n) + 1}
\end{aligned}$$

and then

$$\begin{aligned}
P(J_2 \subset \hat{J}) & \leq \beta + \varphi_n + 4r_n \left(\sqrt{2 \log(p_n)} + \sqrt{-2 \log(\beta + \varphi_n)} \right) \sqrt{\log(p_n) + 1} + \rho_{n,J} \\
& \quad + P(c(\beta, J) < c_0(\beta + \varphi_n, J)) \\
& \quad + P \max_{1 \leq j \leq p_n} |(\max\{\hat{\sigma}_j, \xi\} / \max\{\sigma_j, \xi\}) - 1| > r_n/2 \quad .
\end{aligned}$$

By Lemmas 10 and 13, this lemma is established. \square

Proof of Theorem 2

First, I show

$$\begin{aligned}
P(T(J) > c(\gamma, J)) &\leq \gamma + \zeta_{n2} + \nu_n + 8\zeta_{n1}(\sqrt{\log(|J|)} + 1) + \rho_{n,J} \\
&\quad + P(c_0(\gamma + \varphi_n, J) > c(\gamma, J)) \\
&\quad + P(|\bar{T}(J) - T_0(J)| > \zeta_{n1})
\end{aligned} \tag{21}$$

for every $J \subset \{1, \dots, p_n\}$. Since $T(J) \leq \bar{T}(J)$,

$$\begin{aligned}
P(T(J) > c(\gamma, J)) &\leq P(\bar{T}(J) > c(\gamma, J)) \\
&\leq P(T_0(J) > c(\gamma, J) - \zeta_{n1}) + P(|\bar{T}(J) - T_0(J)| > \zeta_{n1}) \\
&\leq P(T_0(J) > c_0(\gamma + \varphi_n, J) - \zeta_{n1}) \\
&\quad + P(c_0(\gamma + \varphi_n, J) > c(\gamma, J)) + P(|\bar{T}(J) - T_0(J)| > \zeta_{n1}).
\end{aligned}$$

Using Lemma 13,

$$\begin{aligned}
P(T_0(J) > c_0(\gamma + \varphi_n, J) - \zeta_{n1}) \\
&\leq P(T_0(J) > c_0(\gamma + \varphi_n + 4\zeta_{n1}(\sqrt{\log(|J|)} + 1), J)) \\
&= P(T_0(J) > c_0(\gamma + \zeta_{n2} + \nu_n + 8\zeta_{n1}(\sqrt{\log(|J|)} + 1), J)) \\
&\leq P \max_{j \in J} (U_{1,j} - U_{0,j}) > c_0(\gamma + \zeta_{n2} + \nu_n + 8\zeta_{n1}(\sqrt{\log(|J|)} + 1), J) + \rho_{n,J} \\
&= \gamma + \zeta_{n2} + \nu_n + 8\zeta_{n1}(\sqrt{\log(|J|)} + 1) + \rho_{n,J}.
\end{aligned}$$

Next, I show

$$P \max_{j \notin J_2} \hat{\mu}_j \leq 0 > 1 - \beta - \varphi_n - \rho_{n,J}. \tag{22}$$

Since $\max_{j \notin J_2} \hat{\mu}_j > 0$ implies $\max_{1 \leq j \leq p_n} \sqrt{n}(\hat{\mu}_j - \mu_j) / \max\{\sigma_j, \xi\} > c_0(\beta + \varphi_n, \{1, \dots, p_n\})$, it follows that

$$\begin{aligned}
P \max_{j \notin J_2} \hat{\mu}_j > 0 &\leq P \max_{1 \leq j \leq p_n} \frac{\sqrt{n}(\hat{\mu}_j - \mu_j)}{\max\{\sigma_j, \xi\}} > c_0(\beta + \varphi_n, \{1, \dots, p_n\}) \\
&\leq P \max_{1 \leq j \leq p_n} (U_{1,j} - U_{0,j}) > c_0(\beta + \varphi_n, \{1, \dots, p_n\}) + \rho_{n,J} \\
&= \beta + \varphi_n + \rho_{n,J}.
\end{aligned}$$

Last, I show that the statement of this theorem $P(T \leq c^{2S}(\alpha)) \geq 1 - (\alpha - 2\beta) - Cn^{-c}$. If the following three statements are true

$$\begin{aligned}
T(J_2) &\leq c(\alpha - 2\beta, J_2) \\
\max_{j \notin J_2} \sqrt{n}\hat{\mu}_j / \max\{\hat{\sigma}_j, \xi\} &\leq 0 \\
J_2 &\subset \hat{J},
\end{aligned}$$

then I have

$$\begin{aligned}
T &= \max_{1 \leq j \leq p_n} \sqrt{n} \hat{\mu}_j / \max\{\hat{\sigma}_j, \xi\} \\
&= \max_{j \in J_2} \sqrt{n} \hat{\mu}_j / \max\{\hat{\sigma}_j, \xi\} \\
&= T(J_2) \\
&\leq c(\alpha - 2\beta, J_2) \\
&\leq c(\alpha - 2\beta, \hat{J}) \\
&= c^{2S}(\alpha).
\end{aligned}$$

By Lemmas 14 and Eq. (21) and (22), I have

$$\begin{aligned}
P(T > c^{2S}(\alpha)) &\leq \alpha - 2\beta + \varphi_n + \rho_{n,J} \\
&\quad + \zeta_{n2} + \nu_n + 8\zeta_{n1}(\sqrt{\log(|J|)} + 1) + \rho_{n,J} \\
&\quad + P(c_0(\gamma + \varphi_n, J) > c(\gamma, J)) \\
&\quad + P(|\bar{T}(J) - T_0(J)| > \zeta_{n1}) \\
&\quad + \varphi_n + 4r_n \left(\sqrt{2 \log(p_n)} + \sqrt{-2 \log(\beta + \varphi_n)} \right) \sqrt{\log(p_n) + 1} \\
&\quad + \rho_{n,J} + 4C(n_1^{-c} + n_0^{-c}).
\end{aligned}$$

Except for $\alpha - 2\beta$, all the terms on the right-hand side converges to 0 uniformly over P .

