University of California, Davis
Department of Economics
**Microeconomics**

Date: June 26, 2017
Time: 5 hours
Reading Time: 20 minutes

**PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE**

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Please answer **any three** of the following four questions
[If you answer all four questions, please indicate which three you want to be graded]

# Question 1[1]

Because of superb graduate education at our Department, you have been handpicked by The President to serve as his special assistant to his hotel business that operates informally out of the South Wing of the White House.[2] Your first task is to derive the profit maximizing inputs and output of his hotel business. You quickly learn that this president strictly prefers simple answers. So you consider a Cobb-Douglas production function with labor and capital as inputs. In terms of notation, output $q$ is given by

$$q = A z_1^{\alpha_1} z_2^{\alpha_2}$$

where $A, \alpha_1, \alpha_2, z_1, z_2 \geq 0$. We denote by $z_1$ and $z_2$ the amount of labor and capital, respectively.

a.) Unfortunately, the parameters $A$, $\alpha_1$, and $\alpha_2$ are only known to the IRS and the Russians. Lacking the right connections to either of them, you set out to estimate the parameters of the production function. How would you estimate it using OLS? Set up the "econometric model" and state the estimator.

b.) After firing up your (illegal) copy of Stata, you estimate parameters $A, \alpha_1$, and $\alpha_2$ and find that $\hat{A}, \hat{\alpha}_1, \hat{\alpha}_2 > 0$ with $\hat{\alpha}_1 + \hat{\alpha}_2 < 1$. Armed with these estimates, you set out to calculate profit maximizing inputs and output. The President has lamented about the competitiveness of the hotel business. Thus you feel safe to assume that his business is a price taker both in the factor and output markets. Denote by $p$ the price of the output and by $w_1$ and $w_2$ the prices of the inputs, respectively. Set up the profit maximization problem and derive step-by-step the conditional factor demand functions (i.e., factor demand conditional on optimal output) and the supply function.

c.) Derive also the profit function (as a function of factor prices and output price). (The President is really interested in profits. Moreover, it will be relevant in the latter parts of the question.)

d.) Very proud of your achievement, you quickly run back to the oval office to cheerfully present your results. After 140 seconds, The President dismisses your results as fake news. He sternly declares that you omitted a very very important variable. He won't tell you what it is, but he knows it is there, and you better get it right otherwise you are fired. In your despair, you ask around what it could be. Not surprisingly, Jean Speiser is unwilling to tell you. In this high-stress environment, you experience nightmares the moment you try to sleep. As always, Professor Schipper

[1] You may find it easier to start with parts g.) and h.).

[2] Names, characters, businesses, places, events and incidents are either the products of the author's imagination or used in a fictitious manner. Any resemblance to actual persons, living or dead, or actual events is purely coincidental.

appears in your dreams partying merrily between convex sets. He laughingly whispers one hint: "McKenzie (1959)". In the morning you awake with a bad headache and vaguely recall that "McKenzie (1959)" might have something to do with the entrepreneurial factor. It suddenly dawns on you that you essentially forgot The President in your calculation. You set up your new equation

$$q = A z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}, \tag{1}$$

where $z_3$ denotes the entrepreneurial factor. Initially you suspect that only the FBI and the Russian have data on $z_3$. Since you don't find it politically opportune to contact either of them at this point of time, you need some other way to figure out $\alpha_3$. With the entrepreneurial factor in mind, what number should you assume for $\alpha_3$? Explain in as much details as possible. Verify that the conditions under which one can apply McKenzie's proposition (see part g.) are satisfied in our case.

e.) Having set $\alpha_3$ to what McKenzie would have probably set it, you seem to struggle to figure out the supply $q$ because you don't know the price of the entrepreneurial factor, $w_3$. Argue that we can set

$$w_3 = (1 - \alpha_1 - \alpha_2) p A^{\frac{1}{1-\alpha_1-\alpha_2}} \left(\alpha_1 \frac{p}{w_1}\right)^{\frac{\alpha_1}{1-\alpha_1-\alpha_2}} \left(\alpha_2 \frac{p}{w_2}\right)^{\frac{\alpha_2}{1-\alpha_1-\alpha_2}}$$

(You should recognize this expression from earlier parts of the question. You are also allowed to make an assumption on $z_3$ but this assumption should be verbally justified.)

