I. Financing a public good

The private financing of public goods is made difficult by the well-known free-rider problem. In this problem we will show that a way to improve the level of a public good financed by private contributions is to use a lottery (in the layman's sense of the word).

Suppose, for the sake of the argument, that an economy includes I identical agents, each endowed with the following *ex post* utility function:

$$u(x_i, y) = x_i + \alpha \ln(y) \tag{1}$$

where x_i is the consumption by agent i of an aggregate private good we will refer to as the numeraire, y is the level of public good, and $\alpha > 0$ is a positive parameter reflecting the intensity of agents' taste for the public good. Each agent enjoys the same level y of public good, and consumption of the public good by one agent does not diminish consumption by another (think of clean air or national security). We assume that the public good is financed through contributions of private good, one unit of private good giving one unit of public good (constant returns to scale). Each agent has $\omega > 0$ units of private good to start with and, when facing uncertain outcomes, satisfies the expected utility hypothesis.

The problem is structured as follows: first, we will derive the socially optimal level of public good for this economy. Second, we will show that if each agent is asked to contribute voluntarily to the financing of the public good, in equilibrium there will be too little public good compared to the social optimal. Finally, we will show that the situation can be improved through a lottery.

1. We define the socially optimal level of public good as the value y^* that solves the following constrained optimization problem, sometimes referred to as the social planner problem:

$$\max_{\substack{x_i \geq 0\ \forall i \\ y \geq 0}} \sum_{i=1}^{I} u(x_i, y) \quad \text{subject to} \quad y + \sum_{i=1}^{I} x_i \leq I\omega. \tag{2}$$

(<u>Note</u>: the symbol $\forall$ means "for all.")

[0.2] (a) Do you recognize the type of preferences represented by the utility function in (1)?

> **Solution:** They are quasi-linear with respect to the private good.

[0.4] (b) Look at program (2) closely and interpret it in words.

> **Solution:** Program 2 maximizes the sum of agents' utilities subject to the technical feasibility constraint that the private good left over from agents' own consumption is used to produce the public good.

[0.3] (c) Do the first-order conditions to program (2) fully characterize its solution? Briefly justify your answer.

Solution: Yes because the objective is concave and the constraint set is convex (the constraint being linear).

[0.6] (d) Ignoring non-negativity constraints, solve program (2) and show that the socially optimal level of public good is $y^* = \alpha I$. What happens to y^* as I increases?

Solution: The first-order condition with respect to x_i gives $1 - \lambda = 0$ for all $i = 1, \ldots, I$, where λ is the Lagrange multiplier on the technical feasibility constraint. The first-order condition with respect to y gives $\frac{I\alpha}{y} - \lambda = 0$, which yields $y^* = \alpha I$. The size of the public good rises linearly with the number of individuals in the economy and with their relative preference for the public good.

2. We now consider a mechanism by which each agent decides to contribute some of his endowment of private good ω towards the financing of the public good, taking as given the contributions of all the other agents. That is, we are considering a Nash equilibrium where the strategy of each agent is how much to contribute to the public good. We denote by z_i the contribution of agent i, and by Z_{-i} the sum of the contributions of all other agents. Agent i chooses z_i in order to solve the following optimization problem:

$$\max_{0 \leq z_i \leq \omega} u(\omega - z_i, z_i + Z_{-i}). \tag{3}$$

We are only interested in symmetric outcomes where all agents' contributions are the same.

[0.3] (a) Look at program (3) closely and interpret it in words.

Solution: Agent i maximizes his own utility subject to the constraint that his private consumption is limited to his endowment minus his contribution to the public good. He correctly anticipates the amount of public good provided to be $z_i + Z_{-i}$.

[0.4] (b) Ignoring non-negativity constraints, solve program (3). That is, find the best response of agent i, denoted $\bar{z}_i(Z_{-i})$.

Solution: The first-order condition gives $\bar{z}_i(Z_{-i}) = \alpha - Z_{-i}$.

[0.4] (c) Using symmetry and your answer to part (b), show that the level of public good that arises in equilibrium is $\bar{y} = \alpha$. Compare to 1(d) and discuss the difference.

Solution: Using symmetry, it must be that $Z_{-i} = (I-1)z_i$ and thus $\bar{y} = I\bar{z}_i = \alpha$. Unless $I = 1$, there is less public good provided under this equilibrium with voluntary contributions than the socially optimal amount.

