Question 1.

Let there be $L$ goods (which may be interpreted as varieties of a particular product, with the understanding that the consumer may wish to consume many varieties), with price vector $(p_1, \ldots, p_L) \in \mathbb{R}^L$. Define the arithmetic mean of prices as

$$P^A(p_1, \ldots, p_L) = \frac{\sum_{k=1}^{L} p_k}{L},$$

and the quadratic mean of prices as

$$P^Q(p_1, \ldots, p_L) = \sqrt{\frac{\sum_{k=1}^{L} [p_k]^2}{L}},$$

a convex function of $(p_1, \ldots, p_L)$.

Consider the function:

$$\theta : \mathbb{R}^L_+ \to \mathbb{R} : \theta(p_1, \ldots, p_L) = P^A(p_1, \ldots, p_L) + c\left[ P^A(p_1, \ldots, p_L) - P^Q(p_1, \ldots, p_L) \right],$$

where the parameter $c$ is a nonnegative real number. Define

$$f : \mathbb{R}^{L+1}_+ \to \mathbb{R} : f(p_1, \ldots, p_L, u) = \theta(p_1, \ldots, p_L)u.$$

1.1. Show that $P^A(p_1, \ldots, p_L) = P^Q(p_1, \ldots, p_L) = p$ and that, in general, $P^Q \geq P^A$. (Remark. Here and in what follows, $p$ denotes a positive real number, not a vector.)

1.2. Show that $f$ displays all the properties of an expenditure function (i.e., of the value function of the consumer’s expenditure minimization problem), except possibly being increasing in $u$ or in some prices.
1.3. Let \( c = 0 \). Argue that in this case \( f \) is a legitimate expenditure function. What type of preferences generate this expenditure function?

1.4. We return to the general case where \( c > 0 \), but we assume that \( c \) and \((p_1, \ldots, p_L)\) are such that \( f \) satisfies all the properties of a legitimate expenditure function. What can you say about the underlying preferences?

1.5. We maintain the assumptions that \( c \geq 0 \), and that \( c \) and \((p_1, \ldots, p_L)\) are such that \( f \) satisfies all the properties of a legitimate expenditure function. Obtain the Walrasian demand function for good \( j \).

1.6. Assume now, as it is often assumed in the monopolistic competition literature, that when expressing her demand for good \( j \), the consumer perceives the price indices \( P^A \) and \( P^Q \) as given, i.e., independent of the single price \( p_j \): such a perception is approximately justified if \( L \) is large and \( p_j \) is not too large relative to the other prices. For given wealth \( w \), graph the resulting Walrasian demand curve for good \( j \) in the \((x_j, p_j)\) space.

1.7. Compute the price elasticity of the demand curve obtained in 1.6 above. What is the value of this elasticity at a point where the prices of all goods are the same (as is often the case at the equilibria of symmetric models of monopolistic competition)?

1.8. Comment, in particular comparing your answers to questions 1.4 and 1.6 above.
Question 2.
Postulate the expected utility hypothesis, and assume the decision maker’s von Neumann-Morgenstern-Bernoulli (vNMB) utility function is

\[ u : X \rightarrow \mathbb{R} : u(x) = b \left[ a + \frac{x}{c} \right]^{1-c} \text{,} \quad (2.1) \]

where \( a, b \) and \( c \) are real numbers, \( c \neq 0 \), satisfying

\[ \frac{b[1-c]}{c} > 0 \text{,} \quad (2.2) \]

and the domain \( X \) is defined as

\[ X := \{x \in \mathbb{R}++ : a + \frac{x}{c} > 0\} \] . \quad (2.3)

2.1. Argue that the decision maker is strictly risk-averse.

2.2. Show that the preferences described by the following vNMB utility functions are special cases of the ones defined by (2.1)

(A). \( u^\eta : X \rightarrow \mathbb{R} : u^\eta(x) = \frac{x^{1-\eta}}{1-\eta}, X = \mathbb{R}++ , \eta > 0 , \eta \neq 1. \) \quad (2.4)

(B). \( u^N : X \rightarrow \mathbb{R} : u^N(x) = \frac{[x+k]^{1-\eta}}{1-\eta}, X = \{x \in \mathbb{R}++ : x > -k\}, k \in \mathbb{R} , \eta > 0 , \eta \neq 1. \) \quad (2.5)

(C). \( u^Q : X \rightarrow \mathbb{R} : u^Q(x) = -(g-x)^2 , X = \{x \in \mathbb{R}++ : x < g\}. \) \quad (2.6)

2.3. Show that the preferences described by the following vNMB utility function are a limit case of the ones defined by (2.4)

(D). \( u^\theta : X \rightarrow \mathbb{R} : u^\theta(x) = \ln x, X = \mathbb{R}++ . \) \quad (2.7)

2.4. For which values of the parameters \( a, b, c \) does the vNMB of (2.1) display:

- Increasing Absolute Risk Aversion?
- Decreasing Absolute Risk Aversion?
• Increasing Relative Risk Aversion?

• Decreasing Relative Risk Aversion?

Illustrate when possible by referring to the utility functions (2.4)-(2.7)

2.5. Postulate that nature randomly chooses one of two “states or the world”, \( s_1 \) or \( s_2 \). State \( s_1 \) is the bad state, which occurs with probability \( \pi \), whereas \( s_2 \) is the good state, which occurs with probability \( 1 - \pi \). For \( j = 1, 2 \), denote by \( x_j \) consumption contingent to state \( j \) and by \( P_j \) the price of consumption contingent to state \( j \). Assume that \( P_1 / P_2 > \pi / [1 - \pi] \). Denote by \( W \) the wealth of the consumer. Consider only the case where the solution to the consumer optimization problem is interior to the consumption set, which is defined by the property that both \( x_1 \) and \( x_2 \) must belong to the domain of the relevant vNMB utility function

2.5.1. For the vNMB utility function (2.1), show that the wealth expansion path is a straight line, with positive slope, in \((x_1, x_2)\) space.

2.5.2. For the vNMB utility functions given by (2.4) to (2.7), does an increase in the wealth of the consumer lead her to bear absolutely more risk? Relatively more risk? Explain.

2.6. Assume a population of individuals that have different wealth levels but are otherwise identical, with identical preferences defined by a vNMB utility function of the form (2.1), facing the same probability \( \pi \) of the bad state, and the same prices \( P_1 \) of \( P_2 \) for the contingent consumption commodities. Does their aggregate demand for the contingent consumption commodities admit a positive representative consumer? Argue your answer.
Question 3

When some agents are either envious or altruistic, their utility depends not only on their own consumption but also on the consumption of other agents. Intuition might suggest that such external effects will lead to inefficiency of the market system. Let us show that this intuition may not always be correct.

(a) Let us first study envy. Consider a two-agent, two-good economy in which the agents have endowments $\omega^1 = (\omega^1_x, \omega^1_y)$, $\omega^2 = (\omega^2_x, \omega^2_y)$. Let $w = (w_x, w_y) = (\omega^1_x + \omega^2_x, \omega^1_y + \omega^2_y)$ denote the aggregate endowment.

(i) Suppose that the utility functions of the two agents are

$$u_1(x_1, y_1, x_2, y_2) = \ln(x_1 y_1) - a \ln(x_2 y_2), \quad a > 0, \quad u_2(x_2, y_2) = \ln(x_2 y_2)$$

Thus agent 1 is envious of agent 2: the higher the utility of agent 2, the lower the utility of agent 1. Agent 2 only cares about her own consumption. Write the constrained maximum problem which gives the Pareto optima of this economy, maximizing the utility of agent 1 subject to a guaranteed utility for agent 2 and the feasibility constraints. Show that the guaranteed utility constraint for agent 2 is always binding. Find the FOCs which must be satisfied at a Pareto optimum.

(ii) Show that the competitive equilibria of this economy, in which each agent takes prices and the action of the other agent as given are the same as the competitive equilibria of the economy in which both agents have Cobb Douglas utility i.e. the utility of agent 2 is the same as in (i), and agent 1’s utility is $\tilde{u}_1(x_1, y_1) = \ln(x_1 y_1)$. [No need to calculate to show this.] Thus competitive equilibria exist for all possible endowments.

(iii) Show that a competitive equilibrium of the economy in (i) satisfies the FOCs for Pareto optimality. Explain why you cannot deduce from this that the competitive equilibrium is Pareto optimal.

(iv) Thus to establish Pareto optimality we need to do a direct proof. Let us show this in a slightly more general setting. Assume that the first agent has a utility function of the form

$$u_1(x_1, y_1, x_2, y_2) = h(\tilde{u}_1(x_1, y_1), u_2(x_2, y_2))$$

where $u_2$ is the utility function of agent 2, and $\tilde{u}_1$ denotes the utility that agent 1 derives from his own consumption. $\tilde{u}_1$ and $u_2$ are increasing. The function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is increasing in the first variable (the own consumption of the agent) and decreasing in the second variable (the consumption of the other agent). Prove that a competitive equilibrium is Pareto optimal. [Hint: Appropriately modify the usual proof by contradiction.] Thus envy by agent 1 does not lead to inefficiency of a competitive equilibrium.
(b) Let us now study *altruism*.

(i) Consider the same two-agent economy as in a(i) but now assume that $a < 0$: since it is easier to work with positive parameters you may use $b = -a$ as the parameter in the utility of agent 1 who is now altruistic. Consider the maximum problem characterizing the Pareto optima of the economy. Show that there exists a utility level $v_2^*$ such that if $v_2 < v_2^*$ then the constraint $u_2(x_2, y_2) \geq v_2$ is not binding. Conclude that there is no Pareto optimal allocation for which $u_2(x_2, y_2) < v_2^*$. Explain the intuition for this result.

(ii) Use (i) to show that the First Theorem of Welfare Economics does not hold in economies in which agents are altruistic. To make the argument, you can use the Edgeworth box for the economy $E(u_1, u_2, \omega_1, \omega_2)$.

(iii) What type of government intervention would increase efficiency when the competitive equilibrium is not Pareto optimal?
Consider an economy with a private and a public good in which agents have different tastes for the public good: we study an example where the tastes give rise to an extreme form of the free-rider problem. To fix ideas suppose there are $I \geq 2$ agents indexed by $i = 1, \ldots, I$. Each agent initially owns one unit of the private good. The private good can be transformed into a public good with a constant returns technology, one unit of private good producing one unit of public good. The preferences of the agents are given by

$$u_i(x_i, y) = x_i - \frac{a_i}{y}, \quad i = \ldots, I$$

where $x_i$ is agent $i$’s consumption of the private good and $y$ is the quantity of the public good in the economy, and the preference parameters $(a_i)_{i=1}^{I}$ satisfy

$$0 < a_I < \ldots < a_2 < a_1 < 1$$

(a) To determine the equilibrium with voluntary contributions we first study each agent’s optimal contribution (from her point of view). Let $z_i$ denote agent $i$’s contribution and let $Z^{-i} = \sum_{j \neq i} z_j$ denote the contribution of all other agents. Derive the optimal contribution of agent $i$ given $Z^{-i}$.

(b) Show that there is a unique voluntary contributions equilibrium in which agent 1 is the only agent contributing to the provision of the public good: all other agents free ride on agent 1’s contribution. [Hint: show this first for the case $I = 2$.]

(c) Find the Pareto optimal level $y^*$ of the public good and show that $\hat{y} < y^*$: explain why the difference $y^* - \hat{y}$ increases with the number of agents.

(d) Suppose instead that the level of the public good is decided by majority voting i.e. $\tilde{y}$ is the level chosen if there is no other level $y'$ that a majority of agents prefer to $\tilde{y}$, each agent contributing $\tilde{y}/I$ to the cost.

(i) Find the most preferred level of the public good for agent $i$.

(ii) Find the level $\tilde{y}$ chosen by majority voting.

(iii) Assume that $I$ is odd. Show that if the median and the mean of the numbers $a_1, \ldots, a_I$ coincide then $\tilde{y} = y^*$ while if the median is smaller (respectively greater) than the mean then $\tilde{y} < (>) y^*$. Compare this result with the outcome with voluntary contributions and comment.
Micro Prelim June 2012
Question 5

Two travelers returning home from a remote island discover that the identical antiques they bought have been smashed in transit. The airline wants to reimburse the travelers in an amount equal to the price they paid. However, this is private information to the travelers. In principle, they could claim that they paid more than they actually did and the airline would have no way of finding out. The airline manager, having taken a course in game theory, sets up the following game to elicit the value of the article. The two travelers are put in separate rooms and asked to independently fill in a compensation claim between $2 and $100 (in increments of $1; thus only integer amounts). The airline will then reimburse each traveler at the smallest of the two claims; in addition, if the claims differ, a reward of $2 will be paid to the person making the smaller claim and a penalty of $2 is deducted from the reimbursement of the larger claimant (for example, if Player 1 claims $78 and Player 2 claims $55 then Player 1 gets 55 – 2 = $53 while Player 2 gets 55 + 2 = $57).

For parts (a)-(e) assume that each player is selfish and greedy in the sense that each player only cares about how much money he gets and prefers more money to less.

(a) Represent this game using a matrix. Since I don't expect you to have the time or inclination to fill in almost 10,000 entries, show the pattern by showing the payoffs when the travelers claim $2, $3, $4, $98, $99 and $100 (and every combination of these). [Hint: exploit the symmetry of the problem.]

(b) Do the players have a dominant strategy?

(c) What do you get by applying the procedure of iterative elimination of weakly dominated strategies?

(d) Find all the pure-strategy Nash equilibria of this game and prove that they are Nash equilibria and that there are no other Nash equilibria.

(e) Assuming that the true value of the damaged article is indeed between $2 and $100, does this mechanism (or game) induce the travelers to reveal the true price they paid?

(f) Given a strategy profile \((m,n)\) define the regret of Player 1, denoted by \(R_1(m,n)\), as the difference between the maximum amount of money he could have got (given that player 2 chose \(n\)) and the amount of money he actually gets at \((m,n)\).

(f.1) Find \(R_1(m,n)\) for all \(m,n \in \{2,\ldots,100\} \times \{2,\ldots,100\}\).

(f.2) What are the maximum possible regret and the minimum possible regret (over the entire set \(\{2,\ldots,100\} \times \{2,\ldots,100\}\))? [Hint: exploit the symmetry of the problem.]

(f.3) Defining player 2’s regret \(R_2(m,n)\) similarly (as the difference between the maximum amount of money she could have got - given that Player 1 chose \(m\) - and the amount of money she actually gets at \((m,n)\)), write the matrix of regrets. As before you only need to show the pattern by showing regrets when the travelers claim $2, $3, $4, $96, $97, $98, $99 and $100 (and every combination of these; please note that, relative to part a, I have added $96 and $97). [Hint: exploit the symmetry of the problem.]

(f.4) Suppose that each player chooses in a MinMax way in the following sense: for each claim the player finds the maximum possible regret and then chooses a claim that minimizes the maximum regret. What strategy profiles are consistent with both players choosing in a MinMax way?

(f.5) Suppose that it is common knowledge between the players that they choose in a MinMax way (as explained in f.4). Apply the following iterative deletion procedure: in each round eliminate all those claims that are weakly dominated in the current matrix. What do you get by applying this iterative deletion procedure?
Let there be $L$ goods (which may be interpreted as varieties of a particular product, with the understanding that the consumer may wish to consume many varieties), with price vector $(p_1,\ldots,p_L) \in \mathbb{R}_+^L$. Define the \textit{arithmetic mean of prices} as
\[ P^A(p_1,\ldots, p_L) = \frac{\sum_{k=1}^{L} p_k}{L}, \]
and the \textit{quadratic mean of prices} as
\[ P^Q(p_1,\ldots, p_L) = \sqrt{\frac{\sum_{k=1}^{L} [p_k]^2}{L}}, \]
a convex function of $(p_1,\ldots, p_L)$.