Armed with this insight, you rush back to the oval office and argue that both The President and you got it right. That is, it is true that you omitted the entrepreneurial factor but your earlier results on the optimal input of labor and capital and optimal supply are still correct. Verify this claim.

f.) The President turns to his trusted trade advisor, Piotr Novato, to verify your analysis. Being educated both in Cobb-Douglas production functions and constant returns to scale, he rejects your analysis claiming that it implies that The President's hotel business makes zero profits. This flies in the face of empirical evidence of strictly positive profits made by The President. As often, he is just confused but The President is already furious over alleged zero profits. How can you clear up the confusion? Explain.

g.) The President seems pleased to hear that after all he makes positive profits. He invites you to teach his staff about your analysis. You start by stating McKenzie's insight in general terms:

"For any convex production set $Y \subseteq \mathbb{R}^L$ with $\mathbf{0} \in Y$, there is a constant returns, convex production set $Y' \subseteq \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$."

Provide a proof of this proposition. (There is a Twitter-like proof in less than 140 characters.) (Hint: Think about the defining property of a production set with constant returns.)

h.) You continue with the following proposition: "If $y \in Y \subseteq \mathbb{R}^L$ is profit maximizing at $p \in \mathbb{R}^L_{++}$ then $(y, -1) \in Y' \subseteq \mathbb{R}^{L+1}$ is profit maximizing at $(p, \pi(p))$. Conversely, if $(y, -1) \in Y'$ is profit maximizing at $(p, p_{L+1})$, then $y \in Y$ is profit maximizing at $p$ and the profit is $p_{L+1}$." Prove each direction of this proposition.

# Question 2

Consider a smooth exchange economy with $I$ consumers and $L \geqslant 2$ goods, $\{\mathcal{I}, (u^i, w^i)_{i \in \mathcal{I}}\}$, but suppose that, unlike in class, for some reason the agents in this economy cannot exchange the commodities directly. Instead, some institutional arrangement forces them to trade *in bundles of commodities.*

There are $N$ bundles of commodities that they can trade, $n = 1, \dots, N$. Bundle $n$ is a vector $b^n = (b_1^n, \dots, b_L^n) \in \mathbb{R}^L$; it contains $b_\ell^n > 0$ units of commodity $\ell$. Denote by $q_n > 0$ the price of bundle $n$, and let $q = (q_1, \dots, q_N)$ be the vector of bundle prices.

Each individual can buy and sell these bundles at the given prices. Let $y_n^i$ denote the number of units of bundle $n$ bought by individual $i$, with the convention that this number is negative if the individual is actually selling the bundle. Denote by $y^i = (y_1^i, \dots, y_N^i)$ the individual's bundle demand. This demand results in consumption of commodities

$$x^i = w^i + \textstyle\sum_n y_n^i b^n,$$

where $w^i \in \mathbb{R}_{++}^L$ denotes the individual's endowment. Her budget constraint is that

$$q \cdot y^i = \textstyle\sum_n y_n^i q^n \leqslant 0,$$

which means that she can only afford positive expenditure in some bundles if she raises enough liquidity from the sales of other bundles.

A *competitive equilibrium in bundles* is a pair $(q, y)$, where $q$ is a vector of bundle prices and $y = (y^1, \dots, y^I)$ a profile of bundle demands such that

i. each individual is individually rational: for each $i$, bundle $y^i$ solves

$$\max_{\hat{y}} \left\{ u^i(w^i + \textstyle\sum_n \hat{y}_n b^n) : q \cdot \hat{y} \leqslant 0 \right\};$$

ii. all markets clear: $\sum_i y^i = 0$.

An *allocation of commodities* is (still) a profile $x = (x^1, \dots, x^I)$ of consumption bundles, such that $\sum_i x^i = \sum_i w^i$. Allocation of commodities $x$ is said to be *first best* if there does not exist an alternative allocation $\hat{x}$ such that $u^i(\hat{x}^i) \geqslant u^i(x^i)$ for all individuals, with strict inequality for some. It is said to be *second best* if there does not exist a profile of bundle demands $\hat{y} = (\hat{y}^1, \dots, \hat{y}^I)$ such that $\sum_i \hat{y}^i = 0$ and

$$u^i(w^i + \textstyle\sum_n \hat{y}_n^i b^n) \geqslant u^i(x^i)$$

for all individuals, with strict inequality for some.

With respect to this model:

1. Argue that if allocation $x \gg 0$ is first best, then

$$\frac{\partial_{x_\ell} u^i(x^i)}{\partial_{x_{\ell'}} u^i(x^i)} = \frac{\partial_{x_\ell} u^{i'}(x^{i'})}{\partial_{x_{\ell'}} u^{i'}(x^{i'})}$$

for all pairs of individuals and commodities.