Proof of Theorem 3

Denote by $j^* = \arg \max_{1 \leq j \leq p_n} \mu_j / \max\{\sigma_j, \xi\}$. If the following four statements are true

$$\frac{\max\{\sigma_{j^*}, \xi\}}{\max\{\hat{\sigma}_{j^*}, \xi\}} - 1 < \delta \tag{23}$$

$$\frac{\sqrt{n}(\hat{\mu}_{j^*} - \mu_{j^*})}{\sigma_{j^*}} < (1 - \delta)\epsilon \sqrt{2 \log(\max\{p_n, \tau_n\}/(\alpha - 2\beta))} \tag{24}$$

$$\frac{\sqrt{n}\mu_{j^*}}{\max\{\sigma_{j^*}, \xi\}} \geq (1 + \delta)(1 + \epsilon) \sqrt{2 \log(\max\{p_n, \tau_n\}/(\alpha - 2\beta))} \tag{25}$$

$$c(\alpha - 2\beta, \{1, \dots, p_n\}) \leq \sqrt{2 \log(p_n)} + \sqrt{-2 \log(\alpha - 2\beta)}, \tag{26}$$

then $T > c^{2S}(\alpha)$, because

$$\begin{aligned}
T &\geq \frac{\sqrt{n}\hat{\mu}_{j^*}}{\max\{\hat{\sigma}_{j^*}, \xi\}} \\
&= \frac{\sqrt{n}\mu_{j^*}}{\max\{\hat{\sigma}_{j^*}, \xi\}} + \frac{\sqrt{n}(\hat{\mu}_{j^*} - \mu_{j^*})}{\max\{\hat{\sigma}_{j^*}, \xi\}} \\
&\geq \frac{1}{1+\delta} \frac{\sqrt{n}\mu_{j^*}}{\max\{\sigma_{j^*}, \xi\}} - \frac{1}{1-\delta} \frac{\sqrt{n}(\hat{\mu}_{j^*} - \mu_{j^*})}{\max\{\sigma_{j^*}, \xi\}} \\
&> \frac{1}{1+\delta} (1+\delta)(1+\epsilon) \sqrt{2 \log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} - \frac{1}{1-\delta} \frac{\sqrt{n}(\hat{\mu}_{j^*} - \mu_{j^*})}{\max\{\sigma_{j^*}, \xi\}} \\
&= (1+\epsilon) \sqrt{2 \log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} - \frac{1}{1-\delta} \frac{\sigma_{j^*}}{\max\{\sigma_{j^*}, \xi\}} \frac{\sqrt{n}(\hat{\mu}_{j^*} - \mu_{j^*})}{\sigma_{j^*}} \\
&\geq (1+\epsilon) \sqrt{2 \log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} - \epsilon \sqrt{2 \log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} \\
&= \sqrt{2 \log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} \\
&\geq \sqrt{2 \log(p_n)} + \sqrt{-2 \log(\alpha-2\beta)} \\
&\geq c(\alpha-2\beta, \{1, \dots, p_n\}) \\
&\geq c(\alpha-2\beta, \hat{J}) \\
&= c^{2S}(\alpha).
\end{aligned}$$

(23) holds at least with probability $1 - C(n_1^{-c} + n_0^{-c})$ from Lemma 10. (24) hold at least with probability approaching to one because, using the Markov inequality,

$$\begin{aligned}
P \frac{\sqrt{n}(\hat{\mu}_{j^*} - \mu_{j^*})}{\sigma_{j^*}} &> -(1-\delta)\epsilon \sqrt{\log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} \\
&= 1 - P \frac{\sqrt{n}(\hat{\mu}_{j^*} - \mu_{j^*})}{\sigma_{j^*}} \geq (1-\delta)\epsilon \sqrt{\log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} \\
&\geq 1 - \frac{1}{(1-\delta)^2 \epsilon^2 \log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} E \frac{\sqrt{n}(\hat{\mu}_{j^*} - \mu_{j^*})^2}{\sigma_{j^*}} \\
&= 1 - \frac{1}{(1-\delta)^2 \epsilon^2 \log(\max\{p_n, \tau_n\}/(\alpha-2\beta))}.
\end{aligned}$$

(25) hold by Lemma 11 and (26) holds by Lemma 9. Therefore

$$P(T > c^{2S}(\alpha)) \geq 1 - \frac{1}{(1-\delta)^2 \epsilon^2 \log(\max\{p_n, \tau_n\}/(\alpha-2\beta))} - C(n_1^{-c} + n_0^{-c}).$$

Since the right-hand side of the above equation does not depend on P , the uniform convergence in Theorem 3 follows.