Consider the function:
\[ \theta: \mathbb{R}_+^L \to \mathbb{R} : \theta(p_1,\ldots, p_L) = P^A(p_1,\ldots, p_L) + c\left[ P^A(p_1,\ldots, p_L) - P^Q(p_1,\ldots, p_L) \right], \]
where the parameter $c$ is a nonnegative real number. Define
\[ f: \mathbb{R}_+^{L+1} \to \mathbb{R} : f(p_1,\ldots, p_L,u) = \theta(p_1,\ldots, p_L)u. \]

1.1. Show that $P^A(p,\ldots, p) = P^Q(p,\ldots, p) = p$ and that, in general, $P^Q \geq P^A$. (Remark. Here and in what follows, $p$ denotes a positive real number, not a vector.)

\textbf{Answer.} Clearly $P^A(p,\ldots, p) = \frac{Lp}{L} = p$, and $P^Q(p,\ldots, p) = \sqrt{\frac{Lp^2}{L}} = p$. 
Consider the minimization problem: Choose \((p_1, \ldots, p_L)\) in order to minimize
\[
P^Q(p_1, \ldots, p_L) = \sum_{k=1}^{L} \left[ \frac{p_k}{L} \right]^2 \sum_{k=1}^{L} p_k \geq B \text{ where } B \text{ is a positive constant. The first order conditions, sufficient because of the convexity of } P^Q, \text{ give the solution } \]
\[
p_k = B = P^A(p_1, \ldots, p_L), \text{ i.e., } \]
\[
P^Q(p_1, \ldots, p_L) = \sqrt{\sum_{k=1}^{L} \left[ \frac{p_k}{L} \right]^2} \geq \frac{\sum_{k=1}^{L} B^2}{L} = B = P^A(p_1, \ldots, p_L).
\]

1.2. Show that \(f\) displays all the properties of an expenditure function (i.e., of the value function of the consumer’s expenditure minimization problem), except possibly being increasing in \(u\) or in some prices.

**Answer.** (i) \(f\) is 1-homogeneous in prices.

**Proof.**
\[
f(tp_1, \ldots, tp_L, u) = \theta(tp_1, \ldots, tp_L)u = \left\{ \frac{\sum_{k=1}^{L} tp_k}{L} + c \left( \frac{\sum_{k=1}^{L} tp_k}{L} - \sqrt{\frac{\sum_{k=1}^{L} [tp_k]^2}{L}} \right) \right\} u
\]
\[
= t \left( \frac{\sum_{k=1}^{L} p_k}{L} + c \left( \frac{\sum_{k=1}^{L} p_k}{L} - \sqrt{\frac{\sum_{k=1}^{L} p_k^2}{L}} \right) \right) u = tf(p_1, \ldots, p_L, u).
\]

(ii) \(f\) is concave in prices.

**Proof.** It follows immediately from the fact that \(P^A\) is linear, \(P^Q\) is convex and \(c \geq 0\).

(iii) Obviously, \(f\) is continuous.

The presence of the (possibly negative) term \(cp^Q u\) suggests that \(\theta\) may be negative or decreasing in some prices.

1.3. Let \(c = 0\). Argue that in this case \(f\) is a legitimate expenditure function. What type of preferences generate this expenditure function?
If $c = 0$, then 

$$f(p_1, \ldots, p_L, u) = \frac{\sum_{k=1}^{L} p_k u}{L},$$

increasing in $u$ and in each price, which together with the statement in Question 1.2 above implies that 

$$\frac{\sum_{k=1}^{L} p_k u}{L}$$

is a legitimate expenditure function. By duality, the indirect utility function is 

$$v(p_1, \ldots, p_L, w) = \frac{L}{\sum_{k=1}^{L} p_k} w,$$

which corresponds to the perfect-complements direct utility function 

$$\bar{u}(x) = L \min\{x_1, \ldots, x_L\}.$$

1.4. We return to the general case where $c \geq 0$, but we assume that $c$ and $(p_1, \ldots, p_L)$ are such that $f$ satisfies all the properties of a legitimate expenditure function. What can you say about the underlying preferences?

**Answer.** Because now we assume that $f$ is increasing in $u$, $\theta(p_1, \ldots, p_L) > 0$. An expenditure function of the form 

$$e(p_1, \ldots, p_L, u) = \theta(p_1, \ldots, p_L) u$$

or, equivalently, an indirect utility function of the form 

$$v(p_1, \ldots, p_L, w) = \frac{1}{\theta(p_1, \ldots, p_L)} w$$

corresponds to an 1-homogeneous direct utility function. Hence, the underlying preference relation is homothetic.

1.5. We maintain the assumptions that $c \geq 0$, and $c$ and $(p_1, \ldots, p_L)$ are such that $f$ satisfies all the properties of a legitimate expenditure function. Obtain the Walrasian demand function for good $j$.

**Answer.** By duality, 

$$v(p_1, \ldots, p_L, w) = \frac{w}{\theta(p_1, \ldots, p_L)}.$$

Now we apply Roy’s identity to obtain
\[ \tilde{x}_j(p_1,\ldots,p_L,w) = -\frac{\partial v / \partial p_j}{\partial v / \partial w} = -\frac{\partial \left( \frac{w}{\theta(p_1,\ldots,p_L)} \right) / \partial p_j}{\partial \left( \frac{w}{\theta(p_1,\ldots,p_L)} \right) / \partial w} \]

\[ \frac{-w \frac{\partial \theta(p_1,\ldots,p_L)}{\partial p_j}}{\theta(p_1,\ldots,p_L)} = \frac{w \frac{\partial \theta(p_1,\ldots,p_L)}{\partial p_j}}{\theta(p_1,\ldots,p_L)} = \frac{w}{\theta L} \left[ 1 + c - c \frac{p_j}{PQ} \right]. \]

1.6. Assume now, as it is often assumed in the monopolistic competition literature, that when expressing her demand for good \( j \), the consumer perceives the price indices \( P^A \) and \( PQ \) as given, i.e., independent of the single price \( p_j \); such a perception is approximately justified if \( L \) is large and \( p_j \) is not too large relative to the other prices. For given wealth \( w \), graph the resulting Walrasian demand curve for good \( j \) in the \((x_j, p_j)\) space.

**Answer.** Because \( \theta = P^A + c[P^A - PQ] \), taking \( P^A \) and \( PQ \) as given implies taking \( \theta \) as given as well, with Walrasian demand for given \( w \) now understood as the affine function of \( p_j \)

\[ \tilde{x}_j(p_j) = \frac{w}{\theta L} \left[ 1 + c - c \frac{p_j}{PQ} \right]. \]

If \( p_j = \frac{1+c}{c} PQ \), then \( x_j = 0 \). On the other hand, if \( p_j = 0 \), then \( x_j = \frac{w}{\theta L} [1+c] \).
1.7. Compute the price elasticity of the demand curve obtained in 1.6 above. What is the value of this elasticity at a point where the prices of all goods are the same (as is often the case at the equilibria of symmetric models of monopolistic competition)?

**Answer.** We compute

\[
\frac{d\hat{x}_j(p_j)}{dp_j} \frac{p_j}{\hat{x}_j(p_j)} = -\frac{w}{\partial L} \frac{1}{P^q} \left[ \frac{1}{w} \frac{p_j}{1 + c - c \frac{p_j}{P^q}} \right] = -c \left[ \frac{1}{P^q} \frac{p_j}{1 + c - c \frac{p_j}{P^q}} \right],
\]

which if evaluated at \((p_1,\ldots,p_L) = (p,\ldots,p)\) becomes (recalling Question 1.1 above)

\[
\frac{d\hat{x}_j(p_j)}{dp_j} \frac{p}{\hat{x}_j(p)} = -c \left[ \frac{1}{p} \frac{p}{1 + c - c \frac{p_j}{P^q}} \right] = -c \left[ \frac{1 + c - c \frac{p_j}{p}}{p} \right].
\]

1.8. Comment, in particular comparing your answers to questions 1.4 and 1.6 above.

**Answer.** Neither homotheticity of preferences nor the linearity (more precisely, affinity) of the demand curve are particularly realistic, but they are popular because they greatly simplify the analysis of the demand sector in economic models.

The postulated expenditure function displays both features. From Question 1.4 we know that preferences are homothetic. Homothetic preferences typically yield nonlinear demand curves, yet Question 1.6 shows that, when \(L\) is large, the consumer demand curve can be perceived as a straight line. Question 1.7 shows that, in addition, the price elasticity of demand adopts a particularly simple form at a symmetric point where the prices of all varieties are the same, a focal point in many models of monopolistic competition.
Question 2.
Postulate the expected utility hypothesis, and assume the decision maker’s von Neumann-Morgenstern-Bernoulli (vNMB) utility function is

\[ u : X \rightarrow \mathbb{R} : u(x) = b \left[ a + \frac{x}{c} \right]^{1-c} , \quad (2.1) \]

where \( a, b \) and \( c \) are real numbers, \( c \neq 0 \), satisfying

\[ \frac{b[1-c]}{c} > 0 , \quad (2.2) \]

and the domain \( X \) is defined as

\[ X := \{ x \in \mathbb{R}_+ : a + \frac{x}{c} > 0 \} . \]

(2.3)

2.1. Argue that the decision maker is strictly risk-averse.

**Answer.** We compute

\[ u'(x) = b[1-c] \left[ a + \frac{x}{c} \right]^{-c} \frac{1}{c} , \quad (A2.1) \]

positive by (2.1) and (2.2), and

\[ u''(x) = b[1-c][-c] \left[ a + \frac{x}{c} \right]^{-c-1} \frac{1}{c} = -b \frac{1-c}{c} \left[ a + \frac{x}{c} \right]^{-c-1} , \quad (A2.2) \]

negative by (2.1) and (2.2).

2.2. Show that the preferences described by the following vNMB utility functions are special cases of the ones defined by (2.1)

(A). \( u^\eta : X \rightarrow \mathbb{R} : u^\eta(x) = \frac{x^{1-\eta}}{1-\eta} , X = \mathbb{R}_+, \eta > 0, \eta \neq 1. \)

(4.4)

(B). \( u^\nu : X \rightarrow \mathbb{R} : u^\nu(x) = \frac{[x+k]^{1-\eta}}{1-\eta} , X = \{ x \in \mathbb{R}_+ : x > -k \} , k \in \mathbb{R}, \eta > 0, \eta \neq 1. \)

(2.5)
(C). \( u^0 : X \to \mathbb{R} : u^0(x) = -[g - x]^2, X = \{ x \in \mathbb{R}_{++} : x < g \} \). \hspace{1cm} (2.6)

**ANSWER.**

(A). Set \( a = 0 \) and \( c = \eta \), i.e., \( c > 0 \) and \( c \neq 1 \), and \( b = \frac{\eta^{1-\eta}}{1-\eta} \).

(B). Set \( c = \eta \) and \( a = \frac{k}{\eta} \) and \( b = \frac{\eta^{1-\eta}}{1-\eta} \).

(C). Set \( a = g \), \( b = -1 \) and \( c = -1 \).

2.3. Show that the preferences described by the following vNMB utility function are a limit case of the ones defined by (2.4)

\[ u^0 : X \to \mathbb{R} : u^0(x) = \ln x, X = \mathbb{R}_{++} \]. \hspace{1cm} (2.7)

**ANSWER.**

Add a constant to (2.4) so that it becomes \( \frac{x^{1-\eta} - 1}{1-\eta} \), \( X = \mathbb{R}_{++}, \eta > 0, \eta \neq 1 \).

Then the limit \( \lim_{\eta \to 1} \frac{x^{1-\eta} - 1}{1-\eta} \) is of the \(0/0\) type. Apply l'Hôpital rule to obtain

\[
\lim_{\eta \to 1} \frac{x^{1-\eta} - 1}{1-\eta} = \lim_{\eta \to 1} \frac{(d/d\eta)(x^{1-\eta} - 1)}{(d/d\eta)(1-\eta)} = \lim_{\eta \to 1} \frac{x^{1-\eta} \ln x}{-1} = \ln x.
\]

2.4. For which values of the parameters \( a, b, c \) does the vNMB of (2.1) display:

- Increasing Absolute Risk Aversion?
- Decreasing Absolute Risk Aversion?
- Increasing Relative Risk Aversion?
- Decreasing Relative Risk Aversion?

Illustrate when possible by referring to the utility functions (2.4)-(2.7)
\textbf{ANSWER.}

From (2.4) and (2.5), we compute the coefficient of absolute risk aversion

\[ r_A(x) := \frac{-u''(x)}{u'(x)} = -\frac{b(1-c)}{c} \left[ a + \frac{x}{c} \right]^{-c-1} \left[ a + \frac{x}{c} \right]^{-1}. \tag{A2.3} \]

Hence

\[ \frac{d}{dx} r_A(x) = \begin{cases} > 0 & \text{if } c < 0, \\ < 0 & \text{if } c > 0. \end{cases} \tag{A2.4} \]

For example, in the quadratic case of (2.6), \( c = -1 < 0 \), and accordingly, \( r_A \) is (unrealistically) increasing.

On the other hand, in the cases of (2.4) and (2.5) \( c = \eta > 0 \) and \( r_A \) is decreasing.

The coefficient of relative risk aversion can be computed from (A2.3) as

\[ r_R(x) := x r_A(x) = x \left[ a + \frac{x}{c} \right]^{-c-1}, \tag{A2.5} \]

with

\[ \frac{d}{dx} r_R(x) = \begin{cases} > 0 & \text{if } a > 0, \\ = 0 & \text{if } a = 0, \\ < 0 & \text{if } a < 0. \end{cases} \tag{A2.6} \]

For example, in the case of (2.4), \( a = 0 \), \( r_R \) is constant. (This is the CRRA case.) Of course, this coincides with the vNMB of (2.5) for \( k = a = 0 \).

For the vNMB of (2.5) and \( k = a \eta > 0 \), \( r_R \) is increasing.

For the vNMB of (2.5) and \( k = a \eta < 0 \), \( r_R \) is decreasing. (Note that in this case the domain restriction requires \( x \) to be bounded away from zero.)
2.5. Postulate that nature randomly chooses one of two “states or the world”, \( s_1 \) or \( s_2 \). State \( s_1 \) is the bad state, which occurs with probability \( \pi \), whereas \( s_2 \) is the good state, which occurs with probability \( 1 - \pi \). For \( j = 1, 2 \), denote by \( x_j \) consumption contingent to state \( j \) and by \( P_j \) the price of consumption contingent to state \( j \). Assume that \( P_j / P_2 > \pi / [1 - \pi] \). Denote by \( W \) the wealth of the consumer. Consider only the case where the solution to the consumer optimization problem is interior to the consumption set, which is defined by the property that both \( x_1 \) and \( x_2 \) must belong to the domain of the relevant vNMB utility function.

2.5.1. For the vNMB utility function (2.1), show that the wealth expansion path is a straight line, with positive slope, in \((x_1, x_2)\) space.

**ANSWER.**

The wealth expansion path is implicitly defined by the first order condition

\[
\frac{\pi u'(x_1)}{[1 - \pi] u'(x_2)} = \frac{P_1}{P_2},
\]

i.e., using (A2.1)

\[
\frac{\pi [a + \frac{x_1}{c}]^{-c}}{[1 - \pi] [a + \frac{x_2}{c}]^{-c}} = \frac{P_1}{P_2},
\]

i.e.,

\[
\left[ \frac{\pi}{1 - \pi} \right]^{\frac{1}{c}} \left[ \frac{a + \frac{x_1}{c}}{a + \frac{x_2}{c}} \right] = \left( \frac{P_1}{P_2} \right)^{\frac{1}{c}}.
\]

i.e.,

\[
\left[ \frac{P_1}{P_2} \right]^{\frac{1}{c}} \left[ \frac{\pi}{1 - \pi} \right]^{\frac{1}{c}} \left[ a + \frac{x_1}{c} \right] - a = \frac{x_2}{c},
\]

or

\[
x_2 = \left[ \frac{P_1}{P_2} \right]^{\frac{1}{c}} \left[ \frac{\pi}{1 - \pi} \right]^{\frac{1}{c}} \left[ ca + x_1 \right] - ca = ca \left( \left[ \frac{P_1 [1 - \pi]}{P_2} \pi \right]^{\frac{1}{c}} - 1 \right) + \left[ \frac{P_1 [1 - \pi]}{P_2} \pi \right]^{\frac{1}{c}} x_1,
\]
The slope \( \frac{P_1}{P_2} \frac{1}{\pi} \) is obviously positive. By assumption, for \( c > 0 \), then the slope is greater than one, whereas if \( c < 0 \), then the slope is less than one.

