State Prices of Conditional Quantiles: 
New Evidence on Time-Varying Expected Returns

Konstantinos Metaxoglou and Aaron Smith*

October 27, 2011

Abstract

We develop statistics to represent the option implied stochastic discount factor for S&P 500 returns between 1990 and 2008. Our statistics, which we call State Prices of Conditional Quantiles (SPOCQ), estimate the market’s willingness to pay for insurance against outcomes in various quantiles of the return distribution. We demonstrate that the market applied deep discounts to large negative returns during the recessionary periods of 1990-91, 1998-2003, and 2008 and minimal discounts in the boom periods of 1995-96 and 2006-07. We show how SPOCQ relates directly to expected returns. SPOCQ has strong predictive power for S&P 500 returns at the two-year horizon. It generates an R-squared of 0.27 for two-year returns and implies an annualized standard deviation of two-year expected returns equal to 7%. The predictive power of SPOCQ can only be partially explained by the dividend yield and the consumption wealth ratio (CAY) because SPOCQ is more closely tied to the business cycle than are these variables.

JEL codes: C5, G12, G13. Keywords: Conditional Quantiles, Forecasting, Returns, Stochastic Discount Factor, State Prices.

*Corresponding address: Smith: Department of Agricultural and Resource Economics, University of California, Davis, One Shields Avenue, Davis, 95616, California. Email:adsmith@ucdavis.edu Metaxoglou: Bates White LLC, 1300 Eye Street NW, Washington, D.C., 20005. Email:konstantinos.metaxoglou@bateswhite.com Metaxoglou gratefully acknowledges financial support from Bates White LLC during this project. We thank the participants at the April 2011 Applied Econometrics Conference at the Federal Reserve Bank of St. Louis for useful comments.
In a present value model, stock price changes stem either from changes in expected future dividends or changes in the discount rate. If the discount rate is constant, then price changes must reflect changes in expected future dividends. Shiller (1981) and LeRoy and Porter (1981) first showed that stock prices vary much more than can be explained by rational expectations about future dividends. This result suggests that discount rate variation is an important source of price volatility. Moreover, because the term discount rate is synonymous with expected return, it implies substantial variation in expected returns. Following this point to its logical conclusion, time variation in expected returns implies that future returns should be predictable. Thus, return predictability has become a focal point in the finance literature; Cochrane (2011) writes that “discount rate variation is the central organizing question of current asset pricing research.”

A typical paper in the predictability literature proceeds by first proposing a variable that may drive the risk premium. This proposal may be motivated by a particular utility function, a particular specification for the dynamics of the state variables in the model economy, or an accounting identity. The researcher then assesses whether the variable of choice predicts returns. In this article, we do not impose a model of optimizing behavior nor do we specify the distribution and dynamics of the state variables. Rather, we infer the risk premium and its components directly from options prices. We develop a statistic that estimates at a point in time the discount rate applied to returns in a particular part of the conditional distribution. Put another way, we estimate the price of an Arrow-Debreu security that would pay one dollar in the event that the return falls in, say, the bottom quartile of the conditional distribution and zero otherwise. We call our statistics State Prices of Conditional Quantiles (SPOCQs). By focusing on quantiles, we produce a different and more easily interpretable representation of preferences over the return distribution than those produced by moment-based statistics such as skewness and kurtosis. We connect SPOCQ to expected returns and show that it has exhibited strong predictive ability for S&P 500 returns at one-to-two-year horizons since 1990.

Figure 1 shows our main SPOCQ predictor alongside the subsequent two-year returns on the S&P 500 index. This predictor (denoted SPOCQD) is suggested by our theory and equals a SPOCQ term multiplied by a measure of conditional dispersion in returns, the conditional interquartile range. The SPOCQ term equals the difference between the price of an Arrow-Debreu security that pays one dollar in the event that the return falls in the bottom quartile of the return distribution, SPOCQ(0.25), and the price of an Arrow-Debreu security that pays one dollar in the event that the return falls in the upper quartile, SPOCQ(75,100). We show that this predictor captures the component of the risk premium that follows from relative risk aversion; it measures preferences about returns in the lower tail relative to the upper tail. The figure shows that our SPOCQ predictor exhibits significant time variation throughout the sample and tracks subsequent returns closely. Our statistic was highest in 1992-1997 and in 2002-2003, periods that preceded the highest two-year returns in the sample period and came right after recessions. It was lowest from 1999-2001 and 2006-2007, which precede the lowest two-year returns in the sample period.
and which coincide with the peaks of the dotcom and housing booms.

Numerous papers propose variables to capture the predictable component of stock returns. Examples include the price-dividend ratio (Campbell and Shiller (1988a); Campbell and Shiller (1988b); Fama and French (1988); Cochrane (2008)), the consumption-wealth ratio (CAY, Lettau and Ludvigson (2001)), interest rates (Campbell (1987); Fama and French (1988)), and volatility (Bansal and Yaron (2004); Bollerslev, Tauchen, and Zhou (2009)), to name only a few. These predictors have received considerable empirical support but have also attracted skepticism, much of which originates with the high persistence in the predictor variables. Such high persistence can bias coefficient and standard error estimates in predictive regressions and thereby produce spurious results. Nelson and Kim (1993), Stambaugh (1999), Ang and Bekaert (2007), Ferson, Sarkissian, and Simin (2003), and Volkanov (2003) each address this issue and conclude that proper accounting for dependence weakens considerably the evidence of return predictability.

If predictive regressions are spurious, then one may expect forecasts produced by these regressions to perform poorly out of sample. Welch and Goyal (2008) investigate this implication and find that the standard predictors forecast worse out of sample than the historical average return. Campbell and Thompson (2007) and Cochrane (2008) argue that persistent predictors produce imprecise coefficient estimates in short samples, so out-of-sample predictive performance will be poor even if a predictor is economically important. Campbell and Thompson improve estimation efficiency and out-of-sample predictive performance by imposing mild restrictions on the estimated coefficients. Cochrane (2008) generates more powerful inference by using information from the predictability of dividend growth and concludes that the evidence of return predictability is strong. Out of sample evaluation also addresses snooping bias, which arises as economists troll data for regressions that produce a t-statistic greater than two and high $R^2$. Pesaran and Timmermann (1995) address this point directly by simulating the returns of an investor who begins with a base set of potential predictors and searches each period for the best model specification upon which to trade. They find significant predictive power that varies over time and across predictors but no model that performs well consistently.

Our SPOCQ predictor forecasts well both in and out of sample, so it adds to the evidence that stock returns are predictable. Moreover, our estimates imply that two-year expected returns have exhibited a standard deviation of 7% (annualized) since 1990. This variation accounts for more than a quarter of actual return variation. To understand the economic factors that underlie this result, we compare the predictive ability of SPOCQ to that of other predictors in the literature, such as CAY, interest rates, volatility, and the price-dividend and price-earnings ratios. We find that only CAY has significant and robust predictive ability in our sample, but it explains a small part of the risk premium captured by the SPOCQ predictor in Figure 1.

SPOCQ also reveals substantial fluctuations in volatility aversion, i.e., the extent to which the market is willing to pay to receive a dollar in the event that returns fall in either tail of the distribution. The volatility aversion implied by our SPOCQ times series was greatest in the periods
2005-2006, which coincide with the middle of two sustained bull markets. Apart from 2007-08, our
SPOCQ volatility-aversion series moves together with the volatility premium of Bollerslev et al,
which equals the difference between implied volatility and realized volatility. Unlike the relative
risk aversion component, we find that volatility aversion does not forecast S&P 500 returns at the
one and two year horizons.

We construct our SPOCQ series by evaluating the risk-neutral return distribution of the S&P
500 at the conditional quantiles of the physical return distribution. We estimate the conditional
quantiles of the physical return distribution using the MQ-CAViaR model of White, Kim, and
Manganelli (2010), which is a multivariate extension of the CAViaR model in Engle and Manganelli
(2004). This model imposes no parametric structure on the shape of the conditional distribution at
each point in time, but it does specify a parametric model for the evolution of the quantiles. We use
a mixture of logistic distributions that is both flexible and parsimonious to recover the risk-neutral
distribution from the first derivative of the option pricing curves. Past research has focused on
the risk-neutral density, which requires the second derivative of the options pricing curve and is
necessarily estimated less precisely(Figlewski (2010)).

The remainder of the paper is organized as follows. We first provide a framework for SPOCQs
in Section 1 and then in Section 2 we describe how we estimate the components of SPOCQ.
We present the MQ-CAViaR model to estimate the conditional quantiles of the physical return
distribution in Section 2.1. Our curve-fitting exercise based on a mixture of logistic distributions
to infer the risk-neutral distribution takes place in Section 2.2. The data for the MQ-CAViaR and
the logistic-fit exercises are discussed in Section 2.3, and the results for these two exercises are
presented in Section 2.4. We discuss the SPOCQs and their relation to volatility premia in Section
3. The results of our forecasting regressions are discussed in Section 4. Conclusions follow. The
tables and figures are available following the main body of the text. We provide the details for
some of our SPOCQ derivations in the Appendix at the end of the paper.

1 State Prices of Conditional Quantiles

1.1 Framework

In dynamic equilibrium models, the price of a financial asset equals the expected value of dis-
counted future payoffs on the asset (e.g., Cochrane (2001)). Let $S_t$ denote the price of the asset
at time $t$, $S_{t+1}$ the payoff on that asset in $t + 1$, and $M_{t,t+1}$ denote the stochastic discount factor
(SDF) between periods $t$ and $t + 1$. Suppose the state of the economy at time $t$ can be described
by a vector $X_t$, which implies that $M_{t,t+1}$ and $S_{t+1}$ are functions of $X_{t+1}$. In equilibrium, the asset
price is given by

$$S_t = E_t [M_{t,t+1}S_{t+1}], \tag{1}$$
where the notation $E_t[\cdot]$ is short hand for the expectation conditional on the natural filtration of $X_t$. Equivalently to (1), we can express $S_t$ as the expected future payoff under the risk-neutral probability measure (Ross (1976)) discounted at the risk-free rate. Based on the fundamental theorem of asset pricing, the risk-neutral measure always exists if the market is arbitrage-free (Harrison and Kreps (1979)) and is unique if the market is complete (Harrison and Pilska (1981)) with respect to the state space defined by $X_t$. Our SPOCQ statistics represent the risk-neutral distribution evaluated at conditional quantiles of the asset return distribution.

At time $t$, the researcher observes $S_t$ and the prices of any derivatives defined by payoffs on the asset, but not $X_t$. Thus, we focus on the risk-neutral distribution implied by the observed asset and derivative prices, which is the risk-neutral distribution of returns on the asset after integrating out the unobserved parts of the state space. Using the law of iterated expectations, equation (1) becomes

$$1 = E_t \left[ M_{t,t+1} \frac{S_{t+1}}{S_t} \right] = E_t \left[ E_t [M_{t,t+1} | r_{t+1}] \frac{S_{t+1}}{S_t} \right] = \int E_t [M_{t,t+1} | r_{t+1}] \frac{S_{t+1}}{S_t} dF_t (r_{t+1})$$

(2)

where $r_{t+1} = \ln \left( \frac{S_{t+1}}{S_t} \right)$ denotes the log return.\(^1\) Then, multiplying and dividing by $E_t [M_{t,t+1}]$ produces

$$1 = E_t [M_{t,t+1}] \int \frac{E_t [M_{t,t+1} | r_{t+1}]}{E_t [M_{t,t+1}]} \frac{S_{t+1}}{S_t} dF_t (r_{t+1}) = E_t [M_{t,t+1}] \int \frac{S_{t+1}}{S_t} dF_t^* (r_{t+1})$$

(3)

Thus, the risk-neutral conditional distribution of the asset return is given by

$$F_t^* (r_{t+1}) = \frac{E_t [M_{t,t+1} | r_{t+1}]}{E_t [M_{t,t+1}]} F_t (r_{t+1}) = M_t^* (r_{t+1}) F_t (r_{t+1}),$$

(4)

where we define as the standardized SDF

$$M_t^* (r_{t+1}) = \frac{E_t [M_{t,t+1} | r_{t+1}]}{E_t [M_{t,t+1}]} = M_{t,t+1}$$

(5)

The subscript $t$ in the function $M_t^* (r_{t+1})$ accounts for the fact that the standardized SDF may change over time with the state variable $X_t$. At time $t$, i.e., taking the current state of the world $X_t$ as given, $M_t^* (r_{t+1})$ is the discount applied to the return outcome $r_{t+1}$. This development of the standardized SDF draws from Engle and Rosenberg (2002), who define the term $E_t [M_{t,t+1} | r_{t+1}]$ as the projected pricing kernel function. They develop methods to estimate the projected pricing kernel and label their estimate the empirical pricing kernel.