2. Suppose that $b^1_\ell \geqslant 0$ for all $\ell$, with strict inequality for some. Argue that, given that all preferences are strictly increasing, if $(q, y)$ is a competitive equilibrium in bundles, then the resulting allocation of commodities, with

$$x^i = w^i + \textstyle\sum_n y^i_n b^n,$$

is second best.

3. You are now going to argue that the previous result cannot be extended to first best. Suppose that $L = 3$, and there are only two bundles that can be traded: $b^1 = (1, 0, 0)$ and $b^2 = (0, 1, 1)$.

   (a) Argue that at any (interior) competitive equilibrium allocation

   $$\begin{pmatrix} \partial_{x_1} u^i(x^i) \\ \partial_{x_2} u^i(x^i) + \partial_{x_3} u^i(x^i) \end{pmatrix} = \lambda^i \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

   where $\lambda^i > 0$ is a Lagrange multiplier, for every individual.

   (b) Argue that at any (interior) competitive equilibrium allocation

   $$\frac{\partial_{x_2} u^i(x^i)}{\partial_{x_1} u^i(x^i)} + \frac{\partial_{x_3} u^i(x^i)}{\partial_{x_1} u^i(x^i)} = \frac{\partial_{x_2} u^{i'}(x^{i'})}{\partial_{x_1} u^{i'}(x^{i'})} + \frac{\partial_{x_3} u^{i'}(x^{i'})}{\partial_{x_1} u^{i'}(x^{i'})}$$

   for all pairs of individuals.

   (c) *Intuitively*, argue that the latter condition is insufficient to guarantee that the allocation of commodities is first best.

# Question 3

Consider a standard production economy

$$\{\mathcal{I}, \mathcal{J}, (u^i, w^i)_{i\in\mathcal{I}}, (Y^j)_{j\in\mathcal{J}}, (s^{i,j})_{(i,j)\in\mathcal{I}\times\mathcal{J}}\}$$

where at least one of the agents has strictly monotone preferences. Recall that an *allocation* is a pair

$$(x, y) = (x^1, \dots, x^I, y^1, \dots, y^J) \in (\mathbb{R}^L_+)^I \times (\mathbb{R}^L)^J$$

such that $y^j \in Y^j$ for each $j$, and $\sum_i x^i = \sum_i w^i + \sum_j y^j$; and that allocation $(x, y)$ is *Pareto efficient* if there does not exist another allocation $(\hat{x}, \hat{y})$ such that $u^i(\hat{x}^i) \geqslant u^i(x^i)$ for all consumers, with strict inequality for some.

A profile of production plans $y = (y^1, \dots, y^J)$ is *feasible* if $y^j \in Y^j$ for each $j$; a feasible profile of production plans is *technically efficient* if there does not exist an alternative feasible plan $\hat{y}$ such that $\sum_j \hat{y}^j > \sum_j y^j$. Also, given a profile $y$ of production plans, a profile $x = (x^1, \dots, x^I)$ of consumption bundles is *feasible* if $\sum_i x^i = \sum_i w^i + \sum_j y^j$. Finally, feasible profile $x$ is *allocatively efficient*, given $y$, if there does not exist an alternative profile $\hat{x}$ that is also feasible given $y$ and such that $u^i(\hat{x}^i) \geqslant u^i(x^i)$ for all consumers, with strict inequality for some.

Given these definitions:

1. Argue that if $(x, y)$ is Pareto efficient, then profile $y$ is technically efficient (since one agent has strictly monotone preferences).
2. Argue that if $(x, y)$ is Pareto efficient, then profile $x$ is feasible and allocatively efficient given $y$.
3. In what follows you will argue that, even together, technical and allocative efficiency don't suffice to guarantee Pareto efficiency. Suppose that there are two commodities, two individuals and one firm. Both individuals have smooth utility functions, while the technology of the firm is

$$Y = \{(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_- \mid y_1 \leqslant f(-y_2)\},$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing and differentiable production function that transforms (input of) commodity 2 into units of (output of) commodity 1.

   (a) Argue that *any* pair $(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_-$ such that $y_1 = f(-y_2)$ is technically efficient.

(b) Argue that, given $(y_1, y_2)$, allocation $(x^1, x^2)$ is allocatively efficient only if

$$\frac{\partial_{x_1} u^1(x^1)}{\partial_{x_2} u^1(x^1)} = \frac{\partial_{x_1} u^2(x^2)}{\partial_{x_2} u^2(x^2)}.$$

(c) Argue that Pareto efficiency of $(x^1, x^2, y^1, y^2)$ requires that

$$\frac{\partial_{x_1} u^1(x^1)}{\partial_{x_2} u^1(x^1)} = \frac{\partial_{x_1} u^2(x^2)}{\partial_{x_2} u^2(x^2)} = \frac{1}{f'(y_2)},$$

and conclude that the fact that $(x^1, x^2, y^1, y^2)$ is both technically and allocatively efficient does not suffice for it to be Pareto efficient.