**2.5.2.** For the vNMB utility functions given by (2.4) to (2.7), does an increase in the wealth of the consumer lead her to bear absolutely more risk? Relatively more risk? Explain.

**ANSWER.** From (A2.4),

\[
\frac{d}{dx} r_a(x) = \begin{cases} 
1 & > 0 \text{ if } c < 0, \\
\frac{a}{c} & < 0 \text{ if } c > 0.
\end{cases}
\]

Hence, as her wealth increases, the decision maker bears absolutely less risk if \( c < 0 \) (as in (2.6), and absolutely more risk if \( c > 0 \) (as in (2.4), (2.5) and (2.7)).

On the other hand, from (A2.6),

\[
\frac{d}{dx} r_R(x) = \begin{cases} 
\frac{a}{c} & > 0 \text{ if } a > 0, \\
\frac{a}{c} & = 0 \text{ if } a = 0, \\
\frac{a}{c} & < 0 \text{ if } a < 0.
\end{cases}
\]

i. e., as her wealth increases, the decision maker bears relatively more risk if \( a < 0 \) (as in (2.5) for \( k < 0 \), relatively less risk if \( a > 0 \) (as in (2.5) for \( k > 0 \), or trivially as in (2.7)) and unchanged relative risk for \( a = 0 \), as in the CRRA function of (2.4) and its limit case (2.7).

**2.6.** Assume a population of individuals that have different wealth levels but are otherwise identical, with identical preferences defined by a vNMB utility function of the form (2.1), facing the same probability \( \pi \) of the bad state, and the same prices \( P_1 \) of \( P_2 \) for the contingent consumption commodities. Does their aggregate demand for the contingent consumption commodities admit a positive representative consumer? Argue your answer.
**Answer.** Yes. They have identical (and hence same slope) wealth expansion paths which are straight lines, and therefore the Gorman positive representative consumer theorem applies.
Question 3

(a) (i) $\max \log x_i + \log y_i - a \left( \log x_i + \log y_i \right)$

$\log x_i + \log y_i \geq \nu_z$  
$\nu_i \geq \nu_z$  
$p_x$

$y_i + \frac{1}{2} z \leq \omega_y$  
$p_y$

\[
\frac{1}{a_i} = p_x \quad \frac{1}{y_i} = p_y 
\]

$(a + d_e) \frac{1}{x} = p_x \quad (a + d_e) \frac{1}{f_i} = p_y$

$p_x$ and $p_y > 0$ since $\log (x, y)$ is increasing. Thus $-a + d_e > 0 \Rightarrow d_e > a$. The constraint $u_e(x_i, y_i) = \nu_i$ is binding. Giving consumption to agent 2 decreases the utility of agent 1 for two reasons: it takes resources which are no longer available to agent 1 and it decreases agent 1's utility by increasing "envy".

(ii) CE equilibrium: Since agent 1 takes the actions of agent 2 as given, he maximizes $u_i$ subject to the budget constraint. So the equilibrium is the same as if the utility of agent 2 was $\bar{u} (x_i, y_i) = \log (x_i y_i)$.

(iii) Since the equilibrium is the same as in the standard Cobb-Douglas economy, at a competitive equilibrium there exist $d > 0$, $\nu_i > 0$ such that...
\[
\frac{1}{x_i} = \lambda_i P_{x_i} \quad \frac{1}{y_i} = \lambda_i P_{y_i} \quad \frac{1}{x_j} = \lambda_j P_{x_j} \quad \frac{1}{y_j} = \lambda_j P_{y_j}
\]

\[\tilde{P}_x = \lambda_i \tilde{x}_i, \quad \tilde{P}_y = \lambda_i \tilde{y}_i \quad \Rightarrow \quad \lambda_i > 0 \quad \Rightarrow \quad \alpha_e = a + \frac{\tilde{a}}{\lambda_i} > 0
\]

Thus the FOCs in Pareto optimality derived in (i) hold.

The objective of the maximum problem in (i) is not quasi-concave (a log utility is a convex function) so that satisfying the first-order conditions in Pareto optimality does not guarantee that an allocation is Pareto optimal.

(iv) Let \((\tilde{x}_i, \tilde{y}_i, \tilde{x}_j, \tilde{y}_j), (\bar{P}_x, \bar{P}_y)\) be a competitive equilibrium of the economy \(\mathcal{E}(u_1, u_2, w, w')\). Note that, since agent 1 takes \((\tilde{x}_i, \tilde{y}_i)\) as given, maximizing \(u_1(x_i, y_i, \tilde{x}_i, \tilde{y}_i)\) subject to the budget constraint is equivalent to maximizing \(\tilde{u}_1(x_i, y_i)\) subject to the b.e.\(\tilde{x}_i\).

Thus \((\tilde{x}_i, \tilde{y}_i, \tilde{x}_j, \tilde{y}_j), (\bar{P}_x, \bar{P}_y)\) is also an equilibrium of the economy \(\mathcal{E}(\tilde{u}_1, u_2, w, w')\). Suppose that there exist an allocation \((\tilde{x}_i, \tilde{y}_i, \tilde{x}_j, \tilde{y}_j)\) which is feasible and such that

\[
u_1(x_i, y_i, \tilde{x}_i, \tilde{y}_j) \geq u_1(x_i, y_i, \tilde{x}_i, \tilde{y}_j), \quad u_2(x_i, y_i) \geq u_2(x_i, y_i)
\]

with at least one strict inequality. If \(u_2(\tilde{x}_j, \tilde{y}_j) = u_2(\tilde{x}_j, \tilde{y}_j)\)
then the inequality must be strict in each, which is possible only if \( \tilde{u}_i (\tilde{x}_i, \tilde{y}_i) > \tilde{u}_i (\bar{x}_i, \bar{y}_i) \). If \( u_z (\tilde{x}_e, \tilde{y}_e) > u_z (\bar{x}_e, \bar{y}_e) \), then necessarily \( \tilde{u}_i (\tilde{x}_i, \tilde{y}_i) > \tilde{u}_i (\bar{x}_i, \bar{y}_i) \) to compensate for the increased envy. Thus by a reasoning analogous to that of the proof of the First Theorem of Welfare Economics, it must be that

\[
\bar{P}_x \tilde{x}_1 + \bar{P}_y \tilde{y}_1 > \bar{P} \cdot w \quad \text{and} \quad \bar{P}_x \tilde{x}_2 + \bar{P}_y \tilde{y}_2 > \bar{P} \cdot w^2.
\]

Adding up the inequalities implies

\[
\bar{P}_x (\tilde{x}_1 + \tilde{x}_2) + \bar{P}_y (\tilde{y}_1 + \tilde{y}_2) > \bar{P} \cdot w
\]

which contradicts the feasibility of the bile allocation.

(b) a P.O allocation is solution to the problem

\[
\max \log x_1 + \log y_1 + b \left( \log x_2 + \log y_2 \right) \quad \text{subject to}
\]

\[
\log x_2 + \log y_2 \geq u_2 \quad \text{and} \quad \alpha_2
\]

\[
x_1 + x_2 \leq w_x \quad \text{and} \quad \bar{P}_x
\]

\[
y_1 + y_2 \leq w_y \quad \text{and} \quad \bar{P}_y
\]

Pareto:

\[
\frac{1}{x_1} = \frac{1}{y_1} = \frac{1}{y_2} = \frac{b + \alpha_2}{x_2} = \frac{b + \alpha_2}{y_2} = \frac{b + \alpha_2}{y_2}
\]

\[
\alpha_2 \left( \log x_2 y_2 - u_2 \right) = 0.
\]
\( p_x \) and \( p_y \) are necessary positive since agent 1's utility is increasing in \( x, y \). \( c_2 \) is not necessary for the because it may be that increasing the utility of agent 2 above \( v_2 \) also increases the utility of agent 1 because of the "altruistic" term \( b \log x_2 y_2 \).

\[
\alpha_2 = 0 \quad \frac{1}{\alpha_2} = \frac{b}{c_2} \Rightarrow x_2 = b \alpha_2, \quad \frac{1}{y_2} = \frac{b}{c_2} \Rightarrow y_2 = by_2.
\]

Feasibility implies \( \alpha = \frac{w_2}{1+b} \), \( y_1 = \frac{w_2}{1+b} \), \( \alpha_2 = \frac{b}{1+b} w_2, \quad y_2 = \frac{b}{1+b} w_2 \)

\( u(x_1, y_1) = \log w_2 + \log y_1 + 2 \log b = v_2 \star \)

If \( v_2 \leq v_2 \star \), the FOCs are satisfied, and the allocation is P.O. since the maximum problem is convex.

\[
\alpha_2 > 0 \quad \frac{1}{\alpha_2} = \frac{b+\alpha_2}{c_2} \quad \frac{1}{y_2} = \frac{b+\alpha_2}{y_2}
\]

Feasibility implies: \( \alpha_1 = \frac{w_2}{1+b+\alpha_2} = \frac{w_2 x_2}{1+b+\alpha_2} \)

\[
\alpha_2 = \frac{(b+\alpha_2)w_2}{1+b+\alpha_2}, \quad y_2 = \frac{(b+\alpha_2)w_2}{1+b+\alpha_2}
\]

\( u(x_2, y_2) = \log w_2 + \log y_2 + 2 \log \frac{b+\alpha_2}{1+b+\alpha_2} = v_2 \)

which has a positive solution \( \alpha_2 \) if \( v_2 > v_2 \star \).

At a P.O. the utility of agent 2 cannot be too small because transferring goods from agent 1 to agent 2 would increase the utility of both agents. To agent 1 because of added consumption, and to
agent 1 becomes the increase of the term $b \log z_{1.1}$ more than compensate the loss in the own consumption term $\log z_{1.1}$.

(ii) If the endowment of agent 2 is too small, the competitive equilibrium (which is the same as that of the economy $E(\tilde{u}_1, \tilde{u}_2, w', w^2)$) gives too low a utility level $\tilde{u}_2(x, t, t) = b$ to agent 2.

![Edgeworth Box for the economy $E(\tilde{u}_1, \tilde{u}_2, w', w^2)$](image)

The level of utility of agent 2 in equilibrium is lower than $u_2^*$: the equilibrium is not Pareto optimal in the economy $E(\tilde{u}_1, \tilde{u}_2, w', w^2)$.

(iii) In the case where the endowment of agent 2 is too low, redistributive taxation — transfering income from agent 1 to agent 2 — would improve the efficiency of the economy.
4. (a) \[ \max \lambda - z_i - \frac{a_i}{z_i + z_i} \quad \text{subject to} \]

\[ 0 \leq z_i \leq \lambda \]
\[ z_i \leq 1 \]

\[ \text{FOC: } -1 + \frac{a_i}{(z_i + z_i)^2} + \lambda_i - \mu_i = 1 \]
\[ \lambda_i z_i = 0 \]
\[ \mu_i (z_i - 1) = 0 \]

- \[ z_i = 1 \quad \lambda_i \geq 0 \quad \lambda_i = 0 \quad \frac{a_i}{(1 + z_i)^2} = \frac{1}{1 + \mu_i} \geq 1 \]
- Since \( a_i < 1 \) and \( z_i \geq 0 \), \( \frac{a_i}{(1 + z_i)^2} < 1 \) so that this case is not possible.

- \[ z_i = 0 \quad \lambda_i \geq 0 \quad \mu_i = 0 \implies \frac{a_i}{(z_i - z_i)^2} = 1 - \lambda_i \leq 1 \]
\[ \implies (z_i - z_i) \geq a_i \quad z_i \geq \sqrt{a_i} \]

- If \( 0 < z_i < 1 \), \( \lambda_i = \mu_i = 0 \implies \left( z_i + z_i \right)^2 = a_i \implies z_i = \sqrt{a_i} - z_i \)

Thus the optimal contribution of agent \( i \) is:

\[ \begin{cases} 
0 & \text{if } z_i \geq \sqrt{a_i} \\
\sqrt{a_i} - z_i & \text{if } z_i \leq \sqrt{a_i} 
\end{cases} \]

(b) First note that if agent \( i \) does not contribute, then all agents \( i' > i \) do not contribute either. For suppose that agent \( i \) does not contribute and agent \( i' \) contributes. This implies \( z_i = \sqrt{a_i} \) and \( z_i' + z_i = \sqrt{a_i} \)
Since agent $i$ does not contribute $Z^{-i} = Z^{-i} + z_i = \sqrt{a_i} < \sqrt{a_i}$, which contradicts $Z^{-i} = \sqrt{a_i}$. Thus agent 1 to $i$ contribute and agent $i+1$ to $I$ do not contribute.

Suppose $i=2$, i.e. agents 1 and 2 contribute. Then from (1)

$$Z_1 + Z^{-1} = z_1 + z_2 = \sqrt{a_1}$$

and $z_2 + Z^{-2} = z_2 + z_1 = \sqrt{a_2}$. Since $\sqrt{a_1} > \sqrt{a_2}$, this is impossible. This reasoning holds for any $i \geq 2$.

Thus the only solution is $i = 1$. Only agent 1 contributes $Z_1 = \sqrt{a_1}$. For all the other agents, $Z^{-i} = \frac{z_i}{\sqrt{a_i}} > \sqrt{a_i}$ and they free ride on agent 1.

(C) Since utilities are quasi linear, the P.O. allocation in which all agents have positive consumption of the private good are solutions to

$$\max \sum_{i=1}^{I} \left( x_i - \frac{a_i}{y} \right) \quad s.t. \quad \sum_{i=1}^{I} x_i + y = I$$

FOEs: $i=1 \quad \frac{\delta x_i}{\delta y} = 1 \Rightarrow y^* = \sqrt{\sum_{i=1}^{I} a_i}$

The optimal level of public good increases with the number of agent and their parameters $(a_i)$, i.e. $(a_i)_{i=1}^{I}$. The inefficiency of the voluntary contribution equilibrium comes from the fact that each agent choose his/her contribution to maximize private, instead of social, benefit. The more agents there are, the larger the discrepancy between "private" and "social" benefit, and the larger the difference between $y^*$ and $y$ (which is fixed).
(a) (i) the familiar tax rule of agnostic solves
\[
\max_{2; > 0} 1 - 2; - \frac{a;}{1 + 2;} \quad (2; \text{ is necessarily positive})
\]

F.O.C.: \(-1 + \frac{a;}{1 + 2;} = 0\) \(2;^* = \sqrt{\frac{a;}{1}} \quad y;^* = \frac{1}{1 + \sqrt{\frac{a;}{1}}} = \sqrt{\frac{a;}{1}}\)

(b) If the utilities \(2; \rightarrow 1 - 2; - \frac{a;}{1 + 2;}\) are quasi-concave (single peaks) then the median voter theorem applies, and the only level of public good which cannot be achieved by a majority of votes in \(\bar{y} = \sqrt{\sum a; \text{med}}\) where \(\text{med} \in \text{such that}\)

\[
\left\{ \# i \mid a; \leq a; \text{med} \right\} \geq \frac{1}{2} \quad \left\{ \# i \mid a; > a; \text{med} \right\} \geq \frac{1}{2}
\]

\(f(2) = 1 - 2; - \frac{a;}{1 + 2;} \quad f'(2) = -1 + \frac{a;}{1 + 2;} \quad f''(2) = -\frac{2a;}{(1 + 2;)^3} < 0\)

Thus \(f\) is concave, and quasi-concave and single peaked. The equilibrium with majority vote in \(\bar{y}\)

(c) \(\bar{y} \leq y^* \iff \sum a; \text{med} \leq \sum a; \iff a; \text{med} \leq \frac{\sum a;}{1}\)

If the median of the number \((a;): e; i\) is equal to the mean the median voter "perfectly" represents the economy and the level of public good is optimal. Even if this condition is not satisfied, with a large number of agents, \(\text{med} \) will typically be much higher than \(a;\), so that majority vote gives a better result than voluntary contribution. This comes from the fact that the marginal benefit of one more dollar of contribution to the median voter is much higher — since every agent will also contribute one more dollar — than the m.b. of one more dollar of voluntary contribution of any agent.
(a) The strategy spaces and payoff functions are as follows:

<table>
<thead>
<tr>
<th>Strategy space</th>
<th>{2,3,4\ldots, 100}</th>
</tr>
</thead>
</table>
| Payoff function | \begin{align*}
\Pi_i(i, j) &= \begin{cases} 
\min\{i, j\} - 2, & i > j \\
\min\{i, j\}, & i = j \\
\min\{i, j\} + 2, & i < j
\end{cases}
\end{align*} |

The corresponding matrix is as follows:

\[
\begin{array}{cccc|ccc}
& 2 & 3 & 4 & 98 & 99 & 100 \\
\hline
2 & (2,2) & (4,0) & (4,0) & (4,0) & (4,0) & (4,0) \\
3 & (0,4) & (3,3) & (5,1) & (5,1) & (5,1) & (5,1) \\
4 & (0,4) & (1,5) & (4,4) & (6,2) & (6,2) & (6,2) \\
98 & (1,5) & (2,6) & (98,98) & (100,96) & (100,96) & (100,96) \\
99 & (0,4) & (1,5) & (2,6) & (96,100) & (99,99) & (101,97) \\
100 & (0,4) & (1,5) & (2,6) & (96,100) & (97,101) & (100,100) \\
\end{array}
\]

(b) Neither player has a dominant strategy.