Equation (4) reveals two sources of time variation in the risk-neutral conditional distribution of the asset returns $F_t^* (r_{t+1})$. Changes in the distribution of future returns $F_t (r_{t+1})$ comprise

\(^1\)For convenience later in the paper, we write the conditional distribution of the period $t + 1$ return, $F_t (r_{t+1})$, as a function of the log return $r_{t+1}$ rather than the return $R_{t+1} = S_{t+1}/S_t - 1$.\]
the first source of variation and changes in the price of risk comprise the second. We focus on
the second source, which we isolate by evaluating $F_t^* (r_{t+1})$ at conditional quantiles of the asset
returns. Specifically, we define the conditional quantile $q_t(\theta_j)$, such that $F_t(q_t) = \theta_j$. The state
price of the event $r_{t+1} \leq q_t$, which occurs with fixed probability $\theta_j$, is given by

$$F_t^* (q_t(\theta_j)) = M_t^* (q_t(\theta_j)) \theta_j$$

Equation (6) is an expression for a state price reflecting the market’s willingness to pay for insur-
ance against a state with a fixed probability.

We now provide the statistic that is our main focus in this paper, the state price of conditional
quantiles (SPOCQ)

$$SPOCQ_t (\theta_{j-1}, \theta_j) \equiv \int_{q_t(\theta_{j-1})}^{q_t(\theta_j)} dF_t^*(r_{t+1}) = F_t^* (q_t(\theta_j)) - F_t^* (q_t(\theta_{j-1})),$$

where $\theta_j > \theta_{j-1}$. Figure 2 illustrates how we obtain SPOCQ. We invert the physical distribution of
returns to find the quantiles $q_t(\theta_{j-1})$ and $q_t(\theta_j)$, and then we evaluate the risk-neutral distribution
at these quantiles. The SPOCQ in (7) is the market’s willingness to pay to receive a dollar in
the event that next period’s return falls between the $\theta_{j-1}$ and $\theta_j$ quantiles. The time variation
in SPOCQ is driven entirely by the willingness to pay for insurance against this event because
we hold the probability of a return in this interval constant. Under risk neutrality, this state
price would never change; it would always equal the probability of the event occurring, which is
$\theta_j - \theta_{j-1}$.

In general, SPOCQ may change because preferences change in a way that makes investors
more or less risk averse, or because the event associated with the $\theta_{j-1}$ and $\theta_j$ quantiles becomes
ostensibly better or worse. For example, when return volatility is large, returns in the bottom
quartile imply greater losses in wealth than when volatility is low. This difference would prompt
risk averse investors to pay more for a security that pays a dollar if returns fall into the bottom
quartile. If we were to evaluate the risk-neutral distribution at fixed points rather than conditional
quantiles, then we would confound the willingness to pay for a dollar in a particular state with
the probability of that event occurring.

Next, we relate the analytical framework developed above to existing asset pricing models
using two examples. In so doing, we find it convenient to use (4) to write (7) as

$$SPOCQ_t (\theta_{j-1}, \theta_j) = \int_{q_t(\theta_{j-1})}^{q_t(\theta_j)} M_{t+1}^* dF_t(r_{t+1}) = (\theta_j - \theta_{j-1}) E_t \left[ M_{t,t+1}^* | \mathcal{R}_{t+1}^{j-1,j} \right]$$

where we use $\mathcal{R}_{t+1}^{j-1,j}$ to denote the states of the world in time $t + 1$ for which $q_t(\theta_{j-1}) \leq r_{t+1} \leq q_t(\theta_j)$. First, we discuss a three-moment extension of CAPM. We then present a discrete-time
representative agent model with Epstein-Zin preferences. In the empirical part of this paper, we use the three terms $SPOCQ_t(0,25)$, $SPOCQ_t(25,50)$, and $SPOCQ_t(75,100)$ to explain expected returns\(^2\), so we use these same terms in our two examples.

### 1.2 Two Examples

**Example 1: Three-Moment Conditional CAPM (Harvey and Siddique (2000)).** This model allows conditional skewness to affect expected returns and implies the following expression for the standardized SDF as a function of the return on the market portfolio $R_{t+1}$

$$M_{t+1}^* = 1 - \beta_t (R_{t+1} - \mu_t) + \lambda_t \left( (R_{t+1} - \mu_t)^2 - \sigma_t^2 \right)$$

(9)

where $\mu_t \equiv E_t [R_{t+1}]$ and $\sigma_t^2 \equiv var_t [R_{t+1}]$. This expression is the same as equation (6) of Harvey and Siddique, except parameterized to impose $E_t [M_{t+1}^*] = 1$. The first term is the standard CAPM term and implies that the market discounts negative returns more heavily than positive returns; the second term implies an additional discount on large returns of either sign. If $\lambda_t = 0$, this model reduces to the standard conditional CAPM.\(^3\)

We show in Appendix A that, if $R_{t+1}$ is conditionally normally distributed, then

$$SPOCQ_t(0,25) = 0.25 + 0.318 \beta_t \sigma_t + 0.214 \lambda_t \sigma_t^2$$

$$SPOCQ_t(25,75) = 0.5 - 0.428 \lambda_t \sigma_t^2$$

$$SPOCQ_t(75,100) = 0.25 - 0.318 \beta_t \sigma_t + 0.214 \lambda_t \sigma_t^2$$

(10)

These expressions show the two sources of variation in SPOCQ. The parameters $\beta_t$ and $\lambda_t$ measure risk preferences and $\sigma_t$ measures volatility. Specifically, $\beta_t$ reflects relative risk aversion because it represents the decline in the value of a dollar as returns increase. This is the standard CAPM term. According to Harvey and Siddique, $\beta_t$ quantifies relative risk aversion, which is $-W_t U'' (W_t) / U' (W_t)$, where $W_t$ denotes wealth and $U (W_t)$ is the investor’s utility function. The parameter $\lambda_t$ captures volatility aversion because it represents the increase in the value of a dollar in states of the world in which returns are far from the mean. Harvey and Siddique show that $\lambda_t$ depends on the third derivative of the utility function. If positive, it implies non-increasing absolute risk aversion, which is an important property of the preferences of a risk-averse individual (Arrow (1964)). Volatility aversion is increasing in the variance of returns, $\sigma_t^2$. High volatility implies that the events associated with returns in the lower and upper quartiles produce large changes in wealth. Volatility-averse individuals are therefore willing to pay more for a dollar in the event that returns fall in the lower or upper quartiles when volatility is high than when volatility is low.

---

\(^2\)For brevity, we opt for the notation $SPOCQ_t(0,25)$, $SPOCQ_t(25,75)$ and $SPOCQ_t(75,100)$, as opposed to $SPOCQ_0(0.25)$, $SPOCQ_0(0.25,0.75)$ and $SPOCQ_0(0.75,1)$ for the remainder of the discussion in the paper.

\(^3\)See Cochrane (2001), Section 6.3.
To separate the effects of $\beta_t$ and $\lambda_t$, we write

$$SPOCQ_t(0, 25) - SPOCQ_t(75, 100) = 0.636\beta_t\sigma_t$$  \hspace{1cm} (11)

$$SPOCQ_t(0, 25) + SPOCQ_t(75, 100) = 0.5 + 0.428\lambda_t\sigma_t^2$$  \hspace{1cm} (12)

An increase in $\beta_t$ raises $SPOCQ_t(0, 25)$ and lowers $SPOCQ_t(75, 100)$. Thus, in the standard CAPM, the risk premium is determined by the market’s willingness to pay for a dollar in the event that the return falls in the bottom quartile minus the market’s willingness to pay for a dollar in the event that the return falls in the upper quartile of the distribution. This term reflects the declining marginal utility of wealth due to the difference in the value of a dollar between low and high wealth states.\(^4\)

An increase in $\lambda_t$ raises the two tail SPOCQ terms relative to the central term. When $\lambda_t > 0$, the market is willing to pay a premium to receive a dollar in the event that returns are large in either direction, i.e., the market is volatility averse. Volatility aversion only affects the risk premium on a particular asset if its return exhibits coskewness with the market, i.e., if its return is correlated with the square of the market return.\(^5\) If an asset is likely to yield large negative returns when the absolute value of the market return is large, then investors demand a risk premium because they are averse to volatile market returns and therefore dislike assets that reduce wealth in volatile market states of the world. In contrast, if an asset return is uncorrelated with the absolute value of the market return, then that asset is just as likely to yield above average as below average returns in volatile market states of the world. Investors would not demand a risk premium to hold such an asset. This latter case describes the market portfolio in this example because we imposed a conditional normal distribution on market returns. However, individual assets in this economy may exhibit coskewness with the market and thereby have a nonzero volatility risk premium.

**Example 2: Representative agent model with Epstein-Zin preferences** (e.g., Bansal and Yaron (2004), Bollerslev, Tauchen, and Zhou (2009)). Recursive preferences of the form in Epstein and Zin (1989) and Weil (1989) imply that the logarithm of the SDF is

$$m_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1}$$  \hspace{1cm} (13)

where $g_{t+1}$ denotes the log growth rate of aggregate consumption, $r_{a,t+1}$ is the log return on an asset that pays aggregate consumption as its dividend, and $\theta = (1 - \gamma) (1 - 1/\psi)^{-1}$. The three preference parameters are the time discount factor $\delta$, the intertemporal elasticity of substitution $\psi$, and the coefficient of risk aversion $\gamma$. When $\theta = 1$ the model collapses to the power utility specification of, for example, Mehra and Prescott (1985).

\(^4\)See equation (9) in Harvey and Siddique.

\(^5\)See equation (7a) in Harvey and Siddique.
Assuming that $g_{t+1}$ is conditionally normally distributed, we show in Appendix B that

\[
SPOCQ_t(0, 25) = \Phi \left( -\frac{\sigma_{mr}^t}{\sigma_t^r} - 0.674 \right)
\]

\[
SPOCQ_t(25, 75) = \Phi \left( \frac{\sigma_{mr}^t}{\sigma_t^r} + 0.674 \right) - \Phi \left( \frac{\sigma_{mr}^t}{\sigma_t^r} - 0.674 \right)
\]

\[
SPOCQ_t(75, 100) = \Phi \left( \frac{\sigma_{mr}^t}{\sigma_t^r} - 0.674 \right)
\]

where $\Phi(z)$ denotes the standard normal distribution, $\sigma_{mr}^t \equiv \text{cov}_t [m_{t+1}, r_{t+1}]$, $\sigma_t^r \equiv (\text{var}_t [r_{t+1}])^{1/2}$, and $r_{t+1}$ denotes the log return on an asset in the economy. Specifically, $r_{t+1}$ is the log return on the asset on which the SDF is projected in (4) to define SPOCQ. Noting that $\sigma_{mr}^t/\sigma_t^r \leq 0$, we see that, in this model, SPOCQ depends on the covariance between the log SDF and the standardized log return. As this covariance becomes more negative, $SPOCQ_t(0, 25)$ and $SPOCQ_t(25, 75)$ increase and $SPOCQ_t(75, 100)$ decreases.

SPOCQ is a function of a single term in this model but, unlike the standard CAPM in which $M^*_t, t+1$ is linear in $R_{t+1}$, the SPOCQ term that covers the interquartile range is nonconstant and depends on the model parameters. Thus, the model permits some volatility aversion. To see this, we separate the SPOCQ components into a term that captures relative risk aversion and a term that captures volatility aversion as in (11)

\[
SPOCQ_t(0, 25) - SPOCQ_t(75, 100) \approx 0.636 \beta_t \sigma_t^r
\]

\[
SPOCQ_t(0, 25) + SPOCQ_t(75, 100) \approx 0.5 + 0.214 (\beta_t \sigma_t^r)^2.
\]

where $\beta_t \equiv \sigma_{mr}^t / (\sigma_t^r)^2$. We obtain the approximations from a second-order Taylor expansion of $\Phi(z)$ around the point $z = -0.674$.\(^6\) These expressions have a very similar form to those for the CAPM in (11), except that the relative risk aversion term and the volatility aversion term are both driven by $\beta_t \sigma_t^r$.

To give a more specific example, the long-run risk model of Bansal and Yaron (2004) implies

\[
\beta_t = \frac{(\lambda_m, \eta \varphi_d + \lambda_{m,e} \beta_{m,e}) \sigma_t^2 + \lambda_{m,w} \beta_{m,w} \sigma_w^2}{(\varphi_d^2 + \beta_{m,e}^2) \sigma_t^2 + \beta_{m,w}^2 \sigma_w^2}
\]

where $\sigma_t$ represents time-varying consumption volatility and the remaining items denote parameters that are determined by preferences and the dynamics of consumption growth.\(^7\) Similarly, the

\(^6\)Note that $\Phi^{-1}(0.25) = -\Phi^{-1}(0.75) = -0.674$.