3. Now suppose that the government decides to organize a lottery to help finance the public good. The government issues as many lottery tickets as agents demand, and the price of one ticket

is one unit of numeraire. The government uses the receipts from the lottery to cover the cost of a prize equal to R units of numeraire, with $R < \alpha I$. The rest is used to finance the public good. (That is, if there are Z lottery tickets sold there will be $Z - R$ units of public good produced.) Finally, if one agent purchases z_i lottery tickets and other agents purchase Z_{-i} tickets in aggregate, the probability that agent i wins the prize is simply $\frac{z_i}{z_i+Z_{-i}}$. If he wins, agent i's consumption of numeraire is $\omega - z_i + R$. If he loses, agent i's consumption of numeraire is $\omega - z_i$.

[0.6] (a) Modify program (3) to reflect the new opportunities afforded by the lottery. (Hint: your program should now be an expected utility maximization program. Simplify the objective function as much as you can.)

> **Solution:** The new program is
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> $$\max_{0\leq z_i\leq\omega} \frac{z_i}{z_i+Z_{-i}} u(\omega - z_i + R, z_i + Z_{-i} - R) + \left(1 - \frac{z_i}{z_i+Z_{-i}}\right) u(\omega - z_i, z_i + Z_{-i} - R),$$
>
> that is,
>
> $$\max_{0\leq z_i\leq\omega} \omega - z_i + \alpha \ln(z_i + Z_{-i} - R) + \frac{z_i}{z_i+Z_{-i}} R.$$

[0.6] (b) Assuming that $0 < z_i < \omega$, show, using the first-order conditions of the program identified in part (a), that the new best response $\tilde{z}_i$ satisfies the following equality:

$$-1 + \frac{RZ_{-i}}{(\tilde{z}_i + Z_{-i})^2} + \frac{\alpha}{\tilde{z}_i + Z_{-i} - R} = 0. \tag{4}$$

> **Solution:** The first-order condition becomes $-1 + \frac{\alpha}{z_i+Z_{-i}-R} + \frac{RZ_{-i}}{(z_i+Z_{-i})^2} = 0$.

[0.6] (c) Using equation (4) and symmetry, show that the equilibrium level of public good $\tilde{y}$ satisfies the following polynomial equation:

$$-(\tilde{y})^2 I + (\alpha I - R)\tilde{y} + \alpha I R = 0. \tag{5}$$

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> **Solution:** We have that $y = z_i + Z_{-i} - R$, and in addition symmetry implies that $Z_{-i} = (I-1)z_i$. Therefore, the first-order condition can be rewritten as $-1 + \frac{R(I-1)(\tilde{y}+R)}{I(\tilde{y}+R)^2} + \frac{\alpha}{\tilde{y}} = 0$, that is, $-1 + \frac{R(I-1)}{I(\tilde{y}+R)} + \frac{\alpha}{\tilde{y}} = 0$. Multiplying through by $I\tilde{y}(\tilde{y}+R)$, we get $-I(\tilde{y})^2 - I\tilde{y}R + R(I-1)\tilde{y} + \alpha I(\tilde{y}+R) = 0$, that is, $-I(\tilde{y})^2 - R\tilde{y} + \alpha I\tilde{y} + \alpha IR = 0$, which gives the result.

[0.6] (d) Using equation (5), show that $\bar{y} < \tilde{y} < y^*$ and provide intuition for this result. (Hint: Try to plot the polynomial function in equation (5) on a graph with y on the horizontal axis.)

Solution: The function in equation (5) is an inverted parabola in y. There is only one positive root (equal to $\tilde{y}$) given the sign of the coefficients on the polynomial. If $y = 0$, its value is αIR which is positive. If $y = \bar{y} = \alpha$, its value is $-\alpha^2 I + \alpha^2 I - \alpha R + \alpha IR = \alpha(I-1)R$ which is positive. If $y = y^* = \alpha I$, its value is $-\alpha^2 I^3 + \alpha^2 I^2 - \alpha IR + \alpha IR = -\alpha^2 I^2 (I-1)$ which is negative. Therefore, we must have $\bar{y} < \tilde{y} < y^*$. That is, the lottery provides an additional incentive to contribute relative to the voluntary contribution. This is because the marginal benefit of contributing to the public good includes not only the effect on public good size but also an increase in the probability of winning the prize and thus of increasing the ultimate consumption of private good. However, this additional "push" falls short of resulting in a socially optimal level of public good.