**Microeconomics Prelim** June 26, 2017

## QUESTION 4

A total of $n \geq 2$ car drivers wish to travel from *Start* to *End*. There are two routes (see Figure 1 below): one North, through city A and the other South, through city B. The travel time in minutes on the road from *Start* to city A is equal to the number of drivers who have chosen that route ($T_A$) divided by 10, while on the road from *Start* to city B it is a constant 45 minutes; then the road times switch from A and B to *End*: it is a constant 45 minutes from city A to *End*, while from city B to *End* it is equal to the number of drivers who have chosen that route ($T_B$) divided by 10. View this situation as a simultaneous game where each of the $n$ drivers simultaneously chooses whether to take the A route or the B route and each driver's objective is to **minimize** the amount of time it takes to go from *Start* to *End*.

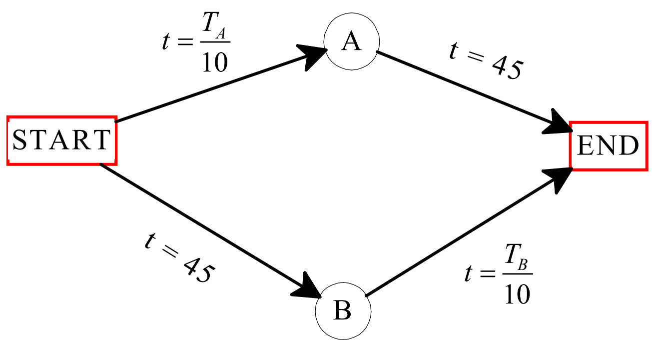


**Figure 1**

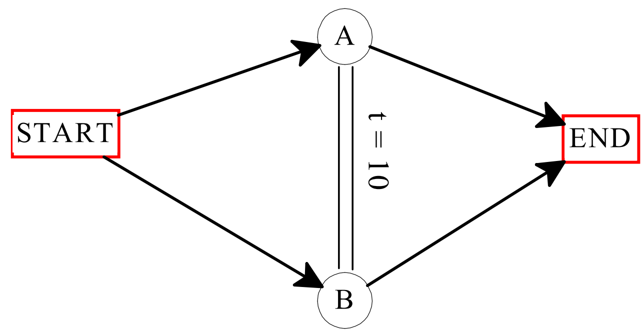


**Figure 2**

**(a)** **(a.1)** Find **all** the pure-strategy Nash equilibria of this game when $n$ is an odd number. [Prove your claim.]
**(a.2)** List all the pure-strategy Nash equilibria of this game when $n = 3$.

**(b)** Find **all** the pure-strategy Nash equilibria of this game when $n$ is an even number. [Prove your claim.]

**(c)** What is the average travel time (or the travel time per driver) at a Nash equilibrium when $n = 400$?

**From now on assume that $n = 400$.**

A two-way road has just been added between city A and city B and it takes 10 minutes to travel from one city to the other along this new road (see Figure 2 above). Thus each driver has now four choices: (1) go to A and continue on from there to *End* (call this route A), (2) go to A then drive to B along the new road and then continue from B to *End* (call this route AB), (3) go to B and continue from there on to *End* (call this route B), (4) go to B then drive to A along the new road and then continue from A to *End* (call this route BA). Travel times are as explained above; thus, for example, taking the AB route takes $\frac{T_A+T_{AB}}{10}+10+\frac{T_B+T_{AB}}{10}$ minutes, where $T_x$ is the number of drivers who take route $x$). Suppose that each driver has to irrevocably decide at the beginning (that is, at *Start*) which route to take (e.g. these are driverless cars that have to be programmed at the beginning of the journey).

**(d)** Are there any pure strategies that are strictly dominated?

**(e)** Find a pure-strategy Nash equilibrium (one is enough for this part).

**(f)** Comparing the average travel time at the Nash equilibrium of part (e) with the average travel time found in part (c), has the new road between cities A and B made the drivers better off ?

**(g)** Write a system of inequalities that are necessary and sufficient for finding all the pure-strategy Nash equilibria.

**(h)** Find **all** the pure-strategy Nash equilibria (that is, find all the solutions to the system of inequalities of part (g)).