\(^7\)See equations (A10) and (A12) in Bansal and Yaron (2004). We express the parameters in the notation of Bansal and Yaron for easy comparison. We do the same for Bollerslev et. al.’s model in (18) below.
model of Bollerslev, et al. yields
\[
\beta_t = \frac{-\gamma \sigma_{g,t}^2 - (1 - \theta) \kappa_1^2 (A_{\sigma}^2 + A_q^2 \varphi_q^2) q_t}{\sigma_{g,t}^2 + \kappa_1^2 (A_{\sigma}^2 + A_q^2 \varphi_q^2) q_t}
\] (18)

where \( \sigma_{g,t}^2 \) denotes the volatility of consumption growth, \( q_t \) denotes the volatility of volatility, which follows an AR(1) process in their model. In both models, dynamic components of consumption volatility change the risk premium by shifting mass between low-quantile SPOCQs and high-quantile SPOCQs. Although these models are rich in parameters, they affect SPOCQ only through the ratio \( \sigma_{mr,t}^2 / \sigma_r^2 = \beta_t \sigma_r^2 \).

These examples show that the relative risk aversion and volatility aversion components of the risk premium are parameterized separately in CAPM-type models and are encapsulated in a single ratio in the Epstein-Zin class of models. Both models assume a distribution for the state variable in the economy and a functional form for the SDF, which is implied by a particular choice of utility function. In the subsequent sections of the paper, we estimate SPOCQ non-parametrically and thus impose neither a distribution of the state variables or a utility function on the data.

1.3 SPOCQ and the Risk Premium

Equilibrium in a representative agent economy implies that the risk premium is
\[
E_t \left[ R_{t+1} - R_{t+1}^f \right] = -cov_t \left[ M_{t,t+1}^*, R_{t+1} \right] = -E_t \left[ (M_{t,t+1}^* - 1) (R_{t+1} - \mu_t) \right] = - \sum_{j=1}^{J+1} (\theta_j - \theta_{j-1}) E_t \left[ M_{t,t+1}^* (R_{t+1} - \mu_t) | R_{t+1} \right] \right],
\] (19)

where \( R_{t+1}^f = 1 / E_t [M_{t+1}] \) denotes the risk free return and \( \mu_t \equiv E_t [R_{t+1}] \) (e.g., see Cochrane (2001)). We define \( \theta_0 = 0 \) and \( \theta_{J+1} = 1 \). To show the connection between SPOCQ and the risk premium, we connect (8) and (19) by decomposing the function \( M_{t,t+1}^* \) into a piece that is linear in \( R_{t+1} \) and a remainder. Because the risk premium in (19) is a function of the covariance between \( M_{t,t+1}^* \) and \( R_{t+1} \), only the linear relationship between them affects the risk premium; higher-order terms are irrelevant.

We perform this decomposition separately for each interval \( \mathcal{R}_{t+1}^{j-1,j} \) using a linear projection of \( M_{t,t+1}^* \) onto \( R_{t+1} \). This projection provides the best linear predictor of \( M_{t,t+1}^* \) as a function of \( R_{t+1} \) over the range defined by \( \mathcal{R}_{t+1}^{j-1,j} \), and we write it as
\[
M_{t+1}^* = \alpha_{jt} \left( 1 + \gamma_{jt} (R_{t+1} - \mu_{jt}) \right) + e_{jt+1},
\] (20)

\( ^8 \)Given that \( r_{t+1} \) and \( R_{t+1} \) exhibit the one-to-one relationship, \( r_{t+1} = ln(1 + R_{t+1}) \), conditioning on \( \mathcal{R}_{t+1}^{j-1,j} \) is identical to conditioning on \( \mathcal{R}_{t+1}^{j-1,j} = \{ R_{t+1} | exp(q_t(\theta_j-1)) \leq 1 + R_{t+1} \leq exp((q_t(\theta_j))) \}. \)
The projection is defined such that the following hold

\[ E_t [e_{j,t+1} R_{t+1} | \mathcal{R}_{t+1}^{j-1}] = 0 \]  
\[ E_t [M_{t+1}^* | \mathcal{R}_{t+1}^{j-1}] = \alpha_{jt} \]  
\[ E_t [R_{t+1} | \mathcal{R}_{t+1}^{j-1}] = \mu_{jt} \]  

Using (20), we write the expectation term in (19) as

\[ E_t [M_{t,t+1}^* (R_{t+1} - \mu_t) | \mathcal{R}_{t+1}^{j-1}] = E_t [(\alpha_{jt} + \alpha_{jt} \gamma_{jt} (R_{t+1} - \mu_{jt}) + e_{j,t+1}) (R_{t+1} - \mu_t) | \mathcal{R}_{t+1}^{j-1}] = \alpha_{jt} (\mu_{jt} - \mu_t + \gamma_{jt} \sigma_{jt}^2), \]  

where \( \sigma_{jt}^2 = E_t [(R_{t+1} - \mu_{jt})^2 | \mathcal{R}_{t+1}^{j-1}] \) is the return variance in the \( j \)th segment of the distribution. Substituting (24) into (19), the risk premium is

\[ E_t [R_{t+1} - R_f] = -\sum_{j=1}^{J+1} (\theta_j - \theta_{j-1}) \alpha_{jt} (\mu_{jt} - \mu_t + \gamma_{jt} \sigma_{jt}^2) \]

\[ = -\sum_{j=1}^{J+1} SPOCQ_t (\theta_{j-1}, \theta_j) (\mu_{jt} - \mu_t + \gamma_{jt} \sigma_{jt}^2) \]  

To obtain (25), we use the fact that \( \alpha_{jt} = E_t [M_{t,t+1}^* | \mathcal{R}_{t+1}^{j-1}] \), which implies that \( SPOCQ_t (\theta_{j-1}, \theta_j) = (\theta_j - \theta_{j-1}) \alpha_{jt} \). In (25), we express the risk premium as a weighted sum of SPOCQ terms. The weights depend on \( \mu_{jt} - \mu_t \), which is the location of the \( j \)th segment relative to the mean return, and a scale term, \( \gamma_{jt} \sigma_{jt}^2 \). More specifically, the scale term is the product of the standardized SDF function slope \( \gamma_{jt} \) in (20) and \( \sigma_{jt}^2 \), the variance of returns conditional on the segment defined by \( \mathcal{R}_{t+1}^{j-1} \). It captures the slope of the standardized SDF within each segment of the return distribution, as opposed to the level, which is captured by the location term. Thus, the scale term is of second-order importance relative to the location term.

We define the constant \( \psi_j \approx - \left( \mu_{jt} - \mu_t + \gamma_{jt} \sigma_{jt}^2 \right) / \sigma_t \), where \( \sigma_t \) is a measure of the dispersion in the conditional distribution of returns, and we write the risk premium and Sharpe ratio, respectively, as follows

\[ E_t [R_{t+1} - R_{t+1}^f] \approx \sum_{j=1}^{J+1} SPOCQ_t (\theta_{j-1}, \theta_j) \sigma_t \psi_j \]

\[ E_t [R_{t+1} - R_{t+1}^f] / \sigma_t \approx \sum_{j=1}^{J+1} SPOCQ_t (\theta_{j-1}, \theta_j) \psi_j \]  

The approximations in (26) and (27) are exact if \( (\mu_{jt} - \mu_t + \gamma_{jt} \sigma_{jt}^2) / \sigma_t \) is time invariant. If all the
time variation in the conditional return distribution is due to its first two moments, then the ratio $(\mu_{jt} - \mu_t)/\sigma_t$ would be a constant that depends only on $j$, so there would be no approximation error from this term. As the number of segments $J$ increases, the within-segment return variance decreases causing the second-order term to decline toward zero. Thus, if most of the variation in the conditional return distribution emanates from the first two moments and if the second-order term is small, then the approximation error will be small.

Equations (26) and (27) suggest a linear regression of future returns on SPOCQ to estimate the risk premium. For quantiles below the median, $\mu_{jt} - \mu_t$ will usually be negative, which implies a positive coefficient $\psi_j$ on $SPOCQ_t(\theta_{j-1}, \theta_j)$ in such a regression. Correspondingly, we expect negative coefficients for quantiles above the median. Thus, the risk premium increases as SPOCQ increases at lower quantiles and decreases at higher quantiles. In other words, greater willingness to pay to receive a dollar in negative return outcomes implies a larger risk premium.

2 Estimating the Components of SPOCQ

2.1 Conditional quantiles of physical distribution: MQ-CAViaR

We estimate the conditional quantiles of the S&P 500 return distribution over one-week and one-month time horizons. We define these horizons to match those in our options data. For example, at the one-week horizon, we observe option prices at the close of the trading day on a Friday that will pay based on the index value when the market opens the following Friday morning. Therefore, the appropriate return from which to extract physical quantiles is the return from the close of business today (Friday) until the open one week ahead (the following Friday). Similar reasoning holds in the case of the one-month horizon, for which we use four weeks, i.e., the return from the close of business today (Friday) until the open four weeks ahead (Friday, four weeks later). Thus, we model the conditional quantiles of the log return series for time $t = 1, \ldots, T$ defined by

$$r_t = 100 \times \ln(SP_t^{open}/SP_{t-1}^{close})$$

(28)

where $SP_t$ denotes the value of the S&P 500 index on date $t$.

We obtain the quantiles of the physical return distribution using the MQ-CAViaR model of White, Kim, and Manganelli (2010), which is a multivariate extension of the CAViaR model in Engle and Manganelli (2004). This model imposes no parametric structure on the shape of the conditional density at each point in time, but it does specify a parametric model for the evolution of the quantiles. Using $q_{jt}$ to denote the $jth$ quantile of $r_t$, the MQ-CAViaR model is

$$q_t = Z_t^i \phi + q_{t-1}^i \gamma,$$

(29)

Throughout the paper, we use the term month to mean four weeks.
where \( q_t = (q_{1t}, \ldots, q_{5t})' \) and \( Z_t = [1, \tilde{r}_{t-1}]' \) with \( \tilde{r}_t \equiv |100 \times ln(SP_{t}^{close}/SP_{t-1}^{close})| \). The associated coefficients of the two vectors are denoted by the \( 5 \times 2 \) matrix \( \phi \) and the \( 5 \times 5 \) matrix \( \gamma \). We use the close-to-close return on the right hand side because it spans the entire period from \( t - 1 \) to \( t \). We fit the model to five quantiles: 2.5%, 25%, 50%, 75%, and 97.5%. Therefore, \( j = 1 \) indicates the 2.5% quantile and \( j = 5 \) indicates the 97.5% quantile. Similarly to a GARCH model, the MQ-CAViaR model expresses each quantile as a function of actual returns and lagged quantiles. The univariate CAViaR of Engle and Manganelli emerges when \( \gamma \) is a diagonal matrix. The model in (29) could be enriched to include further lagged returns and other explanatory variables.

Following White et al., a Quasi-Maximum Likelihood (QML) estimator \( \hat{\delta} \) for our CAViaR model is given by

\[
\min_{\delta} \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{5} [\theta_j - I(r_t \leq q_{j,t}(\delta))][r_t \leq q_{j,t}(\delta)],
\]

(30)

where \( \theta_1 = 0.025, \ldots, \theta_5 = 0.975 \), \( I(\cdot) \) is the indicator function, and \( \delta \) contains the elements of \( \phi \) and \( \gamma \). Using results in White (1994), \( \sqrt{T} \)– consistency and asymptotic normality of the estimator is established such that

\[
\sqrt{T} \left( \hat{\delta} - \delta_0 \right) \overset{d}{\to} N \left( 0, \hat{A}^{-1} \hat{B} \hat{A}^{-1} \right),
\]

(31)

with a consistent estimator of the asymptotic covariance matrix constructed using

\[
\hat{A} = (2\tilde{c}_T T)^{-1} \sum_{t=1}^{T} \sum_{j=1}^{5} I(-\tilde{c}_T < \hat{\varepsilon}_{jt} < \tilde{c}_T) \nabla \hat{q}_{jt} \nabla \hat{q}_{jt}', \quad \hat{B} = T^{-1} \sum_{t=1}^{T} \hat{\eta}_t \hat{\eta}_t',
\]

(32)

\[
\hat{\varepsilon}_{jt} = r_t - \hat{q}_{j,t}, \quad \hat{\eta}_t = \sum_{j=1}^{5} [\nabla \hat{q}_{jt}][\theta_j - I(r_t \leq \hat{q}_{j,t})],
\]

where \( \hat{\Delta} \) and \( \hat{B} \) denote consistent estimates of \( A_0 \) and \( B_0 \), respectively. We use \( \nabla \hat{q}_{jt} \) to denote the gradient of the \( j \)th conditional quantile evaluated at \( \hat{\delta} \) and \( \tilde{c}_T \) to denote a bandwidth with the properties described in assumption (VC1) of the Appendix in Engle and Manganelli.

