ANSWER KEYS Micro Prelim August 10, 2020

QUESTION 1 ANSWER KEYS

(a) A pure strategy in this game is a triple of votes (a,b,c) where (letting WK = White Knight, PP = Poison Pill, WS= Wait and See) $a \in \{WK, MBO\}$, $b:\{WK, MBO\} \to \{WK, MBO, PP\}$ and $c:\{WK, MBO, PP\} \to \{WK, MBO, PP, WS\}$, that is, (1) a choice for the first vote, (2) a choice for the second vote as a function of the outcome of the first vote and (3) a choice for the third vote as a function of the outcome of the second vote.

(b) The last vote must be Wait & See versus something. In this final vote everyone has an incentive to vote honestly (that is, voting sincerely is a weakly dominant choice), since this will determine the actual outcome. The three possibilities are easy to calculate:

- Wait & See vs. Poison Pill, *Poison Pill* wins 4-1.
- Wait & See vs. MBO, *Wait & See* wins 3-2.
- Wait & See vs. White Knight, *White Knight* wins 3-2.

Now we go back one previous round. The contest will be either Poison Pill vs. White Knight or Poison Pill vs. MBO.

- In the first case, both Poison Pill and White Knight are preferred by a majority to Wait & See. So whatever wins the second round will be implemented. In this vote the majority prefers White Knight to Poison Pill, 3-2.
- In the second case, a vote for MBO is in reality a vote for Wait & See. Board members can anticipate that if MBO beats Poison Pill for the active course, it will lose out in the next comparison with Wait & See. So when deciding between Poison Pill and MBO, board members will act as if deciding between Poison Pill and Wait & See, with the result that Poison Pill wins 4-1.

Thus the first-round comparison is truly between Poison Pill and White Knight. White Knight is chosen by a 3-2 margin and is then selected in each of the subsequent comparisons. The outcome of the voting procedure is thus **White Knight.**

(c) The Founder has two equivalent choices.
(1) Give her vote **to the outside board member**. At first glance this seems nothing short of crazy: the adopted preferences are almost the opposite of her true ones. But look at the effect. The votes will now go as follows:

- Wait & See vs. Poison Pill, *Poison Pill* wins 3-2.
- Wait & See vs. MBO, *Wait & See* wins 4-1.
- Wait & See vs. White Knight, *Wait & See* wins 3-2.

The only active option that can beat Wait & See is Poison Pill. Right from the start the board members should predict that if Poison Pill ever loses, the outcome will be Wait & See. Yet both MBO and White Knight supporters prefer Poison Pill to Wail & See. They are forced to vote for Poison Pill as it is their only viable alternative; thus **Poison Pill** wins.
(2) Give her vote **to the management**. In this case the Manager would have 2 votes, so in the final stage

- if PP vs WS then PP wins 4-1
- if MBO vs WS or if WK vs WS then WS wins 3-2

Thus in the second stage PP means PP while MBO and WK both mean WS, so

- if MBO vs PP (but really WS vs PP) then PP wins 3-2
- if WK vs PP (but really WK vs WS) then WS wins 3-2

Thus in the first stage MBO means PP and WK means WS, so WK vs MBO really means WS vs PP; thus PP wins.

Answer Keys Question 2

Consider a monetary lottery L described by the cumulative distribution function $F_L : \mathbb{R} \longrightarrow [0,1]$. We require that some of the probability mass is on negative outcomes but the expected value is positive.

We say that a decision maker with Bernoulli utility function $u : \mathbb{R} \longrightarrow \mathbb{R}$ and wealth level w is indifferent between accepting and rejecting lottery L if and only if

$$\int u(x+w)dF_L(x) = u(w).$$

Now I introduce a concept that we did not discuss in ECN200A. The goal of this part of the preliminary exam is to relate this new concept to concepts that we do know from ECN200A. Aumann and Serrano (Journal of Political Economy, 2008) axiomatized an index of risk that is defined independently from any utility function. The Aumann-Serrano index of risk of lottery L is defined as the unique number $R(L)$ such that

$$\int e^{-\frac{x}{R(L)}} dF_L(x) = 1.$$

(a) Consider the Bernoulli utility function

$$u(x) = -e^{-ax} \text{ for } a > 0.$$

Show that the Arrow-Pratt coefficient of risk aversion is a for all x.

`See MWG Example 6.C.4.`

(b) Consider the class of CARA Bernoulli utility functions (i.e., constant absolute risk aversion). Show that the Aumann-Serrano index of risk of lottery L is the reciprocal of the Arrow-Pratt coefficient of risk aversion of a decision maker with CARA Bernoulli utility who is indifferent between accepting and rejection lottery L. So even though the Aumann-Serrano index of risk is defined independently of any utility function, we can use CARA utility to provide an interpretation.