(c) The iterative elimination of dominated strategies leads to (2,2): 99 dominates 100 for each player; eliminating 100 for both, we get that 98 dominates 99, etc.

(d) There is only one Nash equilibrium: (2,2). Given that the other player is reporting $2, you get $2 if you also report $2 and 0 if you report more than $2.

(m,n) with m = n > 2 is not a Nash equilibrium because Player 1 can increase his payoff from m to (m+1) by switching his strategy to (m−1).

(m,n) with m > n is not a Nash equilibrium because Player 1 can increase his payoff from (n−2) to n by switching his strategy to n.

(e) Since the solution is (2,2), no matter what the true value of the item, the answer is No.
(f.1) If player 2 chooses $n > 2$ the best reply of player 1 (if he wants to get as much money as possible) is to choose $n - 1$ (if $n = 2$ player 1’s best reply is to choose 2 also). Let $B_r(n)$ denote the maximum amount of money that player 1 can get when player 2 chooses $n$. Then

$$B_r(n) = \begin{cases} 2 & \text{if } n = 2 \\ n + 1 & \text{if } n > 2 \end{cases}.$$  

Thus $R_r(m, 2) = \begin{cases} 0 & \text{if } m = 2 \\ 2 & \text{if } m > 2 \end{cases}$ and, for $n > 2$,

$$R_r(m, n) = \begin{cases} n - m - 1 & \text{if } m \leq n - 1 \\ 1 & \text{if } m = n \\ 3 & \text{if } m > n \end{cases}.$$  

(f.2) The minimum regret is 0 (e.g. when $n = m = 2$) and the maximum is 97 (when $n = 100$ and $m = 2$).

(f.3) The matrix is as follows:

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>96</th>
<th>97</th>
<th>98</th>
<th>99</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0,0</td>
<td>0,2</td>
<td>1,2</td>
<td></td>
<td>93</td>
<td>94</td>
<td>95</td>
<td>96</td>
<td>97</td>
</tr>
<tr>
<td>3</td>
<td>2,0</td>
<td>1,1</td>
<td>0,3</td>
<td></td>
<td>92</td>
<td>93</td>
<td>94</td>
<td>95</td>
<td>96</td>
</tr>
<tr>
<td>4</td>
<td>2,1</td>
<td>3,0</td>
<td>1,1</td>
<td></td>
<td>91</td>
<td>92</td>
<td>93</td>
<td>94</td>
<td>95</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>2,93</td>
<td>3,92</td>
<td>3,91</td>
<td></td>
<td>1,1</td>
<td>0,3</td>
<td>1,3</td>
<td>2,3</td>
<td>3,3</td>
</tr>
<tr>
<td>97</td>
<td>2,94</td>
<td>3,93</td>
<td>3,92</td>
<td></td>
<td>3,0</td>
<td>1,1</td>
<td>0,3</td>
<td>1,3</td>
<td>2,3</td>
</tr>
<tr>
<td>98</td>
<td>2,95</td>
<td>3,94</td>
<td>3,93</td>
<td></td>
<td>3,1</td>
<td>3,0</td>
<td>1,1</td>
<td>0,3</td>
<td>1,3</td>
</tr>
<tr>
<td>99</td>
<td>2,96</td>
<td>3,95</td>
<td>3,94</td>
<td></td>
<td>3,2</td>
<td>3,1</td>
<td>3,0</td>
<td>1,1</td>
<td>0,3</td>
</tr>
<tr>
<td>100</td>
<td>2,97</td>
<td>3,96</td>
<td>3,95</td>
<td></td>
<td>3,3</td>
<td>3,2</td>
<td>3,1</td>
<td>3,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

(f.4) Let $M_r(m) = \max_n R_r(m, n)$. Then $M_r(m) = \max \{3, 99 - m\}$. Thus MinMax requires choosing from the set $\{96, 97, 98, 99, 100\}$.

(f.5) If the MinMax criterion is common knowledge, then it is common knowledge that each player will choose from the set $\{96, 97, 98, 99, 100\}$. By referring to the lower right portion of the matrix in part (f.3), it can be seen that 100 is weakly dominated by 99. After deleting 100 for both players, 99 is weakly dominated by 98, etc. Thus the iterative deletion procedure leads to each player choosing 96 and getting $96.
You have 20 minutes to read the exam, and 4 hours to write your answers. Please answer all three problems. The three problems will be equally weighted in the total score. The maximum possible score for each problem is 5 points. It is recommended that you read each problem entirely before attempting any of the parts. Good luck!
1. Consider a household $i$ that lives in village $v$ and has $W_i$ units of wealth that it can allocate to either a stochastic production process or save. Assume that the household has access to a risk-free savings instrument, but that it does not otherwise have access to financial markets (i.e., it can neither borrow, nor purchase insurance). In this setting, a household’s expected utility maximization problem can be written as:

$$\max_{W_s, W_p} E[u(c_i)]$$

subject to:

$$c_i \leq (1 + r)W_s + \theta f(W_p)$$

$$W_s + W_p = W_i$$

$$W_s, W_p \geq 0$$

where $u(\cdot)$ is the household’s utility function; $c_i$ is consumption; $r$ is the certain rate of return on wealth allocated to savings, $W_s$; $f(\cdot)$ is a household production (or net-revenue) function which exhibits positive but diminishing returns to wealth invested in production, $W_p$; and, $\theta_i > 0$ is a strictly positive random variable for which $E[\theta_i] = 1$.

(a) [0.5 points] Assuming that the household is risk neutral (i.e., that $u(\cdot)$ is linear), how would it divide its wealth between safe savings and investment in the risky production process?

(b) [1.5 points] Assuming that the household is risk averse (i.e., that $u(\cdot)$ is concave), how would the household allocate its wealth between savings and investment? Be sure to explicitly compare your answer to this question to your answer to part (a) above. Denote your answers to this problem as $W_s^*$ and $W_p^*$.

(c) [1.5 points] How would the resource allocation of the risk averse household change as its total wealth increases? Specifically, what would be the relationship between expected income ($(1 + r)W_s^* + f(W_p^*)$) and total wealth, $W_i$? In answering this question, please be explicit about whatever assumptions you want to make about the nature of the risk aversion and the utility function.

(d) [1.5 points] Finally, and more speculatively, are endowments fate? To answer this question, imagine now that village $v$ is comprised of $n$ households that are uniformly distributed along a wealth continuum stretching from $W$ to $W$. Under the assumptions laid out in this problem, what would you expect to happen in this village over time in terms of both its GDP and its income distribution? If you were a grand social planner, would you propose any interventions in this village (if so, what)? Alternatively, would you expect new institutions to endogenously evolve in this village that might alter its history compared to that implied by the model?
2. Immiserizing growth

In this problem we investigate a paradoxical situation, formalized by Bhagwati in 1958, where a country is made worse off by the growth of its own production sector. There are two goods, exportables \( X \) and importables \( Y \), and the domestic economy is represented by:

(i) a production frontier \( Y = F(X, \alpha) \), where \( F \) is of class \( C^2 \), decreasing in \( X \) and increasing in \( \alpha \) and concave in \( X \);

(ii) a community utility function \( U(X_d, Y_d) \) that is of class \( C^2 \), increasing and quasi-concave.

Domestic growth is here represented by increases in the parameter \( \alpha \).

The domestic country can trade with the rest of the world (ROW) according to the foreign offer curve \( Y_e = O(X_e) \), where \( X_e \) are the exports and \( Y_e \) are the imports and \( O \) is of class \( C^2 \) and increasing. In the entire problem, we assume that there is free trade between the domestic country and ROW. We use subscripts to denote partial derivatives.

First, assume that the domestic country is “small.” In this case, imports are a linear function of exports.

[0.5] 1. Use a simple graph to argue that domestic growth always leads to an increase in domestic welfare in this case.

We now turn to the more interesting case where the domestic country is “large,” that is, it faces deteriorating terms of trade.

[0.4] 2. What property of the function \( O \) ensures that the terms of trade are deteriorating? Interpret.

[0.4] 3. Express the price of exportables relative to that of importables (that is, the terms of trade) as a function of \( X_e \) and \( Y_e \).

[0.8] 4. Deduce from the previous question the two first-order conditions that hold in the domestic economy at the free trade equilibrium. (Hint: You do not need to solve any optimization program, just write down the first-order conditions.)

[0.5] 5. Write down the set of four equations in four unknowns that implicitly define the free trade equilibrium as a function of the exogenous parameter \( \alpha \). (Hint: there are several ways to write these conditions depending on which variables you decide to drop. I recommend that you keep the following endogenous variables: \( X, X_e, Y, Y_e \).)

[1.0] 6. By totally differentiating \( U(X_d, Y_d) \) with respect to \( \alpha \) and using your answer to the previous question, show that

\[
\frac{dU}{d\alpha} = U_Y F_\alpha + U_X \left[ -1 + \frac{X_e}{Y_e} O' \right] \frac{dX_e}{d\alpha}.
\]

(Hint: You do not need to apply the implicit function theorem.)

[0.6] 7. Is the result in part 6 consistent with your answer to part 1? Explain.

[0.8] 8. Deduce from part 6 necessary conditions for domestic growth to be immiserizing, that is, utility-reducing.
3. Power can be produced by an incumbent industry and an alternative industry, and is currently being provided by the incumbent industry only. The incumbent industry has a private cost structure given by

\[ C_i(q_i; w) = k_i + \gamma_1 \cdot q_i + 0.5 \cdot \gamma_2 \cdot q_i^2 \]

with \( \gamma_1, \gamma_2 > 0 \), and the industry receives subsidies of \( s \) per unit of power produced. It also generates an externality

\[ e_i = b \cdot q_i \]

which is costly to society, with damages \( D \) per unit of \( e_i \). This externality is not regulated and its costs are borne by consumers. The inverse demand facing the incumbent industry is \( p_i(q_i) = \alpha_i - \alpha_2 \cdot q_i \). To simplify things, assume that the subsidy and external cost are such that when the incumbent industry prices monopolistically, it produces more than the socially optimal quantity.

The alternative industry, not currently producing power, has a private cost structure

\[ C_a(q_a; w) = k_a + (\gamma_1 + \Delta_1) \cdot q_a + 0.5 \cdot (\gamma_2 - \Delta_2) \cdot q_a^2 \]

with \( \Delta_1, \Delta_2 > 0 \) and \( \Delta_1 > D \cdot b \). This industry generates no externality and receives no subsidy.

Since consumers care about the production methods used for goods they purchase, the demand for power produced by the alternative industry is \( p_a(q_a) = \alpha_1 + \beta - \alpha_2 \cdot q_a \), with \( \beta > \Delta_1 - b \cdot D \). [Note that this consumer demand effect is entirely separate from the external costs of the incumbent industry.]

(0.2) (a) Provide graphs of (i) the demand and marginal cost relationships for the incumbent industry; (ii) the demand and marginal cost relationships for the alternative industry; and (iii) a graph that includes all these relationships for both industries. Include the private marginal cost with and without the subsidy, and the social marginal cost that includes external costs.

(0.2) (b) If the incumbent industry prices its product competitively, show the distribution of welfare among consumers, producers, and the government.

(0.2) (c) Provide the distribution of welfare among these same parties if the incumbent industry prices its product monopolistically.

(0.2) (d) Which of the market structures in (b) and (c) above generates the largest net benefits to society? Explain your reasoning and the intuition behind your conclusion, along with any caveats you feel are necessary.

(1.0) (e) Now compare these market outcomes for the incumbent industry (price and quantity levels as well as the distribution of welfare) to the outcomes if power is instead provided by the alternative industry only and is competitively priced. What can you say about the magnitude of social surplus for power provided by the two power industries? The best answer here will involve algebraic support of your conclusion.

(3.2) (f) Discuss how the power market might be expected to develop if the alternative source of power is introduced as a competing source of supply alongside power provided by the incumbent industry. Do you expect that the equilibrium reached will feature both sources of power, or only one? Is it the case that the highest-social value source of power will predominate? If so, explain why you think so and how this will occur. If not, is there a rationale for intervention by the government, and how would this improve things?
2. Immiserizing growth

In this problem we investigate a paradoxical situation, formalized by Bhagwati in 1958, where a country is made worse off by the growth of its own production sector. There are two goods, exportables ($X$) and importables ($Y$), and the domestic economy is represented by:

(i) a production frontier $Y = F(X, \alpha)$, where $F$ is of class $C^2$, decreasing in $X$ and increasing in $\alpha$ and concave in $X$;

(ii) a community utility function $U(X_d, Y_d)$ that is of class $C^2$, increasing and quasi-concave.

Domestic growth is here represented by increases in the parameter $\alpha$.

The domestic country can trade with the rest of the world (ROW) according to the foreign offer curve $Y_e = O(X_e)$, where $X_e$ are the exports and $Y_e$ are the imports and $O$ is of class $C^2$ and increasing. In the entire problem, we assume that there is free trade between the domestic country and ROW. We use subscripts to denote partial derivatives.

First, assume that the domestic country is “small.” In this case, imports are a linear function of exports.

0.5] 1. Use a simple graph to argue that domestic growth always leads to an increase in domestic welfare in this case.

**Solution:** See attached figure.

We now turn to the more interesting case where the domestic country is “large,” that is, it faces deteriorating terms of trade.

0.4] 2. What property of the function $O$ ensures that the terms of trade are deteriorating? Interpret.

**Solution:** The terms of trade deteriorate with the volume of trade as long as $O$ is strictly concave. As the volume of trade expands, Home must consent more and more exports to obtain a unit of imports. Another way of saying this is that the price of imports in terms of exports is increasing with the volume of trade.

0.4] 3. Express the price of exportables relative to that of importables (that is, the terms of trade) as a function of $X_e$ and $Y_e$.

**Solution:** The price of exportables relative to that of importables is the average rate of exchange of importables against exportables on the world market, that is, $\frac{Y_e}{X_e}$.

0.8] 4. Deduce from the previous question the two first-order conditions that hold in the domestic economy at the free trade equilibrium. (Hint: You do not need to solve any optimization program, just write down the first-order conditions.)
Solution: At the free trade equilibrium, Home’s marginal rate of substitution must be equal to the world price ratio: \( \frac{U_X}{U_Y} = \frac{X_e}{Y_e} \). In addition, production must happen at a point on the production frontier where the marginal rate of transformation is equal to the world price ratio, that is, \( -F_X = \frac{X_e}{Y_e} \). We can rewrite these two conditions as

\[
\begin{align*}
U_X X_e - U_Y Y_e &= 0 \\
-F_X X_e - Y_e &= 0
\end{align*}
\]

5. Write down the set of four equations in four unknowns that implicitly define the free trade equilibrium as a function of the exogenous parameter \( \alpha \). (Hint: there are several ways to write these conditions depending on which variables you decide to drop. I recommend that you keep the following endogenous variables: \( X, X_e, Y, Y_e \).)