The minimization problem in (30) is challenging for two reasons. First, the QML objective function is highly non-linear in \( \delta \) and involves a non-smooth step function. Second, the non-linear search has to be performed in a high-dimensional space because there are 7 parameters to be estimated in each of the 5 quantile equations in (29). We addressed these challenges in a step-wise approach.

In the first step, we estimate a univariate CAViaR model for each of the 5 quantiles. Each of these univariate models involves only three parameters. In order to obtain the univariate estimates, we employ the Simplex algorithm of Lagarias, Reeds, and Wright (1988), as implemented in the \textit{fminsearch} routine in MATLAB, using 50 draws from a uniform \([0,1]\) distribution as starting values. In the second step, we use the set of starting values that gave rise to the minimum objective function value in each of the 5 univariate CAViaR models to construct the first set
of starting values for 15 of the 35 parameters for the non-linear search in the multivariate case. The remaining 20 parameters were set equal to zero. We constructed 20 more sets of starting values in the multivariate non-linear search by adding small noise drawn from a uniform $[0,0.01]$ distribution. Finally we initialize both the univariate and multivariate CAViaR models using the appropriate empirical quantiles of the first 100 observations.\footnote{We set the tolerances for changes in the objective function value and the parameter vector equal to $1e-4$ ($1e-10$) for the univariate (multivariate) case. The maximum number of function evaluations was set equal to 20,000 for both the univariate and multivariate searches.}

2.2 Risk-neutral distribution implied by options prices: logistic mixture

A large literature, with seminal contributions by Banz and Miller (1978) and Breeden and Litzenberger (1978), has developed methods for estimating the risk-neutral distribution directly from options prices (see Figlewski (2010)). Our approach relies heavily on this literature. Using $X$ and $S_T$ to denote the strike and underlying price at expiration day $T$, the price of a European put option may be written as

$$P_t(X,T) = E_t[M_{t,T} \times max(X - S_T,0)] = E_t[M_{t,T}] \int_{-\infty}^{X} (X - S_T)dF^*_t(S_T) \quad (33)$$

We define the adjusted put option price

$$\tilde{P}_t(X,T) \equiv \frac{1}{E_t[M_{t,T}]} P_t(X,T) \quad (34)$$

and take the derivative with respect to the strike price to get

$$\frac{\partial \tilde{P}_t(X,T)}{\partial X} = \frac{1}{E_t[M_{t,T}]} \frac{\partial P_t(X,T)}{\partial X} = \int_{-\infty}^{X} dF^*_t(S_T) = F^*_t(X). \quad (35)$$

Therefore, it is possible to obtain the risk-neutral distribution $F^*_t(S_T)$ by estimating the first derivative of $\tilde{P}_t(X,T)$ with respect to $X$. Put-call parity produces a parallel expression for call prices. Specifically, put-call parity implies

$$\tilde{P}_t(X,T) = \frac{C_t(X,T) - S_t + E_t[M_{t,T}]X}{E_t[M_{t,T}]}, \quad (36)$$

which leads us to define an adjusted call price as

$$\tilde{C}_t(X,T) \equiv \frac{C_t(X,T) - S_t}{E_t[M_{t,T}]} + X, \quad (37)$$
such that
\[ \frac{\partial \tilde{C}_t (X, T)}{\partial X} = F^*_t (X) \] (38)

As a result, we obtain the risk-neutral distribution \( F^*_t (S_T) \) by estimating the first derivative of the adjusted call and put option price curves, \( \tilde{C}_t (X, T) \) and \( \tilde{P}_t (X, T) \), with respect to \( X \).

We propose a mixture of logistic distributions to approximate the risk-neutral distribution of the adjusted option prices in (34) and (37). Mixture distributions are very flexible as approximations to unknown distributions (Marron and Wand (1992)). Following an approach similar to ours, Yuan (2009) proposes a mixture of lognormals to obtain state price densities. Our curve-fitting approach also follows the literature that assumes a functional form for the risk-neutral distribution. The use of double log-normals by Bahra (1997) is an early example of such an approach. The use of a parametric distribution, such as the logistic here, avoids the problems that other curve-fitting methods (e.g., splines) have in getting the estimated distribution to integrate to one and in estimating the tails (see the discussion in Figlewski (2010)). The logistic distribution is appealing because its integral exists in closed form, which enables us to work with the observed options pricing curve directly.

Instead of fitting a distribution to the derivatives of the adjusted option prices, we fit the integral of a distribution to the adjusted options prices themselves. Fitting the curve before differentiating the option pricing curve is important because it avoids arbitrary assignment of the point \( X \) at which the derivative applies. By fitting the function first, we can obtain the derivative at any point of interest. Thus, we fit a flexible function to the adjusted option prices that imposes the restriction that the derivative is a distribution. Specifically, using \( X_T \) to indicate the exercise price of an option that expires at \( T \), our objective is to fit the following function

\[ F^*_t (X_T) = \sum_{j=1}^{J} \omega_{jt} \Lambda_{jt} (X_T; \mu_{jt}, \sigma_{jt}) , \] (39)

where

\[ \Lambda_{jt} = \exp \left( \frac{X_T - \mu_{jt}}{\sigma_{jt}} \right) / \left( 1 + \exp \left( \frac{X_T - \mu_{jt}}{\sigma_{jt}} \right) \right) \] (40)

is the logistic distribution. We do so by fitting the options pricing curve to the integral of \( F^*_t (X_T) \) given by

\[ \tilde{F}^*_t (X_T) = \sum_{j=1}^{J} \omega_{jt} \tilde{\Lambda}_{jt} (X_T; \mu_{jt}, \sigma_{jt}) , \] (41)

\[ \tilde{\Lambda}_{jt} (X_T; \mu_{jt}, \sigma_{jt}) = \int \Lambda_{jt} dX_T = \sigma_{jt} \ln \left( \exp \left( \frac{X_T}{\sigma_{jt}} \right) + \exp \left( \frac{\mu_{jt}}{\sigma_{jt}} \right) \right) \] (42)

We impose \( \sum_{j=1}^{J} \omega_{jt} = 1 \), with \( \omega_{jt} \geq 0, \ j = 1, \ldots, J \). In our empirical exercise below, we set \( J = 2 \), which implies that we fit the distribution using five parameters at each point in time. This
number of parameters provides considerable flexibility when compared to parametric distributions such as the normal (two parameters), skew-normal and student-t (three parameters), and skew-t (four parameters; see Azzalini and Capitanio (2003)).

At each date of interest $t$, we have adjusted options prices $\tilde{C}_t (X,T)$ and $\tilde{P}_t (X,T)$ for a set of strike prices. Collecting these prices in the $N_t \times 1$ vector of option prices $o_t$, we specify the model

$$o_{it} = \sum_{j=1}^{J} \omega_{jt} \tilde{\Lambda}_{jt} (X_T; \mu_{jt}, \sigma_{jt}) + \epsilon_{it}$$

where the subscript $i$ denotes the $i^{th}$ element of $o_t$. As a result, we solve the constrained non-linear least squares (CNLS) problem

$$\min_{\omega_{it}, \mu_{it}, \sigma_{it}} \frac{1}{N_t} \sum_{i=1}^{N_t} \left( o_{it} - \sum_{j=1}^{J} \omega_{jt} \sigma_{jt} \ln \left( \exp \left( \frac{X_tT}{\sigma_{jt}} \right) + \exp \left( \frac{\mu_{jt}}{\sigma_{jt}} \right) \right) \right)^2,$$

s.t. $\sum_{j=1}^{J} \omega_{jt} = 1$, $\omega_{jt} \geq 0$, $j = 1, \ldots, J$.

We estimate $F^*_t (X,T)$ by taking the parameters estimated from (44) and plugging them into (39). We solve this least squares problem separately for each date $t$ that occurs either a week or a month ahead of an option expiration.

Two practical difficulties arise in the setup described above. First, we don’t observe exactly the closing options prices. We observe the end-of-day bid and ask quotes. We use the midpoint between these quotes as our options price. This choice implies that we will have less than a perfect fit in our least squares problem, but we do not expect a systematic bias. To improve the accuracy of our option price data, we omit option prices implying midquotes of less than 0.5 and those with no trading volume on day $t$ following Figlewski (2010).

Our second difficulty stems from the fact that option trading closes 15 minutes after the stock market at 3:15pm central time. This institutional fact creates an asynchronouusity when calculating the adjusted call price in (37), which has $S_t$ as an argument. We address the asynchronouusity issue by assuming that information in the last 15 minutes of options trading does not change the shape of the pricing curve, but may shift it up or down by an unknown constant. Because we observe multiple call option prices that would be affected by the same constant, we can estimate this constant. Thus, we adjust our constrained non-linear least squares problem to the following

$$\min_{a_{it}, \omega_{jt}, \mu_{jt}, \sigma_{jt}} \frac{1}{n} \sum_{i=1}^{n} \left( o_{it} - a_{it} - \sum_{j=1}^{J} \omega_{jt} \sigma_{jt} \ln \left( \exp \left( \frac{X_tT}{\sigma_{jt}} \right) + \exp \left( \frac{\mu_{jt}}{\sigma_{jt}} \right) \right) \right)^2,$$
\[ s.t. \sum_{j=1}^{J} \omega_{jt} = 1, \; \omega_{jt} \geq 0, \; j = 1, \ldots, J \]

where \( d_{it} \) is equal to one for the call prices and zero otherwise and \( a_t \) is an unknown shift of the pricing curves that we estimate.\(^{11}\)

The error term in (43) captures noise in the data that may result from stale quotes, our use of the bid-ask mid point rather than the unobserved fundamental price, or from approximation error. We assess the fit of the logistic mixtures using the following R-squared measure

\[ R_t^2 = 1 - \frac{\sum_{i=1}^{N_t} (o_{it} - \tilde{o}_{it})^2}{\sum_{i=1}^{N_t} (o_{it} - \bar{o}_t)^2} \] \hspace{1cm} (46)

In terms of notation, \( N_t \) denotes the number of observations for the date under consideration and \( \bar{o}_t = (1/N_t) \sum_{i=1}^{N_t} o_{it} \). We solve (45) using the \textit{lsqnonlin} routine in MATLAB that solves constrained non-linear least squares problems implementing a trust-region reflective Newton algorithm (Coleman and Li (1996)).\(^{12}\) We use the S&P index close price on date \( t \) scaled by 0.001 as a starting value for the non-linear search of \( a_t \). We use 0.5 as the starting value for \( \omega_{1t} \). The starting values for \( \mu_t \) are equal to the S&P index close price on date \( t \). Finally, the starting values for \( \sigma_t \) are equal to the standard derivation of the strike prices on date \( t \).\(^{13}\)

2.3 Data

2.3.1 Physical distribution: S&P 500 index

The data for the S&P 500 index are from Yahoo Finance between January 3rd, 1950 and February 27th, 2009 for a total of 15,152 observations. Although our options data begin in 1990, we estimate the parameters of the MQ-CAViaR model using data between 1950 and 2009 to gain precision. Limiting our attention to the period covered by our options data, our sample reduces to 4,827 daily observations.

As Figure 3 illustrates, the S&P 500 is characterized by two pairs of peaks and troughs between January 1990 and the spring of 2009. The closing values of the index at the each of these peaks are 1,527 on March 24, 2000, and 1,565 on October 9, 2007. The first of these dates corresponds to the peak of the dotcom boom, and the second date comes soon after the peak of the housing boom.

\(^{11}\)The adjusted call price that addresses the timing discrepancy may be written as \( \tilde{C}_{it}(X,T) = (1 + r_{t,T})(C_{it}(X,T) - (S_{t} + \delta^{\text{timing}})). \) The term \( a_t d_{it} \) in our estimating equation plays the role of \( \delta^{\text{timing}} \).

\(^{12}\)The idea behind the trust region is the following. Suppose you are at a point \( x \) of an \( n \)-space and you want to move to a point with a lower value of the objective function \( g \). The basic idea is to approximate \( g \) with a simpler function \( g^* \), which reasonably reflects the behavior of function \( g \) in a neighborhood (trust region) around the point \( x \). The main ingredients of the trust-region approach is how to compute the approximation \( g^* \), how to choose and modify the trust region and how to solve the trust-region subproblem.

\(^{13}\)We set the tolerances for changes in the objective function value and the parameter vector equal to \( 1e-4 \). The maximum number of function evaluations was set equal to 100,000.
The corresponding trough values, are 777 on October 9, 2002, around the Enron and WorldCom scandals, and 734, on the last day of the sample.