`The class of CARA Bernoulli utility functions is exactly the class of exponential utility functions of problem (a). Thus, a CARA decision maker is indifferent if and only if`

$$\begin{aligned}
\int u(x+w)dF_L(x) &= u(w) \\
\int -e^{-a(x+w)}dF_L(x) &= -e^{-aw} \\
\int e^{-a(x+w)}dF_L(x) &= e^{-aw} \\
\int e^{-ax}e^{-aw}dF_L(x) &= e^{-aw} \\
e^{-aw}\int e^{-ax}dF_L(x) &= e^{-aw} \\
\int e^{-ax}dF_L(x) &= 1
\end{aligned}$$

It now follows that $a = \frac{1}{R(L)}$.

(c) Require now additionally that lotteries L satisfy $F_L(c) = 0$ for some $c < 0$ and $F_L(d) = 1$ for some $d > 0$. Show that if lottery L' first-order stochastically dominates lottery L, then $R(L') \leq R(L)$.

By definition, L' first-order stochastically dominates L if for every nondecreasing Bernoulli utility function $u : \mathbb{R} \longrightarrow \mathbb{R}$ we have

$$\int u(x) dF_{L'}(x) \geq \int u(x) dF_L(x).$$

CARA Bernoulli utility is certainly nondecreasing. Thus, for any $w > 0$ and $a \geq 0$,

$$\int -e^{-a(x+w)} dF_{L'}(x) \quad \geq \quad \int -e^{-a(x+w)} dF_L(x)$$

From this inequality follows that, if a CARA decision maker with Arrow-Pratt coefficient of risk aversion a accepts L, then she also accepts L'.

If a CARA decision maker with Arrow-Pratt coefficient of risk aversion a accepts L', then also any CARA decision maker with a weakly smaller Arrow-Pratt coefficient of risk aversion, $a' \leq a$, accepts L'.

Consequently, if a is the largest Arrow-Pratt coefficient of risk aversion with which a CARA decision maker accepts L (meaning that any CARA decision maker with Arrow-Pratt coefficient of risk aversion $b > a$ rejects lottery L), then the largest Arrow-Pratt coefficient of risk aversion with which a CARA decision maker accepts L' must be weakly larger than a.

This implies that $R(L') \leq R(L)$.

(d) Again, require additionally that lotteries L satisfy $F_L(c) = 0$ for some $c < 0$ and $F_L(d) = 1$ for some $d > 0$. Show that if a lottery L' second-order stochastically dominates lottery L, then $R(L') \leq R(L)$.

By definition, L' second-order stochastically dominates L if L' and L have the same mean and for every nondecreasing concave Bernoulli utility function $u : \mathbb{R} \longrightarrow \mathbb{R}$ we have

$$\int u(x) dF_{L'}(x) \geq \int u(x) dF_L(x).$$

CARA Bernoulli utility is certainly concave and nondecreasing. Thus, for any $w > 0$ and $a \geq 0$,

$$\int -e^{-a(x+w)} dF_{L'}(x) \quad \geq \quad \int -e^{-a(x+w)} dF_L(x)$$

The remaining argument is as in (d).

(e) Provide a simple argument for why the converse of (c) or (d) cannot hold.

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Note that the Aumann-Serrano index completely orders lotteries.  Yet,
both first- and second-order stochastic dominance are just partial orders
on the set of lotteries.  Hence, the converse to (d) and (e) cannot hold.
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Answer keys Question 3

Fix an exchange economy $\{\mathcal{I}, (u^i, w^i)_{i\in\mathcal{I}}\}$.

Two individuals i and i' are said to be *equals* if $w^i = w^{i'}$ and

$$u^i(x) \geqslant u^i(x') \Leftrightarrow u^{i'}(x) \geqslant u^{i'}(x')$$

for all consumption bundles x and x'.

An allocation $x = (x^i)_{i\in\mathcal{I}}$ satisfies *equal treatment of equals* if $x^i = x^{i'}$ for any two individuals i and i' who are equals. It displays *indifference between equals* if $u^i(x^i) = u^i(x^{i'})$ whenever i and i' are equals.

1. Argue that if x satisfies equal treatment of equals then it displays indifference between equals, but the opposite implication is not true.

   Answer: Suppose that x satisfies equal treatment of equals, $w^i = w^{i'}$ and

   $$u^i(x) \geqslant u^i(x') \Leftrightarrow u^{i'}(x) \geqslant u^{i'}(x').$$

   By equal treatment of equals $x^i = x^{i'}$ so, obviously, $u^i(x^i) = u^i(x^{i'})$.