Solution: The free trade equilibrium is characterized by

\[
\begin{align*}
U_X (X - X_e, Y + Y_e) X_e - U_Y (X - X_e, Y + Y_e) Y_e &= 0 \\
-F_X (X, \alpha) X_e - Y_e &= 0 \\
Y - F(X, \alpha) &= 0 \\
Y_e - O(X_e) &= 0
\end{align*}
\]

which is a system of 4 equations in the 4 unknowns \( X, X_e, Y \) and \( Y_e \).

6. By totally differentiating \( U(X_d, Y_d) \) with respect to \( \alpha \) and using your answer to the previous question, show that

\[
\frac{dU}{d\alpha} = U_Y F_\alpha + U_X \left[ -1 + \frac{X_e}{Y_e} O' \right] \frac{dX_e}{d\alpha}.
\]

(Hint: You do not need to apply the implicit function theorem.)

Solution: We have \( U(X_d, Y_d) = U(X - X_e, Y + Y_e) \), and therefore

\[
\frac{dU}{d\alpha} = U_X \left( \frac{dX}{d\alpha} - \frac{dX_e}{d\alpha} \right) + U_Y \left( \frac{dY}{d\alpha} + \frac{dY_e}{d\alpha} \right) = U_X \left( \frac{dX}{d\alpha} - \frac{dX_e}{d\alpha} \right) + U_X \frac{X_e}{Y_e} \left( F_X \frac{dX}{d\alpha} + F_\alpha + O' \frac{dX_e}{d\alpha} \right) = U_Y F_\alpha + U_X \frac{dX_e}{d\alpha} \left[ 1 + F_X \frac{X_e}{Y_e} \right] + U_X \frac{dX_e}{d\alpha} \left[ -1 + O' \frac{X_e}{Y_e} \right] = U_Y F_\alpha + U_X \left[ -1 + \frac{X_e}{Y_e} O' \right] \frac{dX_e}{d\alpha},
\]

where we have used \( -F_X (X, \alpha) X_e - Y_e = 0 \) to obtain the last equality.
7. Is the result in part 6 consistent with your answer to part 1? Explain.

**Solution:** If the terms of trade are fixed (small country), then the average rate of exchange of imports for exports is always equal to the marginal rate of exchange, that is, \( \frac{Y}{X} = O' \) everywhere on the foreign offer curve. Therefore, the second term in the expression drops out and we have \( \frac{dU}{d\alpha} = U_Y F_\alpha > 0 \). Therefore the expression is consistent with our answer to part 1.

8. Deduce from part 6 necessary conditions for domestic growth to be immiserizing, that is, utility-reducing.

**Solution:** The first term in the expression for \( \frac{dU}{d\alpha} \) is always positive, therefore to obtain the paradoxical result the second term must be sufficiently negative. The multiplicative factor \(-1 + \frac{X_e}{X}O'\) is negative from the concavity of the foreign offer curve under deteriorating terms of trade. We must therefore have \( \frac{dX_e}{d\alpha} > 0 \), that is, the expansion in the production set must result in an increase in the volume of trade. The necessary conditions are thus that the country be large, and that the growth results in a sufficiently large increase in the volume of trade.
Part 1

foreign offer curve

\[ y = F(x, z) \]

\[ u_1 > u_0 \]
Consider an industry with \( J \) firms, each of them producing a distinct variety of a differentiated product, with the understanding that consumers may wish to consume all varieties. Denote by \((p_1, \ldots, p_J)\) the vector of the prices charged by the \( J \) firms.

The demand for variety \( j \), addressed to firm \( j \) by consumers, is given by the expression

\[ f_j(p_1, \ldots, p_J) = a[P(p_1, \ldots, p_J)]^{\sigma-1} p_j^{-\sigma}. \tag{1} \]

where the price index \( P(p_1, \ldots, p_J) \) is defined as

\[ P(p_1, \ldots, p_J) = \left[ \sum_{j=1}^{J} [p_j]^{-\sigma} \right]^{\frac{1}{1-\sigma}}, \tag{2} \]

and the magnitude \( a \), related to the wealth of the consumers in the economy, is taken as given by all firms.

1.1. What do (1) and (2) suggest on the preferences of the consumers?

For \( j = 1, \ldots, J \), firm \( j \)'s cost function is given by

\[ C(y_j) = \begin{cases} 
0, & \text{if } y_j = 0, \\
 p_0 y_j + F, & \text{if } y_j > 0,
\end{cases} \]

where \( p_0 > 0, F > 0, \) and \( y_j \) denotes the amount of output (variety \( j \)) produced by firm \( j \).

1.2. Comment on the cost function. Is this technology consistent with perfectly competitive (price-taking) behavior? Explain.

In what follows, assume that \( \sigma > 1 \), and let firm \( j (j = 1, \ldots, J) \) maximize its profits under the perception that the price index \( P \) is fixed, but that firm \( j \) can freely choose its own price \( p_j \). (In other words, firm \( j \) views the prices of other firms as given and, in addition, considers the price index \( P \) as independent from its own price, a perception that is approximately justified if \( J \) is large and \( p_j \) is not too large relative to other prices.)

1.3. Write and solve firm \( j \)'s profit maximization problem. Interpret the parameter \( \sigma \) in terms of the competitiveness of the industry.
1.4. It is often postulated in monopolistic competition analysis that the entry of firms drives profits to zero. Accordingly, let the number $J$ of firms (and varieties) adjust so that, for each firm, output is positive but profits are zero. Determine the number $J$ of firms as a function of $a$, $F$ and $\sigma$ under this condition. (For simplicity, here and in what follows we let $J$ be a positive real number; we do not restrict $J$ to be an integer.)

1.5. The aim is now to relate the number of firms to the parameter $\sigma$. As indicated above, $a$ is related to the wealth of buyers in the industry. We now assume that, through general-equilibrium effects that are left unspecified, $a$ is actually a function $\tilde{a}(\sigma)$, with elasticity satisfying

$$0 \leq \tilde{a}'(\sigma) \frac{\sigma}{\tilde{a}(\sigma)} < 1,$$  \hspace{1cm} (3)

which admits the possibility that $\tilde{a}'(\sigma) = 0$.

In addition, we consider two cases concerning $F$.

CASE 1. $F$ is a fixed parameter. This is the conventional case.

CASE 2. $F$ is decreasing in $\sigma$, more precisely

$$\tilde{F}(\sigma) = \frac{b}{\sigma},$$

where $b > 0$. The motivation for Case 2 is that $F$ is an advertising cost that varies with $\sigma$.

Does the number of firms increase or decrease with $\sigma$ in Case 1? In Case 2? Argue your answer, and comment.
Question 2

Consider an economy with 2 consumption goods, good 1 and 2, and two factors of production, capital and labor. There is a representative agent who owns the factors of production, normalized to one unit each. Thus $\omega = (1, 1)$. The representative agent’s utility function is $u(x_1, x_2) = \sqrt{x_1 x_2}$. Good 1 is produced by a firm with production function $y_1 = (\min\{k_1, l_1\})^2$. Good 2 is produced by the representative firm 2 with production function $y_2 = \sqrt{k_2 l_2}$. The representative agent and the representative firm 2 actually stand in for a large number of agents and firms producing good 2, so we can assume that consumers take prices as given, and firm 2 maximizes profit taking prices as given. However since there are increasing returns in the production of good 1, it is more efficient to produce it in a single firm, which is thus a “natural” monopoly. Actually even if we assumed that firm 1 maximized profit taking prices as given we could not find a competitive equilibrium. Thus we assume that, to prevent it from using its market power, a planner takes control of firm 1 instructing it to minimize cost (taking factor prices as given) and sell its output at marginal cost. The planner subsidizes the firm for its losses, using lump-sum taxation of the consumer to finance the subsidy. An equilibrium where the consumer and firm 2 behave as in a competitive equilibrium, firm 1 minimizes cost, the price of good 1 is the marginal cost of production, and markets clear, is called a marginal cost pricing equilibrium.

(a) First let us justify some assertions of the text above. Show that there is no solution to maximizing the profit of firm 1 if it takes input and output prices as given. Then show that if the firm sells its output at marginal cost (once the cost is minimized) it makes a loss. Calculate the loss as a function of the quantity produced and the input prices.

(b) Find the Pareto optimal allocation for this economy. (When you assert that a function is maximized, be sure that you justify your assertion.)

(c) Express formally the conditions on the actions of the consumer and firms and the market prices that must be satisfied in order that $\left( (\bar{x}_1, \bar{x}_2), (\bar{y}_1, \bar{k}_1, \bar{l}_1), (\bar{y}_2, \bar{k}_2, \bar{l}_2), (\bar{p}_1, \bar{p}_2, \bar{r}, \bar{w}) \right)$ is a marginal cost pricing equilibrium.

(d) Find the marginal cost pricing equilibrium. Is it Pareto optimal?

(e) Using a simple Robinson Crusoe economy with two goods and increasing returns in production explain as best as you can the normative property of the marginal cost pricing equilibrium.
**Question 3**

Consider an economy with two goods, labor and a consumption good. There are two firms producing the consumption good from labor. Firm 1’s production function is \( y_1 = L_1 \) and firm 2’s production function is \( y_2 = 2\sqrt{L_2} \), where \( y_j \) is the production of firm \( j \) and \( L_j \) is the labor used by firm \( j \), \( j = 1, 2 \). Firm 2’s technology produces chemical emissions in quantity proportional to its level of production. The units are such that the amount \( z \) of pollution associated with a level of production \( y_2 \) is \( z = y_2 \). There are two (types of) agents \((i = 1, 2)\) with the same utility function

\[
u(x, \ell, z) = \sqrt{x\ell} - \frac{z}{8},
\]

where \( x \) is the amount of consumption good, \( \ell \) is the amount of time used as leisure, and \( z \) is the level of pollution. Each agent is endowed with 3 units of time which can be used as labor or leisure. Agent 1 owns firm 1 while agent 2 owns firm 2.

(a) Calculate the competitive equilibrium with externality for this economy. Show that the level of pollution is \( \bar{z} = 2 \).

(b) Suppose the government can limit pollution by imposing a limit \( z \) on the emissions of firm 2. (If firm 2 produces more emissions than it is allowed, the firm is closed by the government and the owner does not receive any profit). Firms and consumers behave like in (a) (except that firm 2 needs to take the limit \( z \) into account). Find the resulting equilibrium as a function of the ceiling \( z \) on emissions. Let \( u_1(z) \) and \( u_2(z) \) the utility of both agents at this equilibrium.

(c) Show the following properties of the equilibrium found in (b)

(i) There is \( z^* \) such that if \( z^* \leq z < 2 \) both agents are better off than in the competitive equilibrium of (a) (the “laisser-faire” equilibrium)

(ii) There is a level \( \hat{z} > z^* \) such that for any \( z \) in \([\hat{z}, 2]\) there is an equilibrium with \( \bar{z} \in [z^*, \hat{z}] \) which is better from the point of view of both agents.

(iii) In the interval \([z^*, \hat{z}]\) there is no consensus among the two agents on the optimal level of pollution. Explain where the disagreement comes from.

[Hint: study carefully the function \( \phi(z) = u_2(z) - u_2(2) \)]

(d) Suppose that in addition to controlling the level of emissions of firm 2 the government can redistribute income among the agents with lump sum taxes and subsidies. Show that the set of achievable utilities can be written as

\[
u_1 + u_2 = \phi(z)
\]

Find the level of emission which maximizes \( u_1 + u_2 \) and relate it to the discussion in question (c).
Question 4.

There are two parties to a potential lawsuit: the owner of a chemical plant and a supplier of safety equipment. The chemical plant owner (from now on called the plaintiff) alleges that the supplier (from now on called the defendant) was negligent in providing the safety equipment. The defendant knows whether or not he was negligent, while the plaintiff does not know. The plaintiff believes that there was negligence with probability \( q \). These beliefs are common knowledge between the parties. The plaintiff has to decide whether or not to sue. If she does not sue then nothing happens and both parties get a payoff of 0. If the plaintiff sues then the defendant can either offer an out-of-court settlement of \( S \) or resist. If the defendant offers a settlement, the plaintiff can either accept (in which case her payoff is \( S \) and the defendant’s payoff is \(-S\)) or go to trial. If the defendant resists then the plaintiff can either drop the case (in which case both parties get a payoff of 0) or go to trial. If the case goes to trial then legal costs are created in the amount of \( P \) for the plaintiff and \( D \) for the defendant. Furthermore (if the case goes to trial), the judge is able to determine if there was negligence and, if there was, requires the defendant to pay \( W \) to the plaintiff (and each party has to pay its own legal costs), while, if there was no negligence, the judge will drop the case without imposing any payments to either party (but each party has to pay its own legal costs). Each party is “selfish and greedy” (that is, only cares about its own wealth and prefers more to less) and is risk neutral.

Assume the following about the parameters: \( 0 < q < 1, \ 0 < D < S, \ 0 < P < S < W - P \).

(a) Represent this situation of incomplete information using states and information partitions (the only two agents are the plaintiff and the defendant). Be clear about what each state represents or describes.

(b) Apply the Harsanyi transformation to represent the situation in part (a) as an extensive-form game. [Don’t forget to subtract the legal expenses from each party’s payoff if the case goes to trial.]

(c) Write down all the strategies of the plaintiff.

(d) Prove that there is no pure-strategy weak sequential equilibrium which (1) is a separating equilibrium and (2) involves suing.

You might want to answer question (f) before question (e) to gain some insight, but you will be duplicating some effort.

(e) For what values of the parameters \( q, S, P, W, D \) are there pure-strategy weak sequential equilibria which (1) are pooling equilibria and (2) involve suing? Consider all types of pooling equilibria and prove your claim.

(f) For the case where \( q = \frac{1}{12}, \ P = 70, \ S = 80, \ W = 100 \) find all the pure-strategy weak sequential equilibria which (1) are pooling equilibria and (2) involve suing.
Question 5.

Is it better for the shareholders to remunerate CEOs with profit-sharing contracts or other types of contracts? One of the issues is, of course, moral hazard on the part of the CEO. In this question we consider instead a different issue, namely a strategic one.

Consider the following situation. There are two individuals, 1 and 2. Each individual is the sole owner of a firm in a Cournot duopoly. The inverse demand function is \( P = 60 - Q \) and both firms have the same cost function characterized by a constant marginal cost equal to 12 and zero fixed cost. Individuals 1 and 2 simultaneously and independently decide whether to appoint a manager with a profit-sharing contract (the manager of firm \( i \) gets the fraction \( \alpha \) of the profit of firm \( i \)) or a revenue-sharing contract (the manager of firm \( i \) gets the fraction \( \alpha \) of the revenue of firm \( i \)), where \( 0 < \alpha < 1 \). The value of \( \alpha \) is given exogenously (is not a choice variable). The contracts are then made public (that is, they become common knowledge among everybody in the industry) and afterwards the managers simultaneously compete in output levels. The objective of each manager is to maximize her own income. The objective of each owner is to maximize his own net income (= profit of the firm minus the payment to the manager).

(a) Sketch the extensive-form game.

(b) Find the pure-strategy subgame-perfect equilibria of this game for every value of \( \alpha \).

(c) In the past, each owner used to run the firm himself and the industry was a Cournot duopoly. Now they have delegated the running of the firms to managers. Suppose that \( \alpha \) is small (say \( \alpha = \frac{1}{100} \)). Has delegation led to an increase or a decrease in the income of the owners of the firms?
Question 1. The number of firms in monopolistic competition

**ANSWER KEY**

Consider an industry with \( J \) firms, each of them producing a distinct variety of a differentiated product, with the understanding that consumers may wish to consume all varieties. Denote by \((p_1, \ldots, p_J)\) the vector of prices charged by the \( J \) firms.