We provide summary statistics for the weekly and monthly returns of the index for the period 1990 onwards in Table I. For all statistics, we use a block bootstrap with block size of 1 year to construct 95% confidence intervals. The entries in this table are based on the index return at date $t$ given by

$$r_t = 100 \times \ln(\frac{SP_t^{\text{open}}}{SP_t^{\text{close}}})$$

where $SP_t^{\text{open}}$ and $SP_t^{\text{close}}$, are the open and close index values that are $h$ days apart depending on the horizon. We calculate returns as the difference in logs of open and close values of the index because the expiration price of the options is the opening price on the third Friday of the month. We use date pairs that are 5 trading days apart to calculate the weekly returns. At the monthly frequency, we wish to match the interval over which the return quantile applies to the last four weeks of the life of an option. However, the intervals between option expirations are not constant; some expirations are four weeks apart and others are five weeks apart.

The irregularity identified above doesn’t fit neatly into the CAViaR model, which requires uniform intervals. Thus, we create four separate time series of four-weekly “Friday-to-Friday” returns, beginning on the 6th, 13th, 20th and 27th of January, 1950, respectively. The last four weeks of each option expiration will have a matching four-week return in one of the four series. We estimate an MQ-CAViaR model for each series and match the conditional quantiles to the associated intervals in the options data. We have no reason to expect systematic differences between these four series (denoted Monthly I-IV in Table I), but nonetheless we do not impose identical MQ-CAViaR parameters across them. Table I shows that sample statistics vary quite a bit across the four series, although this variation is within the bounds of sampling variation.

Based on Table I, weekly returns averaged 0.068% between 1990 and 2008, which corresponds to a 3.5% annualized return. Their monthly counterparts are similar, taking values between 0.29% and 0.41% per four-weeks (3.8%-5.3% annualized) depending on the sample. Although the mean return is not statistically significantly different from zero over this period, the median return is significant. The annualized volatilities are 14.7% on the weekly basis, and between 13.9% and 16.5% on the monthly basis. In the same table, we include both moment- and quantile-based measures of skewness and excess kurtosis. The quantile-based measures utilize unconditional (empirical) quantiles and are calculated following White, Kim, and Manganelli (2010) using the formulae

$$SK = \frac{\hat{q}_4 + \hat{q}_2 - 2\hat{q}_3}{\hat{q}_4 - \hat{q}_2}$$

$$KR = \frac{\hat{q}_5 - \hat{q}_1}{\hat{q}_4 - \hat{q}_2} - 2.91$$

where $\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4,$ and $\hat{q}_5$ denote the empirical quantiles calculated at the 2.5, 25, 50, 75 and 97.5 percentiles, respectively. The skewness statistic $SK$ captures the proportion of the interquartile
range that lies below the median. The kurtosis statistic $KR$ equals the width of a 95% confidence interval relative to the interquartile range. Both measures equal zero for the normal distribution. These quantities are not measured in the same units as their moment-based counterparts, but the quantile-based measures are less sensitive to outliers (White, Kim, and Manganelli (2010)). This reduced outlier sensitivity improves the precision of sample statistics at the cost of missing features of the extreme tails of the distribution. Specifically, the skewness measure is invariant to the shape of the distribution outside the interquartile range and the kurtosis measure is invariant to the shape of the distribution outside the 95% range.

Both the moment- and the quantile-based skewness statistics imply long left tails for the weekly and the monthly returns. The moment-based skewness equals -1.42 for weekly returns and lies between -2.06 and -0.61 for the monthly returns. The quantile-based skewness statistic is -0.13 for weekly returns and for the monthly returns it takes values between -0.23 and -0.05. The moment-based skewness is statistically significant at the 5% level for 4 out of the 5 return series, whereas the quantile-based skewness is statistically significant only for weekly returns and for the monthly II series. The greater significance of the moment-based measures suggests that they are driven in large part by the extreme tails. The moment-based excess kurtosis for the weekly returns is 12.2, and it ranges from 1.04 to 12.10 for the monthly returns, which shows the sensitivity of moment-based kurtosis estimates to extreme values. The quantile-based kurtosis is around 0.98 for weekly returns and ranges from 0.70 to 1.40 for monthly returns.

Regarding the dynamics of the returns, the first-order autocorrelation coefficients for the weekly and monthly returns are -0.1 and -0.09 to 0.08. The correlation coefficients between the squared levels and their lags are 0.28 and 0.05 to 0.33, respectively. These notable autocorrelations between squared returns and their lags are consistent with volatility clustering that characterizes many financial asset returns (e.g., Engle (2004)).

### 2.3.2 Risk-neutral distribution: S&P 500 options

The data for the S&P 500 options are from the Chicago Board Options Exchange (CBOE). The market for S&P 500 options operates between 8:30 am and 3:15pm central time. At any point in time, three near-term expiration months are trading along with three additional months from the March quarterly cycle (March, June, September and December). Currently, the strike price intervals are set at 5 points (25-point intervals for distant expiration months). The expiration date is the Saturday following the third Friday of the expiration month. Trading usually ceases on the business day preceding the day on which the exercise-settlement value is calculated. The options generally may be exercised only on the last business day before expiration (European style) with exercise resulting in delivery of cash on the business day following expiration.\(^\text{14}\)

The data in hand contain information regarding trading volume, as well as bid, ask and exercise

\(^{14}\)Additional information about the market is available at [www.cboe.com](http://www.cboe.com)
prices for calls and puts between January 1990 and November 2008.\textsuperscript{15} Table II provides some contract and price information for weekly and monthly expiration horizons for both call and put options. We have 227 contract expirations available for both horizons, one for each month between January 1990 and November 2008. The average call (put) strikes for options expiring within a week are 955 (1104). Their counterparts for options expiring within a month are 966 (1,060). Call mid prices are on average 121 and 115 for the weekly and monthly horizons, respectively.\textsuperscript{16} The put mid prices for the same horizons are 73 and 57. We calculated the adjusted options prices in (34) and (37), using $E_t [M_{t,T}]^{-1} = (1 + r_{t,T})$, where $r_{t,T}$ denotes the LIBOR rate that covers the interval from $t$ to $T$.\textsuperscript{17}

2.4 Estimation results

2.4.1 Physical distribution: MQ-CAViaR

We present the MQ-CAViaR estimates along with model diagnostics for both the one-week and one-month horizons in Table III. For the purpose of estimation, we use data for returns on the S&P 500 index between January 1950 and February 2009. We don’t limit our estimation sample to the period that overlaps with the options data, January 1990 through November 2008, in order to gain precision in our parameter estimates.

For the one-month horizon, we report results for each of the four-week (monthly) return series. Parameter estimates for each of the five MQ-CAViaR equations associated with the 2.5%, 25%, 50%, 75% and 97.5% conditional quantiles are shown in the rightmost section of the table. The table also reports the in-sample dynamic-quantile (DQ) statistic of Engle and Manganelli (2004), along with an in-sample hit-rate statistic. The in-sample DQ statistic is used to test the null hypothesis of correct specification for the particular CAViaR process studied. The hit-rate statistic calculates the fraction of observations for which $r_t$ lies below the $j^{th}$ quantile under consideration using the sequence of indicator functions $\{I(r_t \leq q_{j,t}(\delta))\}_{t=1}^{T}$. We report both of these statistics for the period following January 1990 rather than over the whole sample because we use only post-1990 quantiles to estimate SPOCQ. Figures 4 and 5 contain time series plots of the 5 conditional quantiles for the one-week and each of the one-month series between 1990 and the end of our sample.\textsuperscript{18}

At the one-week horizon, the coefficients of the autoregressive term (lagged own quantile) all exceed 0.8 in absolute value and also exceed the coefficients of the remaining lagged quantiles by

\textsuperscript{15}We exclude ”mini” contracts identified by the following codes ”SXZ”, ”SPB”, ”LSW”, ”LSX”, ”LSY”, ”LSZ”, ”XSC”, ”XSE”, ”XSK”, ”XSL”, ”XSO”, ”XSP”. Having defined the option price for calls and puts as the mid point of the corresponding bid and ask prices, we drop observations with option prices below 0.5.

\textsuperscript{16}The mid price is the mid-point of the bid and ask prices.

\textsuperscript{17}Available from the British Bankers Association. See also www.economagic.com/libor.htm.

\textsuperscript{18}The time series plots in these figures correspond to dates one and four weeks prior to each of the 227 trading dates of the S&P 500 options.
at least an order of magnitude for 4 out of the 5 equations. Therefore, the autoregressive term seems to be more significant than the remaining conditional quantile terms for the time evolution for each of the five conditional quantiles. The values of the coefficient associated with the lagged return \( \tilde{r}_{t-1} \) lie between -0.32 (2.5% quantile) and 0.18 (97.5% quantile). The constant takes values between -0.14 (2.5% quantile) and 0.4 (50% quantile).

At the one-month horizon, the coefficients of the autoregressive terms largely exceed 0.70 for all the four Friday-to-Friday series with very few exceptions, such as the coefficients of the 2.5% quantile for the first (monthly I) and second (monthly II) series, as well as the coefficient of the 50% quantile for the second series, which are all below 0.5. In a single case, that of the fourth series (monthly IV), the coefficient of autoregressive term for the 25% quantile exceeds one. With the exception of the coefficients associated with the autoregressive terms, the remaining quantile coefficients are almost indistinguishable from zero for all five quantile equations in the case of the fourth series. For the remaining series, a pattern similar to that of the one-week horizon emerges: the coefficients of the lagged own quantiles exceed the remaining lagged quantile by at least an order of magnitude in the majority of the equations. Additionally, the coefficients of the lagged return lie between -0.56 (2.5% quantile, monthly II), and 0.21 (97.5% quantile, monthly III). Finally, the constant values are between -2.96 (2.5% quantile, monthly I) and 1.34 (97.5% quantile, monthly II).

The hit rate statistic, calculated between January 1990 and the end of the sample, shows that the model does reasonably well in describing the evolution of the quantiles of the physical distribution over the post-1990 period, especially for the quantiles lying on the right of the distribution. The in-sample DQ statistic essentially tests whether the hits are autocorrelated. We test for correlation with the first four lags. The DQ statistic is distributed as \( \chi^2(4) \) under the null hypothesis of correct specification. Based on the DQ statistic, we don’t reject the null of correct specification for 22 of the 25 equations in Table III given a critical value for a \( \chi^2(0.05,4) \) equal to 9.49. The three equations for which we reject the null of correct specification are: 50% quantile (weekly) 75% quantile (monthly I) and 25% quantile (monthly II).

2.4.2 Risk-neutral distribution: logistic mixture

We provide an example of the adjusted put and call option pricing curves for the one-week and one-month horizons in Figures 6 and 7. These pricing curves are constructed using the expressions in (34) and (37). In the case of the one-week horizon, we use the contracts traded on November 14, 2008 for expiration on November 22, 2008. In the case of the one-month horizon, the contracts were traded on October 24, 2008 expiring on the same date with those of the one-week horizon.

In our example of the one-week horizon, the index closed at 873 on November 14, 2008. In Figure 6, we observe 184 strike prices with calls for 123 of them and puts for 156 of them. We observe no calls for strikes exceeding 1,215 and no puts for strikes below 540. The average bid-ask
spread is 4.7 for calls and 5.1 for puts. The location and scale parameters associated with the most probable component of the mixture of the two logistic distributions, which is assigned a weight of 0.74, are equal to 889 and 36.¹⁹ The same parameters for the second mixture component, which is assigned a weight of 0.26, are 784 and 41, respectively. The coefficient of the call dummy is -11.95 and the $R^2$ is equal to 0.99 pointing to a tight fit of the logistic mixture.

In the case of our one-month example, the index closed at 877 on October 24, 2008. In Figure 7, we have 169 strike prices with calls for 142 of them and puts for all but one of them. The strikes range from 300 to 1,700 with no calls for those between 1,175 and 1,500. The average bid-ask spreads is 8.7 for calls and 11.4 for puts. The most probable component of the logistic mixture is assigned a weight of 0.51 and its location and scale parameters are 969 and 51, respectively. The second mixture component that is assigned a weight of 0.49 has a location parameter of 762 and a scale parameter of about 100. The call dummy is equal to -10.92 and the $R^2$ is equal to 0.99.

We performed the same curve fitting exercises for each of the remaining 226 trading dates for both horizons. In the case of the one-week horizon, the median numbers of calls, puts and strikes across all the 227 trading dates are 30, 21, and 41, respectively. Across all one-week contracts, the minimum number of strikes is 13 and the maximum is 184 with an average of 43. In the case of the one-month horizon, the minimum number of strikes across all one-month contracts is 12. The average is 41 and the maximum is 177. For all the one-month contract expirations, the median numbers of calls, puts and strikes are similar in magnitude to those of the week-ahead contracts; 31, 29, and 36, respectively.