   The opposite implication is not true: suppose that $u^1(x_1, x_2) = u^2(x_1, x_2) = x_1 + x_2$. If $w^1 = w^2$, indifference between equals requires that $x^1_1 + x^1_2 = x^2_1 + x^2_2$, which does not imply that $x^1_1 = x^2_1$ and $x^1_2 = x^2_2$. □

2. Argue that if x is a competitive equilibrium allocation, then it displays indifference between equals.

   Answer: Let (p, x) be a competitive equilibrium of the economy, and suppose that $w^i = w^{i'}$ and

   $$u^i(x) \geqslant u^i(x') \Leftrightarrow u^{i'}(x) \geqslant u^{i'}(x').$$

   By definition of competitive equilibrium and the first assumption, $u^i(x^i) \geqslant u^i(x^{i'})$ and $u^{i'}(x^{i'}) \geqslant u^{i'}(x^i)$. By the second assumption, the latter implies that $u^i(x^{i'}) \geqslant u^i(x^i)$, and hence that $u^i(x^i) = u^i(x^{i'})$. □

3. Argue that if all individuals in the economy have strictly quasi-concave utility functions and x is a competitive equilibrium allocation, then it satisfies equal treatment of equals.

   Answer: Suppose not, and let (p, x) be a competitive equilibrium of an economy where $w^i = w^{i'}$ and

   $$u^i(x) \geqslant u^i(x') \Leftrightarrow u^{i'}(x) \geqslant u^{i'}(x').$$

   By the previous result, we know that $u^i(x^i) = u^i(x^{i'})$. If $x^i \neq x^{i'}$, by strong quasi-concavity we have that

   $$u^i\left(\tfrac{1}{2}x^i + \tfrac{1}{2}x^{i'}\right) > u^i(x^i),$$

   which is impossible since

   $$p \cdot \left(\tfrac{1}{2}x^i + \tfrac{1}{2}x^{i'}\right) \leqslant p \cdot w^i$$

   and x^i solves

   $$\max_x \{u^i(x) : p \cdot x \leqslant p \cdot w^i\}.$$ □

4. Argue that even if all individuals in the economy have strictly quasi-concave utility functions, there are Pareto efficient allocations that fail to display indifference between equals.

Answer: Consider a two-person economy with

$$u^1(x_1, x_2) = u^2(x_1, x_2) = x_1 x_2$$

and $w^1 = w^2 = (1, 1)$. Allocation $((1,1),(0,0))$ is still efficient. □

5. Prove that

$$\max_{x_1, x_2} \{(1 - x_1)(2 - x_2) : 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 2 \text{ and } x_1 x_2 \geqslant 1\} = 3 - 2\sqrt{2}.$$

Answer: This follows by direct computation. The maximizers are $x_1 = \sqrt{1/2}$ and $x_2 = \sqrt{2}$. □

6. Use the previous result to argue that the allocation

$$\mathbf{x} = ((9/20, 9/20), (11/20, 11/20), (1, 1))$$

is in the core of the following three-person exchange economy: preferences are represented by

$$u^1(x_1, x_2) = u^2(x_1, x_2) = u^3(x_1, x_2) = x_1 x_2,$$

and endowments are $w^1 = w^2 = (1, 0)$ and $w^3 = (0, 2)$.

Proof. It suffices to show that no coalition objects to $\mathbf{x}$. This is obvious for singleton coalitions and for the coalition $\{1, 2\}$. It is also true for the grand coalition, since $\mathbf{x}$ is Pareto efficient: it is interior, and the marginal rates of substitution are equal to 1 for the three individuals. It only remains to show that coalitions $\{1, 3\}$ and $\{2, 3\}$ don't object either. For this, it suffices to consider $\{1, 3\}$.

The largest utility that individual 1 can attain in coalition $\{1, 3\}$, subject to not making individual 3 worse off, is

$$\max_{x_1, x_2 \geqslant 0} \{u^1(1 - x_1, 2 - x_2) : u^3(x_1, x_2) \geqslant 1\},$$

since $w^1 + w^3 = (1, 2)$. From the previous part we know that such utility is $3 - 2\sqrt{2} \approx 0.17$, which is strictly less than $u^1(9/20, 9/20) > 0.2$. □

7. Argue that even if all individuals in the economy have strictly quasi-concave utility functions, there may be allocations in the core of an economy that fail to display indifference between equals.

Answer: In the economy of the previous part, all individuals have strictly quasi-concave utility functions and the allocation given there displays envy.[1] □

[1] Yes, these utility functions fail strict quasi-concavity at the boundary, but the boundary is playing no role in this argument.