The demand for variety \( j \), addressed to firm \( j \) by consumers, is given by the expression

\[
f_j(p_1, \ldots, p_J) = a[P(p_1, \ldots, p_J)]^{\sigma-1} p_j^{-\sigma},
\]

(1)

where the price index \( P(p_1, \ldots, p_J) \) is defined as

\[
P(p_1, \ldots, p_J) = \left[ \sum_{j=1}^{J} [p_j]^{1-\sigma} \right]^{1/(1-\sigma)},
\]

(2)

and the magnitude \( a \), related to the wealth of the consumers in the economy, is taken as given by all firms.

1.1. What do (1) and (2) suggest on the preferences of the consumers?

**Answer.** Demand functions as those given by (1) and (2) are related to CES preferences. If \( a \) were the wealth of a consumer, then (1) would be her Walrasian demand for the utility function

\[
u(x_1, \ldots, x_J) = \left( \sum_{j=1}^{J} x_j^{\sigma-1} \right)^{\frac{\sigma}{\sigma-1}}, \sigma \in (-\infty, 0) \cup (0, 1) \cup (1, \infty).
\]

(A1)

Because a positive representative consumer exists under identical, homothetic preferences, (1) would be the aggregate demand from a consumption section with aggregate wealth \( a \) and individual utility functions given by (A1).
But $a$ is not assumed to coincide with wealth: it could for example be a fraction of aggregate wealth, in which case the preferences of the consumers could also involve goods other than varieties $(1,\ldots,J)$, with a $(A1)$ providing a subutility function for these varieties.

For $j = 1,\ldots,J$, firm $j$'s cost function is given by

$$C(y_j) = \begin{cases} 0, & \text{if } y_j = 0, \\ p_0 y_j + F, & \text{if } y_j > 0, \end{cases}$$

where $p_0 > 0$, $F > 0$, and $y_j$ denotes the amount of output (variety $j$) produced by firm $j$.

1.2. Comment on the cost function. Is this technology consistent with perfectly competitive (price-taking) behavior? Explain.

**Answer.** This technology displays constant marginal cost, at level $p_0$, and a fixed, avoidable (i.e., nonsunk) cost, at level $F$. This implies that the average cost is decreasing. Perfectly competitive behavior is incompatible with positive production with this technology. Trivially, one could have price-taking profit maximization with market price less than or equal to $p_0$ and zero output.

In what follows, assume that $\sigma > 1$, and let firm $j$ ($j = 1,\ldots,J$) maximize its profits under the perception that the price index $P$ is fixed, but that firm $j$ can freely choose its own price $p_j$. (In other words, firm $j$ views the prices of other firms as given and, in addition, considers the price index $P$ as independent from its own price, a perception that is approximately justified if $J$ is large and $p_j$ is not too large relative to other prices.)

1.3. Write and solve firm $j$'s profit maximization problem. Interpret the parameter $\sigma$ in terms of the competitiveness of the industry.

**Answer.** The profit maximization problem of the firm can be written in two steps

**Step 1.** Choose $p_j$ order to maximize $[p_j - p_0][p_j]^{-\sigma}aP^{\sigma-1} - F$.

**Step 2.** If profits are nonnegative at the solution $p_j^*$ to Step 1, then set the price equal to $p_j^*$ and the quantity equal to $[p_j^*]^{-\sigma}aP^{\sigma-1}$. Otherwise, set the quantity equal to zero.

**(Remark.** If profits are zero at the solution $p_j^*$ to Step 1, then both $y_j = [p_j^*]^{-\sigma}aP^{\sigma-1}$ and $y_j = 0$ maximize profits; we will focus on the positive solution in this case.)
The solution to Step 1 satisfies
\[ [p_j]^\sigma aP^{\sigma-1} - \sigma[p_j - p_0][p_j]^{\sigma-1} aP^{\sigma-1} = 0, \]
or dividing through by \([p_j]^{\sigma-1} aP^{\sigma-1},\) \( p_j - \sigma[p_j - p_0] = 0,\) which yields \( p_j = \frac{\sigma}{\sigma-1} p_0,\) same for all firms, i.e.,
\[ p_j = p := \frac{\sigma}{\sigma-1} p_0. \] (A2)

The markup = \([\text{PRICE-MARGINAL COST}]/\text{PRICE}\) is then
\[ \frac{p - p_0}{p} = \frac{1}{\sigma}, \] (A3)
equal to the degree of monopoly, which is the reciprocal of the absolute value of the price elasticity of the demand curve faced by the firm. Hence, the higher \( \sigma,\) the higher the degree of competitiveness in the industry.

1.4. It is often postulated in monopolistic competition analysis that the entry of firms drives profits to zero. Accordingly, let the number \( J \) of firms (and varieties) adjust so that, for each firm, output is positive but profits are zero. Determine the number \( J \) of firms as a function of \( a, F \) and \( \sigma \) under this condition. (For simplicity, here and in what follows we let \( J \) be a positive real number: we do not restrict \( J \) to be an integer.)

**ANSWER.**

Using (2) and (A2), we can write
\[ P(p, \ldots, p) = [Jp^{1-\sigma}]^{1-\sigma} = J^{1-\sigma} p, \]
so that, from (1), the equilibrium output of a firm becomes
\[ a \left[ J^{1-\sigma} p \right]^{\sigma-1} p^{-\sigma} = aJ^{-1} p^{\sigma-1} p^{-\sigma} = \frac{a}{Jp}, \]
and the zero-profit condition becomes \([p - p_0]\frac{a}{Jp} = F,\)
or, using (A3),
\[ J = \frac{a}{\sigma F}. \quad \text{(A4)} \]

1.5. The aim is now to relate the number of firms to the parameter \( \sigma \). As indicated above, \( a \) is related to the wealth of buyers in the industry. We now assume that, through general-equilibrium effects that are left unspecified, \( a \) is actually a function \( \tilde{a}(\sigma) \), with elasticity satisfying

\[ 0 \leq \tilde{a}'(\sigma) \frac{\sigma}{\tilde{a}(\sigma)} < 1, \quad \text{(3)} \]

which admits the possibility that \( \tilde{a}'(\sigma) = 0 \).

In addition, we consider two cases concerning \( F \).

**CASE 1.** \( F \) is a fixed parameter. This is the conventional case.

**CASE 2.** \( F \) is decreasing in \( \sigma \), more precisely

\[ \tilde{F}(\sigma) = \frac{b}{\sigma}, \]

where \( b > 0 \). The motivation for Case 2 is that \( F \) is an advertising cost that varies with \( \sigma \).

Does the number of firms increase or decrease with \( \sigma \) in Case 1? In Case 2? Argue your answer, and comment.

**Answer.** **Case 1.** From (A4), \( \tilde{J}(\sigma) = \frac{\tilde{a}(\sigma)}{\sigma F} \), and

\[ \tilde{J}'(\sigma) = \frac{\tilde{a}'(\sigma) \sigma F - F \tilde{a}(\sigma)}{\sigma^2 F^2} = \frac{\tilde{a}(\sigma)}{\sigma^2 F} \left[ \tilde{a}'(\sigma) \frac{\sigma}{\tilde{a}(\sigma)} - 1 \right] < 0, \]

by (3), which is somewhat counterintuitive: higher competitiveness (higher \( \sigma \)) yields fewer firms. An explanation is that, as \( \sigma \) increases, the markup goes down, making it harder to cover the fixed costs unless the number of firms decreases and the output per firm increases.

**Case 2.** Now \( \tilde{J}(\sigma) = \frac{\tilde{a}(\sigma)}{\sigma F} = \frac{\tilde{a}(\sigma)}{b} \), nondecreasing in \( \sigma \), and actually increasing if \( \tilde{a}'(\sigma) > 0 \). This is more intuitive, but it relies on the assumption that, as competitiveness increases, the fixed cost goes down.
(a) Let \((p_1, p_2, r, w) \geq 0\) denote the prices of good 1, good 2, capital and labor respectively. To maximize profit, firm 1 must first minimize cost which implies \(k_1 = f_1\). The profit of firm 1 is then
\[
\Pi_1 = p_1 f_1 - (w+r) f_1. \quad \frac{d\Pi_1}{dk_1} = 2 p_1 f_1 - (w+r),
\]
which is positive when
\[
l_1 > \frac{w+r}{2 p_1}.
\]
When \(l_1 \to +\infty\), \(\Pi_1 \to +\infty\) and therefore there is no maximum for the profit.

The cost function is \(c(y_1) = (r+w)\sqrt{y_1}\). If the firm sells \(y_1\) at price \(p_1 = \frac{r+w}{2\sqrt{y_1}}\), the profit is
\[
\sqrt{\frac{r+w}{2\sqrt{y_1}}} - (r+w)\sqrt{y_1} = -\frac{1}{2} (r+w)\sqrt{y_1}.
\]
The loss is thus \(\frac{1}{2} (r+w)\sqrt{y_1}\), which increases with the scale of production.

(b) The Pareto optimal allocation must maximize
\[
\sqrt{n_1, n_2}
\]
subject to
\[
\begin{align*}
n_1 & \leq k_1, \\
k_1 & \leq f_1, \\
n_2 & \leq \sqrt{k_1 k_2}, \\
k_1 + k_2 & \leq 1, \\
l_1 + l_2 & \leq 1, \\
l_1 & > 0, l_2 > 0, k_1 > 0, k_2 > 0.
\end{align*}
\]
Since the utility of the representative agent is 0 if one good is not produced while it can be positive, and since the utility is strictly monotonic, the non-negativity constraints are not binding.
and the other constraints are binding, so that the P.O. allocates

\[
\text{maximizes} \quad \sqrt{b^2 \sqrt{(1-b,)/(1-k_1)}}
\]

(since \(k_i = \bar{b}_i, y_i = k^2_i\), \(k = k_i, \bar{k}_2 = k_2 = 1 - k_i\))

or equivalently maximizes \(k^2 (1 - k_i) \equiv \varphi (k_i)\)

\(\varphi' (k_i) = 2k_i (1 - k_i) - k^2_i = 2k_i - 3k_i^2\), which leads to the

variation of \(\varphi_i\) of the form

\[
\begin{array}{ccc}
0 & \frac{2}{3} & 1 \\
\varphi'(k_i) & + & - \\
\varphi(k_i) & \rightarrow & \varphi(2/3)
\end{array}
\]

Thus the maximum is obtained \(b_i^* = \frac{2}{3} = \bar{b}_i^*\), \(y_i^* = \frac{4}{9}\)

\(b_i^* = \bar{b}_i^* = \frac{2}{3}, y_i^* = \frac{4}{9}\).

\(\text{(c) } \quad ((\bar{x}_i, \bar{x}_e), (\bar{y}_i, \bar{b}_i, \bar{p}_i) (\bar{f}_i, \bar{k}_i, \bar{p}_i) (\bar{f}_i, \bar{p}_i, \bar{f}_i, \bar{w}_i))\)

is a marginal cost pricing equilibrium if

1. \((\bar{b}_i, \bar{p}_i)\) minimizes \(c_{\bar{p}, \bar{w}} (\bar{y}_i)\)

2. \(\bar{p}_i = c'_{\bar{p}, \bar{w}} (\bar{y}_i)\) (marginal cost pricing)

3. \((\bar{f}_2, \bar{k}_2, \bar{p}_2)\) maximizes \(\bar{f}_2 y_2 = \bar{f} - k_2 - \bar{p}_2\) s.t. \(y_2 = \sqrt{\bar{k}_2 \bar{p}_2}\)

4. \((\bar{x}_2, \bar{x}_e)\) maximizes \(\sqrt{x_2} x_e\) subject to

\[\bar{p}_2 x_2 + I_{\bar{p}_2} x_e = \bar{f} + \bar{w} + \bar{\pi}_2 - \bar{t}\]

where \(\bar{\pi}_2\) is the subsidy of firm 2 (here 0) and \(\bar{t}\) is the

subsidy to firm 1

5. \(\bar{t} + \bar{p}_2 \bar{y}_2 = \bar{f}_2, \bar{\pi}_2 = 0\) (subsidy = subsidy = loss of firm 1)

6. \(\bar{y}_2 = \bar{x}_2, \bar{f}_2 = \bar{f}_e, \bar{b}_2 + \bar{p}_2 = 1, \bar{b}_2 + \bar{k}_2 = 1\)
(c) \textbf{firm 1:} \quad \text{from (a): } b_i = \bar{p}_i = \sqrt{y_i}; \quad \bar{p}_i = \frac{\bar{p} + \bar{w}}{2\sqrt{\bar{d}_i}}

\text{from 2:} \quad \text{max } \bar{p}_2 \sqrt{b_2 \bar{d}_2} - \bar{w} \bar{p}_2 + \bar{p} \bar{k}_2

\text{A FOC: } \bar{p}_2 \sqrt{\bar{b}_2 \bar{d}_2} = \bar{p} \quad \bar{b}_2 = \frac{\sqrt{b_2 \bar{d}_2}}{2 \sqrt{\bar{d}_2}} = \bar{w}

\Rightarrow \quad \frac{\bar{b}_2}{\bar{d}_2} = \frac{\bar{w}}{\bar{y}_2} \quad \text{and} \quad \bar{p}_2 = 4 \bar{p} \bar{w} \quad (\text{condition on price imposed by constant returns to scale})

the scale of production adapt to the demand and

\bar{d}_2 = \sqrt{\frac{\bar{b}_2}{\bar{w}}} \bar{y}_2, \quad \bar{k}_2 = \sqrt{\frac{\bar{w}}{\bar{b}_2}} \bar{y}_2.

\underline{Consumer income of the consumer: } \bar{r} + \bar{w} - \frac{\bar{r} + \bar{w}}{2} \sqrt{\bar{d}_i},

(\text{the loss of the firm is subsidized by a lumpsum tax on the consumer and the profit of firm 2 is zero}).

\underline{Cobb-Douglas demand:}
\begin{align*}
x_1 &= \frac{\bar{r} + \bar{w} - \frac{\bar{r} + \bar{w}}{2} \sqrt{\bar{d}_i}}{\bar{y}_1} \quad \underline{x_2} = \frac{\bar{r} + \bar{w} - \frac{\bar{r} + \bar{w}}{2} \sqrt{\bar{d}_i}}{2 \bar{p}_2}
\end{align*}

\underline{market clearing:} \quad \bar{x}_i = \bar{y}_i.

\Rightarrow \quad \bar{y}_1 = \frac{\bar{r} + \bar{w} - \frac{\bar{r} + \bar{w}}{2} \sqrt{\bar{d}_i}}{2} \quad \Rightarrow \quad \bar{y}_1 = \sqrt{\bar{d}_i} \left( 1 - \frac{\sqrt{\bar{d}_i}}{2} \right)

\Rightarrow \quad \sqrt{\bar{d}_i} = 1 - \frac{\sqrt{\bar{d}_i}}{2}

\Rightarrow \quad \frac{3}{2} \sqrt{\bar{d}_i} = 1

\Rightarrow \quad \sqrt{\frac{1}{3}} = \frac{4}{9}
Thus \( \bar{r}_1 = \bar{k}_1 = \frac{2}{3} \), \( \bar{k}_2 = \bar{p}_2 = \frac{1}{3} \), \( \bar{r} = \bar{w} = 1 \) (no molotion)

\[
\begin{align*}
\bar{p}_1 &= \frac{2}{3}, \\
\bar{p}_2 &= \frac{3}{2}, \\
\bar{p}_3 &= 2
\end{align*}
\]

Check: \( \bar{a}_2 = \frac{2 - \frac{2}{3} \times \frac{1}{2}}{2 \times 2} = \frac{2 - \frac{1}{3}}{4} = \frac{1}{3} \)

(c) Suppose that there are only 2 goods: labor and good 1 and one consumer providing the labor and consuming good 1. If the price of good 1 is the marginal cost of production, the rate of transformation in production and the marginal rate of substitution in the consumer were equalized, which is a necessary condition for Pareto optimality. However, since the production set is non convex, the condition is not sufficient so that marginal cost pricing equilibrium may or may not be Pareto optimal. In the example above the condition turns out to be sufficient.
(a) competitive equilibrium

- firms profit maximizing with a particular level of production imposes $p = w$

- firm 2: \[ \text{max } 2 \ p \ L_2 - w \ L_2 \Rightarrow L_2 = \frac{p^2}{w} \quad y_2 = 2 \ p \ \frac{L_2}{w} = 2 \]

\[ L_2 \geq 0 \]

since $p = w$ \[ L_2 = 1 \quad y_2 = 2 \Rightarrow 2 \]

- consumer 1: \[ \text{max } \sqrt{x_1^3} \quad \text{tastes as given } \]

\[ p \ x = w (3 - e) + 0 \quad \text{with } p = w \]

\[ s^2 = \frac{3}{2} \quad \frac{p^2}{w} = \frac{3}{2} \quad L_1 = 3 - \frac{5}{2} = 3/2 \]

- consumer 2: \[ \text{max } \sqrt{x_2^3} \]

\[ p \ x = w (3 - e) + \frac{p^2}{w} \]

\[ x_2^2 = \frac{1}{2} \quad 3w + \frac{p^2}{w} = \frac{2}{p} \]

\[ \frac{p^2}{w} = 2 \quad \frac{L_2}{w} = 1 \]

(indices of consumers in superscript, indices of firms in subscript)

At equilibrium firm 2 uses 1 unit of labor and firm 1 uses 3/2 units of labor. The total production is 3/2 + 2, which equals the total demand. The level of pollution is $z = 2$.