We conclude our discussion in this section by commenting on Table IV, which provides summary statistics for the scale parameters and the $R^2$ when either a single or a mixture of logistic distributions is used to infer the risk-neutral distribution. In the case of the mixture, the summary statistics for the scale parameter and the weight are constructed for the most probable component of the mixture. At the one-week horizon, the scale parameter lies between 2.6 and 64.9 with a median value of 11.3 assuming a single logistic distribution. Fitting a mixture of distributions produces a somewhat smaller range for the scale parameter, 1.1 to 49.2; its median value is 9.4. For half of the expiration dates, the most probable component receives a weight of at least 0.92. At the one-month horizon, the scale parameter for the single distribution is between 5.4 and 101.3 with a median value of 25.3. In the case of a mixture of distributions, the range of the scale parameter for the most probable component is tighter, 2.5 to 62.5. Its median value is 18.7. The median value of the weight associated with the most probable component is 0.83, which is smaller than for the weekly horizon and suggests a less symmetric distribution than the weekly horizon. Fitting either a single logistic distribution or a mix of logistic distributions leads to $R^2$ greater than 0.99 on average for both horizons.

¹⁹ In terms of Equation (45), the most probable component $j$ is the one for which $\hat{\omega}_{jt} > 0.5$
3 SPOCQ Over Time: 1990-2008

We provide the plots of representative SPOCQ series in Figures 8 and 9 along with vertical lines to signify events contributing to the volatility of the U.S. stock market and shaded areas corresponding to the U.S. business cycles as identified by the National Bureau of Economic Research.\footnote{See: \texttt{www.nber.org/cycles.html}} In addition to the events highlighted in Bloom (2009), we also identify Alan Greenspan’s famous “irrational exuberance” speech in December 1996, the peak of dotcom boom in March 2000, and the peak of the U.S. housing market in April 2006 as defined by the S&P/Case-Shiller 20-City Composite Index. We show SPOCQ values measured one-week and one-month ahead of each of the 227 option expiration dates from January 1990 to November 2008.

The SPOCQ(0,25) line indicates the investors’ willingness to pay (WTP) for insurance against outcomes in the lowest quartile of the S&P 500 return distribution. This insurance takes the form of a one-dollar payout in the event that returns fall in the bottom quartile. The difference between the SPOCQ(0,75) and SPOCQ(0,25) lines indicates the WTP for insurance against outcomes in the interquartile range of the same distribution, which may be written as SPOCQ(25,75). Finally, the difference between one and the SPOCQ(0,75) line indicates the WTP for insurance against outcomes in upper quartile, the SPOCQ(75,100).\footnote{The SPOCQ series have been smoothed using an average of the 4 most recent observations. For example using \( x_t \) to denote the SPOCQ series under consideration \( x_t = (1/4)\sum_{\tau=0}^{3} x_{t-\tau}. \)}

At the one-week horizon, the average state prices are 26 cents, 45 cents, and 29 cents, for outcomes in the lower 25%, mid 50% and upper 25% of the return distribution, respectively. Therefore, on average, investors’ WTP for insurance against outcomes in the lowest and highest quartiles was higher than their WTP for outcomes in the middle 50% of the return distribution. At the one-month horizon, the average state prices are 26 cents for the high and low quartiles and 48 cents for the middle 50% of the return distribution. The median state prices are essentially indistinguishable from the average ones for both horizons. The only exception is the SPOCQ(0,25) for the monthly horizon whose median is equal to 47 as opposed to 48 cents.

Under risk neutrality, the SPOCQ(0,25) line would be horizontal at 0.25 and the SPOCQ(0,75) line would be horizontal at 0.75. As we discuss in Section 2, the component of the risk aversion that emanates from relative risk aversion manifests in the difference between SPOCQ(0,25) and SPOCQ(75,100). When SPOCQ(0,25) is large relative to SPOCQ(75,100) the market places greater marginal value on a dollar in low-return states of the world relative to high-return states. Figure 9 shows that, at the one-month horizon, SPOCQ(0,25) exceeded SPOCQ(75,100) in the periods 1992-1997 and 2002-2004. The difference, SPOCQ(0,25)-SPOCQ(75,100), was as high as 7.8 cents during these periods with an average of 3 cents. These periods began soon after the business cycle troughs in March 1991 and November 2001. They ended about three years before the ensuing business cycle peaks, with the first period ending soon after Alan Greenspan’s
irrational exuberance speech and the second about two years before the housing market peak. As we scan the vertical lines in Figures 8 and 9, we see that few uncertainty events occurred in these periods. Looking at the conditional quantiles in Figures 4 and 5, we see that these periods were characterized by low volatility. Thus, in the periods immediately following the 1990-91 and 2001 recessions, the one-month ahead SPOCQ series imply larger than average risk aversion through declining marginal utility of wealth. The one-week ahead series shown in Figure 8 display less evidence of this type of risk aversion.

Figures 8 and 9 show that the tail state prices, SPOCQ(0,25) and SPOCQ(75,100) simultaneously exceed 25 cents in several periods, which suggests positive volatility aversion. In these periods, investors were willing to pay a premium to receive a dollar in the event that returns are either large positive or large negative. At the one-week horizon, the periods of positive volatility aversion are 1990 to 1994, late 2000 to 2004, and late 2007 to 2008. More specifically, between January 1990 and December 1994, the average lower-quartile state price was 27 cents and its upper-quartile counterpart was 30 cents. Between October 2000 and December 2004, the average lower-quartile state price was 28 cents and its upper-quartile counterpart was 30 cents. At the one-month horizon, 1990 to 1991, 1997 to 2004 and 2008 are the periods of positive volatility aversion. The state prices between January 1990 and December 1991 are 27 and 30 cents for the lower and upper quartiles, respectively. The 1997-2004 average state prices are highly similar; 27 cents in the lower quartile and 28 cents in the upper quartile. The period between January 2008 and the end-of-sample has the largest volatility aversion, with average state prices equal to 31 cents in each tail.

Each period of volatility aversion encompasses a recession, tends to last much longer than the recession, includes significant uncertainty shocks, and coincides with large stock market volatility (see the conditional quantiles in Figures 4 and 5). These characteristics of the volatility aversion periods are particularly prominent at the one-month horizon. For example, the period of volatility aversion from 1997 to 2004, begins around the Asian crisis in Fall 1997 and includes the Russian Financial Crisis (Fall 1998), September 11 (Fall 2001), the Enron/WorldCom (Summer/Fall 2002) collapse, and Gulf War II (Spring 2003). The most recent period of volatility aversion began with the credit crunch in August 2007 and persisted through the end of our sample, peaking with the collapse of Lehman Brothers in the Fall of 2008. Two instances of negative volatility aversion arise, the first in 1995-96 and the second in 2005-07. The lower- and upper-quartile average state prices are 23 cents and 20 cents for the period 1995-1996, respectively. The same averages during the period 2005-2007 were 23 and 24 cents. These two episodes occurred during periods of increasing stock prices and low volatility and include Greenspan’s irrational exuberance speech in December of 1996 and the peak of the housing market in the Spring of 2006.

To further explore volatility aversion, we plot in Figure 10 the sum of SPOCQ(0,25) and SPOCQ(75,100) for the one-month horizon along with the smoothed annualized volatility premium series from Bollerslev, Tauchen, and Zhou (2009), which equals the difference between
option-implied volatility (VIX) and realized volatility. Both SPOCQ(0,25)+SPOCQ(75,100) and the volatility premium measure dispersion in the risk-neutral distribution relative to the physical distribution. However, they may differ because SPOCQ is a quantile-based measure whereas the volatility premium is moment based. For example, Bollerslev, Tauchen, and Zhou (2009) show that, in a representative agent model with Epstein-Zin preferences, the volatility premium is determined by the volatility of consumption-growth volatility. If shocks to consumption-growth volatility were homoskedastic, then the variance premium would be constant in their model. However, as we show in equation (15), SPOCQ(0,25)+SPOCQ(75,100) depends on the level of volatility as well as the volatility of volatility. Higher volatility will cause volatility averse investors to pay more for the security defined by SPOCQ(0,25)+SPOCQ(75,100) because higher volatility means that larger absolute returns characterize the upper and lower quartiles, which make volatility-averse individuals worse off.

The SPOCQ(0,25)+SPOCQ(75,100) and volatility premium series track each other well until 2007, and their correlation is especially strong prior to 2000. The correlations between the two series are 0.89 between 1990 and 2000 and 0.31 for the entire period. In early 2000, around the dotcom peak, the volatility premium dropped in half and remained at that level, whereas our measure of volatility aversion stayed high. In August 2007, at the beginning of the credit crunch, our volatility aversion statistic increased substantially before levelling off and then increasing again one year later, at the peak of the financial crisis. In contrast, the volatility premium did not change in 2007 and dropped sharply in September 2008. This difference suggests that volatility increased substantially in this period, but the volatility of volatility did not.

To summarize, the SPOCQ series in Figures 8-10 show that the periods 1992-1997 and 2002-2004 were associated with larger than average relative risk aversion. These periods occurred right after the 1990-91 and 2001 recessions and they were also characterized by low volatility of the stock market, as indicated in Figures 4 and 5. During the same periods, the investors placed greater marginal value on a dollar in low-return states of the world relative to high-return states. The high-volatility periods encompassing these two recessions were characterized by high volatility aversion, in which the investors’ WTP against outcomes in the lower and upper quartiles was higher than their WTP for outcomes in the mid-50% of the return distribution increases. The most recent recession is also characterized by high volatility aversion.

\[ \text{In Section 3.2.2 of Bollerslev et al., the volatility premium is defined as } \sqrt{IV} - \sqrt{RV}, \text{ where } IV \text{ and } RV \text{ are the implied and realized variances. The } IV \text{ and } RV \text{ series are available at: } \text{www.federalreserve.gov/econresdata/researchdata.htm}. \text{ After annualizing their volatility premium series, we smoothed it using a four-month trailing moving average as with our SPOCQ series.} \]
4 Forecasting S&P 500 Returns

4.1 Preliminaries

In this section, we examine how well the SPOCQs measured monthly from January 1990 to November 2008 can forecast S&P 500 excess market returns over horizons that vary between one month and 2 years. We thus make 227 forecasts at each horizon and use return data up to Fall 2010 at the longest horizon. Our results are based on linear regressions of the returns on different sets of lagged predictors. These predictors include our SPOCQs and variables that have been employed in the literature before (e.g., Bollerslev, Tauchen, and Zhou (2009)). Following the norm in the literature, we compare the forecasting ability of various predictors on the basis of the adjusted $R^2$, which we denote by $R^2_{full}$. We perform in-sample inference using the standard errors in Hodrick (1992) to account for the dependence induced by overlapping forecast horizons.\[^{23}\] To further account for the possibility of spurious regression, we follow Welch and Goyal (2008) and report an out-of-sample $R^2$, which we denote by $R^2_{out}$. We compute $R^2_{out}$ from forecasts of the last 120 observations of our sample using expanding regressions in which the size of the estimation sample increases progressively.\[^{24}\]

We perform the forecasting regressions for horizon $h$ using the following expression for excess returns

$$r_{t+h} - r^f_t = 100 \times a_h \times \ln(SP_{t+h}^{open} / SP_t^{close}) - r^f_t,$$

(50)

where $SP_t^{close}$ is the closing price of S&P on the trading date $t$ implied by the option contract, $SP_{t+h}^{open}$ is the opening price $h$ days apart, and $a_h = 12, 4, 2, 1, 2/3, \text{ and } 1/2$ is an annualization factor. Since we consider five alternative horizons corresponding to 1 month, 1 quarter, half year, one year, 18 months and two years, the values of $h$ are: 28, 84, 168, 336, 504, and 672 days. For example, the first observation for $r_{t+h}$ is constructed using the S&P closing price on January 19, 1990 and the opening price on February 16, 1990 in the case of the one month ($h = 28$) horizon. In the same spirit, the return in the case of the 6 month horizon ($h = 168$), is calculated using the closing price on January 19, 1990 and the opening price on July 6, 1990. Similar reasoning applies for the remaining observations and horizons. We use the 3-month T-bill rate on date $t$ as a proxy for the risk free return $r^f_t$.

\[^{23}\text{Ang and Bekaert (2007) and Wei and Wright (2010) are examples of recent writings on the issue.}\]

\[^{24}\text{For example, the first one-month out-of-sample forecast refers to the return from Dec-18-1999 to Jan-15-1998 and is constructed using coefficients estimated from the first 107 observations in our sample that span the period January 1990 to November 1998. Similarly, the second out-of-sample forecast refers to the return from Jan-22-1999 to Feb-19-1999 and is constructed using the first 108 in-sample observations, which span the period January 1990 to December 1998, etc. At all horizons, we use the largest estimation sample that has no overlap with the forecast period.}\]
Based on (26), we first predict returns using the following three SPOCQ series

\[
SPOCQ_{25} \equiv SPOCQ_t(0, 25)\sigma_t \\
SPOCQ_{50} \equiv SPOCQ_t(25, 75)\sigma_t \\
SPOCQ_{75} \equiv SPOCQ_t(75, 100)\sigma_t
\]

We use \(\sigma_t = \tilde{q}_t - \tilde{q}_{2t}\), the interquartile range from the MQ-CAViaR, as our measure of the return dispersion. Motivated by the functional form in (26), and because \(SPOCQ_t(0, 25), SPOCQ_t(25, 75),\) and \(SPOCQ_t(75, 100)\) sum to one, we run these regressions without a constant. Using additional SPOCQ series would reduce the size of the second-order term in (25) at the expense of degrees of freedom in the predictive regressions.

Based on Examples 1 and 2 in Section 2, we also estimate models that separate the contribution of SPOCQ into a relative risk aversion term and a volatility aversion term. These models include the following two SPOCQ series as predictors

\[
SPOCQD \equiv (SPOCQ_t(0, 25) - SPOCQ_t(75, 100))\sigma_t \\
SPOCQS \equiv (SPOCQ_t(0, 25) + SPOCQ_t(75, 100))\sigma_t.
\]

The SPOCQD series is the one plotted in Figure 1. Finally, to assess the gain in forecasting ability obtained from evaluating the risk-neutral distribution at the conditional quantiles from the MQ-CAViaR, as opposed to arbitrary points of the return distribution, we use the risk-neutral distribution evaluated at 0, \(F^*_t(0)\) (denoted FSTAR0) as one of our predictors. As shown in Section 2, \(F^*_t(0)\) confounds the willingness to pay for a dollar in the event of positive returns with the probability of that event occurring. In all regressions involving SPOCQ variables, we use as predictors the monthly horizon SPOCQ series, which we smooth with a 4-month trailing moving average as in Figures 9 and 10.

The remaining predictors correspond to those in Bollerslev, Tauchen, and Zhou (2009). They use the VIX index from the CBOE to estimate implied variance (IV), and they estimate realized variance (RV) using the sum of the 78 within-day five-minute squared returns during the normal trading hours 9:30-4:00 p.m. along with the squared close-to-open overnight return. Their variance premium equals IV-RV. We calculate the log price-earning ratio (PE) and the log price-dividend ratio (PD) using quarterly data from Standard and Poor’s. The default spread series (DFSP) is the difference between Moody’s BAA and AAA bond yield indices and is available from the Federal Reserve Bank of St. Louis website (FRED). We obtained the 3-month T-bill rate (TBILL) from the Treasury yield curve.

---

25 We attempted to replicate the data used in Bollerslev, Tauchen, and Zhou (2009) as closely as possible. We thank Hao Zhou for helpful discussions during this process.

26 The Standard and Poor’s data are available on a quarterly frequency (March, June, September, December). We transform the quarterly Standard and Poor’s data into monthly using the most recent quarterly observation to fill in the subsequent three months. For example, we use the PD and PE values of March, to fill in the values for April, May and June. Similarly, we use the values of June, to fill in the values for July, August and September. Analogous reasoning applies for the remaining months.
from the Treasury. We calculate the term spread (TMSP) between the 10-year T-Bond and the 3-month T-bill rates also using data from the Treasury. The stochastically detrended 3-month T-bill rate (RREL) is calculated using monthly FRED data. Finally, we downloaded the most recent version of the consumption-wealth ratio (CAY, see Lettau and Ludvigson (2001)) from Martin Lettau’s website.

We present summary statistics for the returns and the various predictors in Table 6. Among the summary statistics, we report measures of unconditional skewness and excess kurtosis that are both moment and quantile based. The mean annualized excess returns for the S&P 500 vary between 0.05% (3 months) and 2.28% (1 month). The return standard deviation and the moment-based measure of excess kurtosis decrease in magnitude as the forecasting horizon increases. The moment-based skewness exhibits largely identical pattern in absolute value. The only exception in the declining pattern occurs as we move from the 3- to the 6-month horizon. The absolute values of the quantile-based measures of skewness are U-shaped; they decrease between the 1- and 6-month horizons and they increase between the 12- to 24-month horizons. In the case of the 6-month returns, the two measures have opposite signs. The quantile-based measure of excess kurtosis is also U-shaped reaching its minimum at the 3-month horizon. All excess return series become more persistent as the horizon increases. The autocorrelation coefficient, AR(1), lies between 0.02 (1 month) and 0.97 (24 months). High persistence characterizes all the predictors except for the variance premium, which has an autocorrelation coefficient of 0.29. The AR(1) coefficients of the various SPOCQ series range between 0.87 and 0.98.

4.2 Forecasting using SPOCQ

Table VI presents results from regressions using the first set of SPOCQ predictors we described in the previous section across the six forecasting horizons. Aside from coefficient estimates and Hodrick (1992) standard errors, we report the adjusted $R^2$ for the full sample, $R^2_{\text{full}}$, and the out-of-sample $R^2$, $R^2_{\text{out}}$. Panel A shows that the SPOCQ25 and SPOCQ75 series are strongly statistically significant at horizons greater than a year, but they are not statistically significant at horizons less than a year. For horizons exceeding one year, the signs of the coefficients are consistent with equation (26). A greater SPOCQ value in the lower (upper) tail of the return distribution implies an increase (decrease) in the risk premium. In the case of the two-year horizon, $R^2_{\text{full}}$ equals 0.27 and $R^2_{\text{out}}$ is 0.09. Both statistics are similar to the ones for the 18-month horizon, which are 0.21 and 0.07, respectively.

The regressions in Table VI imply that the standard deviation of annualized two-year expected

---

27 We calculate the following expression for RREL: $RREL_t = \text{TBILL}_t - \sum_{r=1}^{12} \text{TBILL}_{t-r}$.

For example the TBILL moving average for January 1990 is calculated using the average TBILL between January 1989 and December 1989.

28 See http://faculty.haas.berkeley.edu/lettau/data_cay.html. This version of the data spans 1951Q4-2010Q2.
excess returns is about 7%, which is large compared to the average return, which was around 1% during this period. It is slightly larger than the standard deviation of expected returns (5.46) reported by Cochrane (2011) in his one-year forecasting regressions of CRSP-value weighted excess returns using the dividend-price ratio as a predictor for the period 1947-2009. As Cochrane points out, these standard deviations are large and are similar in value to the average equity premium. For example, Mehra and Prescott (1985) initiated the equity premium puzzle literature with an estimated equity premium of 6%.

Comparing the $R^2_{full}$ in Panel A with that in Panel B across horizons leads to the conclusion that evaluating the risk-neutral return distribution at the conditional quantiles from the MQ-CAViaR model as opposed to an arbitrary point on the return distribution leads to superior predictive performance as measured by both $R^2_{full}$ and $R^2_{out}$. For example, the $R^2_{full}$ in Panel A is larger than that in Panel B for five of the six forecasting horizons. Additionally, in the case of horizons exceeding 6 months, the $R^2_{full}$ in Panel A is at least twice as large as its analog in Panel B. For example, the $R^2_{full}$ is 0.27 in Panel A and only 0.13 in Panel B for the 2-year horizon. Overall, the model in Panel B has very poor out-of-sample performance generating negative $R^2_{out}$ values for all six horizons considered. The same model generates expected returns with standard deviations ranging from 0.12% for the one-month horizon to 5% for the two-year horizon.

Motivated by Examples 1 and 2 in Section 2, we report in Panel C of Table VI regression results using the SPOCQD and SPOCQS series. The $R^2_{full}$ in this panel is almost identical to that of Panel A for all six horizons. At the same time, there is a significant improvement in the $R^2_{out}$ for horizons exceeding one year moving from Panel A to Panel C: 0.07 to 0.14 (18 months), and 0.09 to 0.13 (two years). Moreover, the coefficient on SPOCQS, which captures volatility aversion, is small and insignificant at all horizons. Thus, we find no evidence that our measure of volatility aversion is priced in the S&P 500. This result may arise because the diversification inherent in the S&P 500 smooths out the skewness in individual stock returns. Harvey and Siddique (2000) and Conrad, Dittmar, and Ghysels (2009) find idiosyncratic skewness to be important in pricing the cross-section of stock returns.

At longer horizons, our finding regarding the lack of forecasting ability for the volatility aversion term SPOCQS is consistent with the results in Bollerslev et al. for the volatility premium. Unlike Bollerslev et al., we see no evidence regarding the forecasting ability of our measure of volatility aversion in shorter horizons. SPOCQS prices a security that pays out in the event that returns fall in the lower or upper quartile of the conditional return distribution. As we show in (11) and (15), the price of this security is a function of both the level of volatility and preferences regarding volatility. In contrast, the volatility premium of Bollerslev et al. isolates the risk premium on a

---

29 An immediate observation from Table VI is that the predictive accuracy of the models increases with the forecasting horizon. This feature should be treated with caution. Boudoukh, Richardson, and Whitelaw (2008) show that $R^2$ in forecasting regressions with highly persistent predictors and overlapping returns increase roughly proportional with the return horizon and the length of the overlap even in the absence of true predictability.
variance swap, which, according to their model, is driven by one component of volatility aversion, i.e., volatility of volatility.

The SPOCQD coefficient is significant and positive in Panel C for the 18-month and two-year horizons. The same coefficient is positive and borderline significant at the one-year horizon. Therefore, the predictive ability of SPOCQ stems entirely from the relative risk aversion term, as represented in Figure 1, which shows that SPOCQD varies widely and strongly predicts returns at the two-year horizon. At the average value of our measure of return dispersion, which is 4.68, a SPOCQD coefficient of 0.4 implies that a 5-cent shift from SPOCQ(75,100) to SPOCQ(0,25) raises annualized expected returns by the substantial amount of 9.3%.\(^{30}\) Figures 1 and 9 show that such WTP shifts from the lowest- to the highest-quartile state occurred multiple times in our sample period, most notably in 1991, 1998, 2002-03, and 2007. As we show in the next section, SPOCQD remains strongly significant even when we include predictors previously used in the literature in multivariate specifications of our forecasting regressions.

4.3 Forecasting using SPOCQ and other predictors

In this section, we investigate the extent to which predictors previously used in the literature can account for the option implied risk premia revealed by SPOCQ. Table VII presents a set of univariate regressions at the one-year horizon, and Table VIII contains the results from the same univariate regressions for the two-year horizon. Looking at the univariate models for the one-year horizon, only the stochastically detrended 3-month T-bill rate (RREL), and the consumption-wealth ratio (CAY) outperform SPOCQD in terms of their \(R^2_{\text{full}}\). The \(R^2_{\text{full}}\) for both of these predictors is 0.16, while the \(R^2_{\text{full}}\) for SPOCQD is 0.09. The slope coefficient is statistically significant at the 5% level in the CAY and SPOCQD models but not in the RREL model. Our SPOCQD series generates an \(R^2_{\text{out}}\) equal to 0.07, while the CAY and RREL series generate an \(R^2_{\text{out}}\) equal to 0.01 and 0.04, respectively. The standard deviation of the expected returns for the three models ranges between 5.45% (SPOCQD) and 7.14% (CAY). Using the average value of our measure of return dispersion, 4.68, a SPOCQD coefficient of 0.30 implies that a 5-cent shift from SPOCQ(75,100) to SPOCQ(0,25) raises annualized expected returns by 7%.

For the same univariate regressions at the two-year horizon, we see in Table VIII a notable increase in predictive performance for CAY, the log price-dividend ratio (PD), the term spread between the 10-year T-Bond and the 3-month T-bill rates (TMSP), as well as for our SPOCQD series. As would be expected based on Boudoukh et. al., the \(R^2_{\text{full}}\) for the two-year horizon is almost double its one-year analog for three out of the four models: CAY, 0.32 vs. 0.16, PD 0.18 vs. 0.08, and SPOCQD, 0.26 vs. 0.11. At the same time, the \(R^2_{\text{full}}\) for TMSP rises to 0.13 from essentially zero. However, the only two models with slope coefficients that are statistically significant at the 5% level are CAY and SPOCQD. The empirical support for pricing of relative

\(^{30}\)The implied increase in expected returns is given by 0.4 \times 0.05 \times 4.68 = 0.093.
risk aversion as measured by SPOCQD remains, which is consistent with our findings in the previous section. The SPOCQD coefficient is very similar to its one-year counterpart, namely 0.38. The next largest $R^2_{full}$ arises in the PD model slope, but the slope coefficient in this model lacks statistical significance. All the remaining models have $R^2_{full}$ that does not exceed 0.13 and are characterized by slope coefficients that are statistically indistinguishable from zero. The $R^2_{out}$ for SPOCQD, 0.17, is almost double the one for CAY, 0.09, while the $R^2_{out}$ for PD is negative. The standard deviation of annualized expected returns equals 7.01% for SPOCQD and 7.58% for CAY.