At equilibrium \[ u' = \sqrt{\frac{3}{2} \times \frac{3}{2} - \frac{2}{8}} = \frac{3}{2} - \frac{1}{4} = \frac{5}{4} \]

\[ u^2 = 2 - \frac{2}{8} = 2 - \frac{1}{4} = \frac{7}{4} \]
(b) equilibrium with pollution limited at $z$.

- Firm 1 still imposes $\rho = \omega$; that we normalize to 1.
- Firm 2: if $z \leq 2$; max $2p\sqrt{L_z} - wL_z$ s.t. $L_z \leq z$.

It is easy to see that it produces at the maximum allowed

$$L_z = \frac{z}{4} \quad y_z = z \quad \pi_z = p \frac{z^2}{2} - \frac{w z^2}{4} = z - \frac{w z^2}{4} \quad (p, w = 1)$$

- Consumer 1: $a^1 = l^1 = \frac{3}{2} \quad L^1 = \frac{3}{2} \quad u^1 = \frac{3}{2} - \frac{w}{8}$
- Consumer 2: $a^2 = l^2 = \frac{1}{2} \left( \frac{3 + z - \frac{z^2}{8}}{4} \right) \quad L^2 = \frac{3}{2} - \frac{z}{2} + \frac{w^2}{8}

At equilibrium firm 2 uses $\frac{z^2}{4}$ units of labor to produce 2 units of goods. Firm 1 uses $\frac{3}{2} + \frac{3}{2} - \frac{w}{8} - \frac{z^2}{8} = 3 - \frac{w}{2} - \frac{z^2}{8}$ to produce the same quantity of good and market clear. The utility of agent 2 is $u^2 = \frac{1}{2} \left( \frac{3 + z - \frac{z^2}{4}}{2} \right) - \frac{w}{8}$

Thus $\begin{cases} u^1(z) = \frac{3}{2} - \frac{w}{8} \\ u^2(z) = \frac{3}{2} + \frac{3 z}{8} - \frac{z^2}{8} - \frac{w^2}{8} \end{cases}$

(c) (i) If $z < 2$ agent 1 is better off than in the laissez faire equilibrium of (a), and the smaller $z$ the better off agent 1 is.

For agent 2 $u^2(z) - u^2(2) = \frac{3}{2} + \frac{3 z}{8} - \frac{z^2}{8} - \frac{w^2}{4}$

$$= -\frac{1}{4} + \frac{3}{8} z - \frac{z^2}{8}$$

Let $\varphi(x) = -\frac{1}{4} + \frac{3}{8} x - \frac{x^2}{8} \quad \varphi'(x) = \frac{3}{8} - \frac{x}{4}$
variations of $\psi$

\[
\begin{array}{cccc}
0 & \quad 2^* & \quad 3/2 & \quad 2 \\
\psi'(x) & + & - & 0 \\
\psi(x) & \to & 1/2 & \to 0
\end{array}
\]

Since $\psi$ is increasing from $-1/4$ to $1/2$ when $x$ varies from 0 to $3/2$, there exists $2^* \in [0, 3/2)$ such that, if $2^* < x < 2$,

$\psi(x) > 0$. Thus if $2^* < x < 2$, both agents are better off than in the laissez-faire equilibrium.

(iii) If $x \in [2^*, 2]$, there exists $\bar{2} \in (2^*, 3/2)$ such that $\psi(\bar{2}) = \psi(2)$. If $\bar{2} < x$, agent 2 is strictly better off than if it is limited at $\bar{2}$, and agent 1 is also better since there is less pollution.

However, in the interval $[2^*, 3/2]$ the utility of agent 2 is increasing in $x$ and the utility of agent 1 is decreasing so that the agent cannot agree on the level of pollution to impose. Agent 1 always wants to reduce $x$ of pollution to impose, but agent 2 has two conflicting interests: as a consumer he/she wants to decrease $x$ but as a owner of the firm he/she wants to increase $x$ to produce more and increase the profit. In the interval $(2^*, 3/2)$ the income effect is stronger than the externality effect.
(d) If the planner can make a transfer from agent 2 to agent 1, then

\[ u_1 = \frac{3 + t}{2} - \frac{z}{8} \]

\[ u_2 = \frac{3 + z - \frac{z^2}{8}}{2} - \frac{z}{8} \]

Eliminate \( t \)

\[ u_1 + u_2 = \frac{6 + z - \frac{z^2}{4}}{2} - \frac{z}{8} = \frac{3}{2} + \frac{z}{4} - \frac{z^2}{8} \]

\[ \psi(z) = \frac{3}{2} + \frac{z}{4} - \frac{z^2}{8} \]

\[ \psi'(z) = \frac{1}{4} - \frac{z}{4} \]

The maximum of \( u_1 + u_2 \) is \( z = 1 \), in which case \( \psi(z) = 0 \). That is, \( u_1 + u_2 \) is maximized for \( z = z^* = 1 \). Thus \( t < 0 \) and the transfer goes from agent 1 to agent 2 to compensate the loss of profit due to a severe reduction in the production of good 2.
ANSWER KEY FOR QUESTION 4

(a) Let $G_1$ and $G_2$ be the following games (in $G_1$ the defendant is negligent and in $G_2$ he is not):

**GAME $G_1$**

<table>
<thead>
<tr>
<th></th>
<th>PLAINTIFF</th>
<th>DEFENDANT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resist</td>
<td>0</td>
<td>PLAINIFF</td>
</tr>
<tr>
<td></td>
<td>resist</td>
<td>offer</td>
</tr>
<tr>
<td></td>
<td>DROP</td>
<td></td>
</tr>
<tr>
<td></td>
<td>trial</td>
<td></td>
</tr>
<tr>
<td></td>
<td>settle</td>
<td></td>
</tr>
<tr>
<td></td>
<td>sue</td>
<td></td>
</tr>
</tbody>
</table>

**GAME $G_2$**

<table>
<thead>
<tr>
<th></th>
<th>PLAINTIFF</th>
<th>DEFENDANT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resist</td>
<td>0</td>
<td>PLAINIFF</td>
</tr>
<tr>
<td></td>
<td>resist</td>
<td>offer</td>
</tr>
<tr>
<td></td>
<td>DROP</td>
<td></td>
</tr>
<tr>
<td></td>
<td>trial</td>
<td></td>
</tr>
<tr>
<td></td>
<td>settle</td>
<td></td>
</tr>
<tr>
<td></td>
<td>sue</td>
<td></td>
</tr>
</tbody>
</table>

Then the situation can be represented as follows (state $\alpha$ is the state where the defendant is negligent and state $\beta$ is the state where the defendant is not negligent):

**$G_1$**

1: $\circ$

2: $\bullet q$ $(1-q) \bullet$

**$G_2$**


(b) The extensive game is as follows:

[Diagram of the extensive game]
(c) Plaintiff’s strategies: (1) (sue; if offer settle; if resist drop), (2) (sue; if offer settle; if resist go to trial), (3) (sue; if offer go to trial; if resist drop), (4) (sue; if offer go to trial; if resist go to trial), (5) (not sue; if offer settle; if resist drop), (6) (not sue; if offer settle; if resist go to trial), (7) (not sue; if offer go to trial; if resist drop), (8) (not sue; if offer go to trial; if resist go to trial). [The last four can be considered as one “plan of action” which we can call “Not sue”.]

(d) There are two possibilities for a separating equilibrium: (1) the defendant’s strategy is “resist if negligent and offer if not negligent” and (2) the defendant’s strategy is “offer if negligent and resist if not negligent”. In both cases we assume that the plaintiff’s strategy involves suing with probability 1.

Consider case (1) first. By Bayes’ rule, at the bottom information set the plaintiff must attach probability 1 to the negligent type and thus, by sequential rationality, must choose “trial” (because \( W - P > 0 \)). Similarly, by Bayes’ rule, at the middle information set the plaintiff must attach probability 1 to the non-negligent type and thus by sequential rationality must choose “settle”. But then the negligent type of the defendant gets \(- (W+D)\) by resisting and would get \(-S\) by offering. Since, by assumption, \( S < W < W + D \), the choice of resisting is not sequentially rational.

Consider now case (2). By Bayes’ rule, at the bottom information set the plaintiff must attach probability 1 to the non-negligent type and thus by sequential rationality must choose “drop”. But then the negligent type of the defendant gets a negative payoff by offering, while he would get 0 by resisting. Hence the choice of offering is not sequentially rational.

(e) There are two candidates for a pure-strategy pooling equilibrium: (1) both types of the defendant choose “offer” and (2) both types of the defendant choose “resist”.

Consider case (1) first (both types of the defendant choose “offer”). In order for “offer” to be sequentially rational for the non-negligent type, it cannot be that the plaintiff’s strategy involves “settle” at the middle information set (the non-negligent type would get either 0 or \(-D\) by resisting and both payoffs are greater than \(-S\)) and/or “drop” at the bottom information set. That is, it must be that the plaintiff chooses “trial” at both information sets. By Bayes’ rule, at the middle information set the plaintiff must attach probability \( q \) to the negligent type and probability \((1-q)\) to the non-negligent type. Hence at the middle information set “trial” is sequentially rational if and only if \( qW - P \geq S \), that is, \( q \geq \frac{S+P}{W} \). In order for “trial” to be sequentially rational at the bottom information set, the plaintiff must attach sufficiently high probability (namely \( p \geq \frac{P}{W} \)) to the negligent type. This is allowed by weak sequential equilibrium because the bottom information set is not reached. Finally, in order for “sue” to be sequentially rational it must be that \( qW - P \geq 0 \), that is, \( q \geq \frac{P}{W} \), which is implied by \( q \geq \frac{S+P}{W} \). Thus

\[
\text{there is a pooling equilibrium of type (1) with ((sue,trial,trial),(offer,offer) ) if and only if } \quad \frac{S+P}{W}.
\]

Now consider case (2) (both types of the defendant choose “resist”). [Note: since in part (f) the restriction \( S < W - P \) does not hold, we will carry out the analysis below at first without imposing the restriction.] If the plaintiff’s strategy involves “drop” at the bottom information set, then it is indeed sequentially rational for both types of the defendant to choose “resist”. Now, “drop” is sequentially rational in this case if and only if \( qW - P \leq 0 \), that is, \( q \leq \frac{P}{W} \). Then “sue” is also sequentially rational, since the Plaintiff’s payoff is 0 no matter whether he sues or does not sue. Thus
there is a pooling equilibrium of type (2) with \(((\text{sue},x,\text{drop}),(\text{resist},\text{resist}))\) if and only if \(q \leq \frac{P}{W}\) and appropriate beliefs as follows (\(p\) is the probability on the left node of the unreached middle information set):

- either \(x = \text{settle} \) and any \(p\) if \(W \leq S + P\) or \(p \leq \frac{S + P}{W}\) if \(W > S + P\)
- or \(x = \text{trial} \) and \(p \geq \frac{S + P}{W}\), which requires \(W \geq S + P\).

Since it is assumed that \(W > S + P\), we can conclude that

\[
\begin{array}{|l|}
\hline
((\text{sue},\text{settle},\text{drop}), (\text{resist},\text{resist})) \text{ is an equilibrium if and only if } q \leq \frac{P}{W} \text{ with } p \leq \frac{S + P}{W} \\
\hline
\end{array}
\]

\[
\begin{array}{|l|}
\hline
((\text{sue},\text{trial},\text{drop}), (\text{resist},\text{resist})) \text{ is an equilibrium if and only if } q \leq \frac{P}{W} \text{ with } p \geq \frac{S + P}{W} \\
\hline
\end{array}
\]

If, on the other hand, \(q \geq \frac{P}{W}\) then \text{“trial” is sequentially rational at the bottom information set. Then, in order for the non-negligent type of the defendant to choose \text{“resist” it must be that the plaintiff’s strategy involves \text{“trial” also at the middle information set, for which we need him to assign probability } p \geq \frac{S + P}{W}\) to the negligent type (which is possible, since the middle information set is not reached); of course, this requires \(W \geq S + P\). Thus,

\[
\begin{array}{|l|}
\hline
((\text{sue},\text{trial},\text{trial}), (\text{resist},\text{resist})) \text{ is an equilibrium if and only if } q \geq \frac{P}{W} \text{ with } p \geq \frac{S + P}{W} \\
\hline
\end{array}
\]

(f) Note that here the restriction \(W - P > S\) does not hold. In this case, \(q < \frac{S + P}{W} = \frac{80 + 70}{100} = \frac{3}{2}\) and thus, by the previous analysis, there is no pooling equilibrium of type (1), namely \(((\text{sue},\text{trial},\text{trial}),(\text{offer},\text{offer}))\).

Also, since \(S + P > W\) there is no pooling equilibrium of type (2) with \(((\text{sue},\text{trial},\text{trial}),(\text{resist},\text{resist}))\) or with \(((\text{sue},\text{trial},\text{drop}),(\text{resist},\text{resist}))\).

However, there is a pooling equilibrium of type (2) with \(((\text{sue},\text{settle},\text{drop}),(\text{resist},\text{resist}))\) with any beliefs at the middle information set, since \text{“settle” strictly dominates \text{“trial” there (and, of course, belief } q = 1/12\text{ on the left node of the bottom information set).}
(a) In the figure below, \( P \) stands for Profit-sharing contract and \( R \) for Revenue-sharing contract. \( Mi \) stands for Manager of firm \( i \).

\[ \begin{align*} 
\text{Cournot game} \quad \text{where } M_1 \text{ maximizes profits and } M_2 \text{ maximizes profits} \\
\text{Cournot game} \quad \text{where } M_1 \text{ maximizes profits and } M_2 \text{ maximizes revenue} \\
\text{Cournot game} \quad \text{where } M_1 \text{ maximizes revenue and } M_2 \text{ maximizes profits} \\
\text{Cournot game} \quad \text{where } M_1 \text{ maximizes revenue and } M_2 \text{ maximizes revenue}
\end{align*} \]

(b) Number the subgames games 1 to 4 from left to right.

**Subgame 1:** \( q_1 \) is chosen to maximize \( \alpha \Pi_1(q_1, q_2) = \alpha (q_1(60 - q_1 - q_2) - 12q_1) \) and \( q_2 \) is chosen to maximize \( \alpha \Pi_2(q_1, q_2) = \alpha (q_2(60 - q_1 - q_2) - 12q_2) \). Solving \( \frac{\partial \Pi_1}{\partial q_1} = 0 \) and \( \frac{\partial \Pi_2}{\partial q_2} = 0 \) gives \( q_1 = q_2 = 16 \). Player 1’s payoff is \( (1 - \alpha) \Pi_1(16, 16) = (1 - \alpha)256 \) and the same is true for player 2.

**Subgame 2:** \( q_1 \) is chosen to maximize \( \alpha \Pi_1(q_1, q_2) = \alpha (q_1(60 - q_1 - q_2) - 12q_1) \) and \( q_2 \) is chosen to maximize \( \alpha \Pi_2(q_1, q_2) = \alpha q_2(60 - q_1 - q_2) \). Solving \( \frac{\partial \Pi_1}{\partial q_1} = 0 \) and \( \frac{\partial \Pi_2}{\partial q_2} = 0 \) gives \( q_1 = 12 \) and \( q_2 = 24 \). Player 1’s payoff is \( (1 - \alpha) \Pi_1(12, 24) = (1 - \alpha)144 \) and player 2’s payoff is \( \Pi_2(12, 24) - \alpha \Pi_2(12, 24) = 288 - \alpha 576 \).

**Subgame 3:** this is the same as subgame 2, with the roles reversed. Thus Player 1’s payoff is \( 288 - \alpha 576 \) and Player 2’s payoff is \( (1 - \alpha)144 \).

(In this game \( q_1 \) is chosen to maximize \( \alpha \Pi_1(q_1, q_2) = \alpha q_1(60 - q_1 - q_2) \) and \( q_2 \) is chosen to maximize \( \alpha \Pi_2(q_1, q_2) = \alpha (q_2(60 - q_1) - 12q_2) \). Solving \( \frac{\partial \Pi_1}{\partial q_1} = 0 \) and \( \frac{\partial \Pi_2}{\partial q_2} = 0 \) gives \( q_1 = 24 \) and \( q_2 = 12 \). Player 1’s is \( \Pi_1(24,12) - \alpha \Pi_1(24,12) = 288 - \alpha 576 \) and player 2’s payoff is \( (1 - \alpha) \Pi_2(24,12) = (1 - \alpha)144 \).

**Subgame 4:** \( q_1 \) is chosen to maximize \( \alpha \Pi_1(q_1, q_2) = \alpha q_1(60 - q_1 - q_2) \) and \( q_2 \) is chosen to maximize \( \alpha \Pi_2(q_1, q_2) = \alpha q_2(60 - q_1 - q_2) \). Solving \( \frac{\partial \Pi_1}{\partial q_1} = 0 \) and \( \frac{\partial \Pi_2}{\partial q_2} = 0 \) gives \( q_1 = 20 \) and \( q_2 = 20 \). Player 1’s is \( \Pi_1(20,20) - \alpha \Pi_1(20,20) = 160 - \alpha 400 \) and similarly for Player 2.
The normal form is

<table>
<thead>
<tr>
<th>Profit contract</th>
<th>Revenue contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit contract</td>
<td>(1−α)256, (1−α)256</td>
</tr>
<tr>
<td>Revenue contract</td>
<td>288−α576, (1−α)144</td>
</tr>
</tbody>
</table>

The Nash equilibria of this game are as follows:

1. If $\alpha < \frac{1}{16}$ then $R$ (a Revenue contract) is a strictly dominant strategy for each player and thus $(R,R)$ is the only Nash equilibrium.
2. If $\alpha = \frac{1}{16}$ then $R$ is a weakly dominant strategy for each player; there are 3 Nash equilibria: $(R,R)$, $(R,P)$ and $(P,R)$.
3. If $\frac{1}{16} < \alpha < \frac{1}{10}$ there are two Nash equilibria: $(R,P)$ and $(P,R)$.
4. If $\alpha = \frac{1}{10}$ then $P$ is a weakly dominant strategy for each player; there are 3 Nash equilibria: $(P,P)$, $(R,P)$ and $(P,R)$.
5. If $\alpha > \frac{1}{10}$ then $P$ is a strictly dominant strategy for each player and thus $(P,P)$ is the only Nash equilibrium.

(c) From the calculations for Subgame 1, we get that in the past each firm had a profit of 256. When $\alpha$ is small, the only equilibrium involves a revenue contract for each manager, yielding an income of at most 160 for each owner. Thus delegation has reduced the owners’ incomes. This is a Prisoners’ Dilemma situation: when one of the firms delegates with a revenue contract then the other must too, giving rise to a Pareto inefficient situation (from the point of view of the owners only).
You have 20 minutes to read the exam, and 4 hours to write your answers. Please answer all three problems. The three problems will be equally weighted in the total score. The maximum possible score for each problem is 5 points. It is recommended that you read each problem entirely before attempting any of the parts. Good luck!
1. Define total income for an agricultural household as:

\[ Y = p\alpha F(L_f, T) + w\phi(L_s) - wL_h \]

where we assume that:

- The agricultural commodity, \( F \), has a price \( p \) and is produced with a constant returns to scale technology using labor, \( L_f \) and land, \( T \). The household’s land is fixed at its inherited level, \( T\bar{} \). The total factor productivity term, \( \alpha > 0 \), varies across households and equals one on average in the farming population. IGNORE this term (set it equal to 1) until part (e) of the problem.

- The household has a total labor stock of labor, \( \bar{L} \), that can be supplied to the off-farm labor market \((L_s)\), and/or allocated to own farm activities \((\bar{L} - L_s)\). Total farm labor is the sum of family and hired labor: \( L_f = L_h + (\bar{L} - L_s) \).

- Hired labor is paid the parametrically given market wage, \( w \). However, the household cannot always find a job when it desires one and its days employed as a function of time supplied to the market is given by the function \( \phi(L_s) \), with \( 1 > \phi' > 0, \phi'' \leq 0 \).

Putting all this together with the assumption that the household acts to maximize its income yields the following choice problem

\[
\max_{L_s, L_h} pF(L_f, T) + w\phi(L_s) - wL_h
\]

subject to:

\[
L_f = L_h + (\bar{L} - L_s)
\]
\[
L_s \geq 0
\]
\[
L_h \geq 0
\]

(a) [0.5 points] Draw a picture of the employment function, \( \phi(L_s) \), and the marginal wage \( w\phi' \). Without doing any mathematical derivation, describe in words what happens to marginal wage as labor supplied to the market increases. Intuitively, what will this function do to the opportunity cost of household labor for households with more land (larger \( \bar{T} \)) and households with less land (smaller \( \bar{T} \))?

(b) [1.0 points] What are the possible Kuhn-Tucker solution regimes for this problem? Would a double interior solution in which both inequality constraints do NOT bind make sense? Justify your answer by analyzing the relevant conditions.

(c) [1.5 points] According to this model, what is the relationship between a household’s land endowment and its land productivity? That is, as a household’s land endowment increases, what happens to land productivity \((F/\bar{T})\)? Besides doing the math, be sure to graph your
solution, being sure to investigate the entire land endowment domain (that is, consider families with larger and smaller values of $T$ and across all relevant Kuhn Tucker regimes).

(d) \textit{1.5 points} Poverty rates are disproportionately high in rural areas of most world regions. Would land redistribution of land from non-poor to poor households be a good public policy response according to this model? In answering this problem, assume that poor families are those who optimally choose $L_s > 0$ and that non-poor families are those who choose $L_h > 0$. Could a land redistribution be designed that is Pareto improving?

(e) \textit{0.5 points} Suppose that you now want to use observational data to test for the relationship you sketched out in part (c). Assume that you have data on a random sample of rural households (some with large land endowments and some with small land endowments) and that the households have varying levels of technical efficiencies (as represented by the term $\alpha$ in the model above). Do you think that the OLS regression of land productivity on land endowment would reliably identify the information you needed to validate your theoretical model? Please explain your answer and be clear on your assumptions.
2. Monopoly and product quality

In this problem we investigate the supply of quality by a monopolist. Assume an economy with one good supplied by a single firm at constant unit cost $c \times \mu^2$, where $\mu$ is an index representing the quality of the good, $\mu \in [0, +\infty)$ and $c$ is a positive constant. There is a mass $M$ of consumers uniformly distributed on the interval $[0, 1]$ according to a taste parameter $\theta$ indicating their sensitivity to the quality of the good. Each consumer purchases at most one unit of the good. More specifically, the preferences of a consumer of type $\theta$ can be represented by the following utility function:

$$U_\theta(\mu, p) = \begin{cases} \theta \mu - p & \text{if the consumer purchases one unit of the good at price } p \\ 0 & \text{if the consumer purchases nothing} \end{cases}.$$ 

We define social welfare as the sum, over consumers who consume one unit of good, of their willingness to pay for the good, minus the costs of producing these goods.

First assume that the quality $\mu$ is exogenous at level $\bar{\mu}$, such that $c\bar{\mu} < 1$.

[0.2] 1. How much is a consumer of type $\theta$ willing to pay for a unit of good of quality $\bar{\mu}$?

[0.6] 2. Derive the allocation that maximizes social welfare. How many units of good are produced in this allocation? (Hint: You do not need to do any derivations. Simple economic reasoning and your answer to part 1 should suffice.)

[1.2] 3. If a monopolist is supplying the good, how many units of good are produced? Comment.

Now assume that the quality of the good $\mu$ is endogenous.

[1.6] 4. Derive the allocation that maximizes social welfare. Characterize it in terms of the the quality level and the total quantity produced. (Hint: You can use your answer to part 2 to simplify the optimization program.)

[1.4] 5. If a monopolist is choosing the quality and supplying the good, what are the resulting quality and quantity? Comment. (Hint: You can use your answer to part 3 to simplify the optimization program.)
3. The market for farm labor can be characterized, to first order, by a model with linear demands and supplies, given in inverse form as

\[(\text{demand}) \quad p^d = \alpha_0 + \alpha_1 \cdot Q\]

\[(\text{supply}) \quad p^s = \beta_0 + \beta_1 \cdot Q\]

where \(p\) is the wage paid and \(Q\) is the number of hours of work hired.

Depending on location and the specific industry, two main types of market power can be observed in this market: (1) monopoly power by farmworkers if they unionize; and (2) monopsony power by growers.

0.25 1. In a graph, show the equilibrium wage and number of hours hired for both of these forms of market power.

0.75 2. Determine the conditions under which the deadweight loss from market power is greater under grower monopsony than it is under farmworker monopoly. You should be able to establish this algebraically.

1.0 3. Another case of market power that can arise is that of a farm labor contractor, who hires farmworkers and supply them to growers, acting as a pure middleman in the market and exercising both monopsony and monopoly power. Show rigorously that the deadweight loss under the farm labor contractor/pure middleman case is always larger than the deadweight loss under monopsony and monopoly.

1.5 4. In a market with a single farm labor contractor (that is, a pure middleman), would subsidizing growers increase or decrease social welfare? Assume that a unit subsidy of \(s\) is given to growers, so that \(s\) dollars of the farmworker wage is paid by the government. Explain the changes in the market and the welfare consequences of the subsidy, supporting your conclusions rigorously with the simple market model above. To support your argument, show graphically how the distribution of surpluses to growers, farmworkers, the farm labor contractor, and the government changes with the subsidy.

1.5 5. Adapt your framework from parts 2. and 3. above to explain the choices made by a farm labor contractor, where a unit subsidy \(s\) may be given to either the growers or the farmworkers. Under what conditions is the change in deadweight loss greater when growers are subsidized than when farmworkers are subsidized?
2. Monopoly and product quality

In this problem we investigate the supply of quality by a monopolist. Assume an economy with one good supplied by a single firm at constant unit cost \( c \times \mu^2 \), where \( \mu \) is an index representing the quality of the good, \( \mu \in [0, +\infty) \) and \( c \) is a positive constant. There is a mass \( M \) of consumers uniformly distributed on the interval \([0,1]\) according to a taste parameter \( \theta \) indicating their sensitivity to the quality of the good. Each consumer purchases at most one unit of the good. More specifically, the preferences of a consumer of type \( \theta \) can be represented by the following utility function:

\[
U_\theta(\mu, p) = \begin{cases} 
\theta \mu - p & \text{if the consumer purchases one unit of the good at price } p \\
0 & \text{if the consumer purchases nothing}
\end{cases}
\]

We define social welfare as the sum, over consumers who consume one unit of good, of their willingness to pay for the good, minus the costs of producing these goods.

First assume that the quality \( \mu \) is exogenous at level \( \bar{\mu} \), such that \( c\bar{\mu} < 1 \).

[0.2] 1. How much is a consumer of type \( \theta \) willing to pay for a unit of good of quality \( \bar{\mu} \)?

**Solution:** He is willing to pay \( \theta \bar{\mu} \).

[0.6] 2. Derive the allocation that maximizes social welfare. How many units of good are produced in this allocation? (Hint: You do not need to do any derivations. Simple economic reasoning and your answer to part 1 should suffice.)

**Solution:** The allocation that maximizes social welfare must be such that all consumers with willingness to pay above the marginal social cost \( c\bar{\mu}^2 \), that is, with taste parameter above \( c\bar{\mu} \) consume one unit. The total number of units produced is thus \( M \int_{c\bar{\mu}}^{1} d\theta = M(1 - c\bar{\mu}) \).

[1.2] 3. If a monopolist is supplying the good, how many units of good are produced? Comment.

**Solution:** The monopolist would set price in order to maximize profit, that is, solves \( \max_p (p - c\bar{\mu}^2)M(1 - \bar{\theta}) \), where \( \bar{\theta} \) is the taste parameter of the marginal consumer. The marginal consumer being indifferent between consuming the good or not, his taste parameter is \( \bar{\theta} = \frac{c}{\bar{\mu}} \). The monopolist’s program is thus \( \max_p (p - c\bar{\mu}^2)(1 - \frac{c}{\bar{\mu}}) \). The objective is a quadratic function, therefore the maximum is obtained at the critical point \( p^* = \frac{\bar{\mu}(1+c\bar{\mu})}{2} \).

The corresponding quantity is \( \frac{M(1-c\bar{\mu})}{2} \). Therefore, the monopolist supplies half the quantity supplied in the social optimum. This is expected because the marginal cost is flat and the demand facing the monopolist is linear.

Now assume that the quality of the good \( \mu \) is endogenous.
4. Derive the allocation that maximizes social welfare. Characterize it in terms of the the quality level and the total quantity produced. (Hint: You can use your answer to part 2 to simplify the optimization program.)

**Solution:** First note that the optimal quality level is such that the marginal cost \( c\mu^2 \) must be lower than the willingness to pay of the consumer with taste parameter \( \theta = 1 \), that is, \( c\mu^2 < \mu \) or \( c\mu < 1 \). The allocation that maximizes social welfare is such that at the optimal quality, the consumers with taste parameter above \( c\mu \) are given one unit of good (see part 2). Therefore, the optimal quality level maximizes \( M \int_{c\mu}^1 (\theta \mu - c\mu^2) d\theta \) or, equivalently, \( \mu(1-c\mu)^2 \) over the range \( c\mu < 1 \). The first-order condition w.r.t. \( \mu \) yields \( \hat{\mu} = \frac{1}{3c} \). The corresponding quantity is \( \frac{2M}{3^2} \).

5. If a monopolist is choosing the quality and supplying the good, what are the resulting quality and quantity? Comment. (Hint: You can use your answer to part 3 to simplify the optimization program.)

**Solution:** The monopolist chooses the quality level and the price that maximize her profit \( (p - c\mu^2)M(1-\theta) \), where \( \theta \) is the taste parameter of the marginal consumer, and thus \( \hat{\theta} = \frac{p}{\mu} \). Conditional on \( \mu \), we know from part 3 that \( p = \frac{\mu(1+c\mu)}{2} \), which we can plug into the objective function. The monopolist’s choice of quality maximizes \( \left( \frac{\mu(1-c\mu)}{2} \right) \left( \frac{1-c\mu}{2} \right) \) and thus maximizes the function \( \mu(1 - c\mu)^2 \). This is the same objective as the social welfare problem and thus the quality level is the same, \( \hat{\mu} = \frac{1}{3c} \). However the quantity supplied is different, \( \frac{M}{3^2} \). We conclude that in this vertical differentiation model the monopolist distorts the quantity compared to the social optimum, but not quality.