Next, we add various combinations of the other predictors with SPOCQD to investigate whether they capture similar features of risk-premium variation. If these predictors capture components of the risk premium represented by SPOCQD, then we would expect them to be statistically significant and to reduce the value of the SPOCQD coefficient. We include CAY, PD, PE, RREL, and TMSP because these are the predictors that achieved an $R^2_{full}$ greater than 0.05 in the univariate regressions for either the one- or the two-year horizon. Tables IX and X report regression results based on these multivariate specifications for the one-year and two-year horizons, respectively.

In spite of some relatively large $R^2_{full}$ values, all the multivariate models for the one-year horizon have worse out-of-sample performance than the univariate SPOCQD model, with four out of nine of them generating negative $R^2_{out}$ values. CAY is statistically significant at 10% level in all the specifications, and RREL is significant at 10% except in the specification that includes all six predictors. The SPOCQD coefficient, which equals 0.30 in the univariate SPOCQD model, reduces to 0.19 when CAY is added to the model, 0.27 when RREL is added and 0.08 when RREL and CAY are added along with TMSP. The two-year horizon regression results show a quite impressive fit for several models in terms of their $R^2_{full}$. Notably, the richest specification that includes SPOCQD and all the other predictors has an $R^2_{full}$ equal to 0.53, while the bivariate specification based on SPOCQD and CAY generates an $R^2_{full}$ of 0.44 both of which than 0.27, which is the $R^2_{full}$ achieved by the univariate SPOCQD specification. However, none of the multivariate specifications has an out-of-sample forecasting performance that is superior to the univariate SPOCQD, and four of them have negative $R^2_{out}$. Moreover, although SPOCQD is significant at 5% level for seven out of nine specifications considered, CAY is the only other predictor that is statistically significant at the 10% level in any specification. The coefficient on SPOCQD reduces from 0.39 in the univariate specification to 0.28 in the bivariate specification that includes CAY to 0.18 in the multivariate specification that includes CAY, RREL, and TMSP. In that latter case, neither RREL or TMSP are significant at the 10% level. Thus, the regressions in Tables IX and VIII indicate weakly that some of the variation in expected returns related to SPOCQD emanates from fluctuations in the consumption-wealth ratio, and possibly short-term interest rates and the term spread.

To investigate further how these variables relate to SPOCQD, we present a temperature plot in Figure 11. In addition to the predictors from Tables IX and VIII and the subsequent 24-month
S&P 500 excess return, we also include 4-quarter backward-looking real GDP growth and a binary variable that tracks the NBER-defined recessions. The temperature plot utilizes dark (light) colors to indicate the negative (positive) values of a variable, with the shading becoming increasingly darker (lighter) as the values become more negative (positive). All variables with the exception of the binary variable that tracks the NBER recessions are standardized to have mean zero and variance one, and the shading is assigned such that any values less than -1 are indicated by black and any values that exceed 1 are indicated by white. The binary variable that tracks the NBER recessions takes the value -1 for the recessionary quarters and 1 otherwise. We plot the negative of the log PD and PE ratios because PD and PE are thought to be negatively related to expected returns. Although the series for the returns and our predictors are recorded on a monthly basis, the temperature plot tracks their values in the first month of each quarter. As we plot in Figure 1, and as is revealed in the regressions, low SPOCQD values correlate with low subsequent returns and high SPOCQD values correlate with high subsequent returns. Here, we are most interested in how the other variables match SPOCQD.

Figure 11 shows that CAY matches the movements of returns and SPOCQD quite well in the period 1994-2002 and in 2007-08. Thus, the 1990s boom and subsequent bust fits the notion, because investors desire to smooth consumption, they exhibit an increased willingness to save when wealth increases. It follows that low values of CAY imply high willingness to save and therefore expected returns. This argument does not fit well in the early 1990s, when CAY was high relative to expected returns and in the mid 2000s when CAY was low relative to expected returns. The PD and PE ratios also match broadly the patterns in SPOCQD and returns in the middle of the sample, but match less well in the first and last few years. In short, SPOCQ and subsequent returns are more closely related to the business cycle than are CAY, PD, and PE.

The short-term interest rate (RREL) tends to be a lagging indicator, reflecting the fact that the Federal Reserve tends to raise short term interest rates when the economy is booming and lower them during recessions. This pattern is further revealed by the correspondence between GDP growth, NBER-defined recessionary periods and RREL. A steep yield curve (i.e., high TMSP) corresponds well to the periods when expected returns were highest (1992-94, 2002-2003), but for both downturns TMSP declines long before SPOCQD and returns. Thus, TMSP was an asymmetric indicator during this period; high risk aversion through SPOCQD tended to return soon after the end of a recession when the yield curve was steepest, but the yield curve flattened a few years before expected returns declined. In sum, Figure 11 suggests that, during the 19-year period from 1990-2008, expected returns were largest just after recessions when the yield curve was steepest. This observation matches the results of Fama and French (1989), who first demonstrated the acyclical nature of expected returns. Although CAY captures some of these business cycle features, there remains scope for asset pricing models explore this connection further. Lochstoer (2009) provides one useful step in this direction by showing that the conditional volatility of luxury good consumption moves with the business cycle and has some predictive power for stock returns.
5 Conclusion

We track the option-implied stochastic discount factor for S&P returns between 1990 and the Fall of 2008. We develop a new series of statistics, the State Prices of Conditional Quantiles (SPOCQ), which reflect the market’s willingness to pay for insurance against outcomes across various quantiles of the return distribution. We construct the SPOCQ series by evaluating the risk-neutral return distribution at the conditional quantiles of the physical return distribution using a multivariate CAViaR model. We show that SPOCQs relate directly to asset returns via two channels. The first channel is volatility aversion – the market’s willingness to pay to avoid outcomes in the tails of the return distribution. The second channel is relative risk aversion; the market places greater marginal value on a dollar in low-return states of the world relative to high-return states.

We find that SPOCQ has strong predictive power for returns at the one-to-two year horizon. It explains 27% of the total return variation at the two-year horizon implying an annualized standard deviation of expected returns equal to 7%. In particular, our regressions imply that a 5-cent shift from the highest- to the lowest-quartile state price raises expected returns by 9.3%. This predictive power emanates from the relative risk aversion component; we find no evidence that volatility aversion predicts S&P 500 returns. It remains an open question whether such predictive power would exist for individual stocks, which may exhibit greater skewness than the index. The predictive power of SPOCQ remains strong even in the presence of other regressors that have been offered in previous literature, such as the price-dividend (PD), the price-earnings (PE) and the consumption-wealth (CAY) ratios. Overall, our findings are consistent with the market applying deep discounts to large negative returns during the recessionary periods of 1990-91, 1998-2003, and 2008, and suggest a renewed emphasis on the relationship between the business cycle and expected asset returns.
Appendix A: SPOCQ for Three-Moment Conditional CAPM

We derive SPOCQ for the three-moment conditional CAPM of Harvey and Siddique (2000) as presented in Example 1 in Section 2. The standardized SDF is

\[ M_{t+1}^* = 1 - \beta_t (R_{t+1} - \mu_t) + \lambda_t ((R_{t+1} - \mu_t)^2 - \sigma_t^2) \]  

(51)

where \( R_t \sim N(\mu_t, \sigma_t^2) \). Standard formulae for the mean and variance of a truncated normal distribution along with the fact that \( q_t(0.25) = -0.674 \) imply that

\[ E_t [R_{t+1}|r_{t+1} < q_t(0.25)] = \mu_t - \sigma_t \frac{\phi(0.674)}{0.25} \]  

(52)

\[ E_t [(R_{t+1} - \mu_t)^2 |r_{t+1} < q_t(0.25)] = \text{var}_t [R_{t+1}|r_{t+1} < q_t(0.25)] + (E_t [R_{t+1}|r_{t+1} < q_t(0.25)] - \mu_t)^2 \]  

(53)

\[ E_t [(R_{t+1} - \mu_t)^2 |r_{t+1} < q_t(0.25)] = \sigma_t^2 \left( 1 - \frac{0.674 \phi(0.674)}{0.25} - \frac{\phi(0.674)^2}{0.25^2} \right) + \sigma_t^2 \frac{\phi(0.674)^2}{0.25^2} \]  

(54)

Applying the same formulae to \( q_t(0.75) \) yields the standardized SDF for the three segments

\[ E_t [M_{t,t+1}^*|r_{t+1} < q_t(0.25)] = 1 + \beta_t \sigma_t \frac{\phi(0.674)}{0.25} + \lambda_t \sigma_t^2 \frac{0.674 \phi(0.674)}{0.25} \]  

(55)

\[ E_t [M_{t,t+1}^*|q_t(0.25) < r_{t+1} < q_t(0.75)] = 1 - 2\lambda_t \sigma_t^2 \frac{0.674 \phi(0.674)}{0.5} \]  

(56)

\[ E_t [M_{t,t+1}^*|r_{t+1} > q_t(0.75)] = 1 - \beta_t \sigma_t \frac{\phi(0.674)}{0.25} + \lambda_t \sigma_t^2 \frac{0.674 \phi(0.674)}{0.25} \]  

(57)

Using the symmetry of the normal distribution, \( \phi(-0.674) = \phi(0.674) = 0.318 \), it follows from (8) that

\[ SPOCQ_t(0.25) = 0.25 + 0.318 \beta_t \sigma_t + 0.214 \lambda_t \sigma_t^2 \]  

\[ SPOCQ_t(25, 75) = 0.5 - 0.428 \lambda_t \sigma_t^2 \]  

(58)

\[ SPOCQ_t(75, 100) = 0.25 - 0.318 \beta_t \sigma_t + 0.214 \lambda_t \sigma_t^2 \]
Appendix B: SPOCQ for Representative-Agent Models with Epstein-Zin Preferences

We derive SPOCQ for the Epstein-Zin model presented in Example 2 in Section 2. Recursive preferences of the form in Epstein and Zin (1989) and Weil (1989) imply that the logarithm of the SDF is

$$m_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1}$$

(59)

where $g_{t+1}$ denotes the log growth rate of aggregate consumption, $r_{a,t+1}$ is the log return on an asset that pays aggregate consumption as its dividend, and $\theta = (1 - \gamma)(1 - 1/\psi)^{-1}$. The three preference parameters are the time discount factor $\delta$, the intertemporal elasticity of substitution $\psi$, and the coefficient of risk aversion $\gamma$. When $\theta = 1$ the model collapses to the power utility specification of, for example, Mehra and Prescott (1985).

Assuming that $g_{t+1}$ is conditionally normally distributed, applying the Campbell and Shiller (1988a) approximation, and solving the model implies that log returns are conditionally normally distributed (e.g., Bansal and Yaron). Standard formulas for linear projections imply that

$$E_t [m_{t+1}|r_{t+1}] = E_t [m_{t+1}] + \frac{\sigma_{mr}^t}{(\sigma_t^r)^2} (r_{t+1} - E_t [r_{t+1}])$$

(60)

$$\text{var} [m_{t+1}|r_{t+1}] = \text{var} [m_{t+1}] - \frac{\sigma_{mr}^t}{(\sigma_t^r)^2},$$

(61)

where $\sigma_{mr}^t \equiv \text{cov}_t [m_{t+1}, r_{t+1}]$, $\sigma_t^r \equiv (\text{var}_t [r_{t+1}])^{1/2}$, and $r_{t+1}$ denotes the log return on an asset in the economy. It follows from the formula for the mean of a lognormal random variable that the standardized SDF is

$$M_{t,t+1}^* = \frac{E_t [M_{t,t+1}|r_{t+1}]}{E_t [M_{t,t+1}]} = \exp \left( \frac{\sigma_{mr}^t}{(\sigma_t^r)^2} (r_{t+1} - E_t [r_{t+1}]) - 0.5 \left( \frac{\sigma_{mr}^t}{\sigma_t^r} \right)^2 \right).$$

(62)

The formulae for the truncated mean of a lognormal random variable imply

$$E_t [M_{t,t+1}^*|r_{t+1} > q_t(\theta_j)] = \frac{E_t [M_{t,t+1}^*]}{1 - \theta_j} \Phi \left( -\frac{\sigma_{mr}^t}{\sigma_t^r} q_t(\theta_j) + \frac{\sigma_{mr}^t}{\sigma_t^r} E_t [r_{t+1}] + \left( \frac{\sigma_{mr}^t}{\sigma_t^r} \right)^2 \right)$$

$$= \frac{1}{1 - \theta_j} \Phi \left( \frac{\sigma_{mr}^t}{\sigma_t^r} - \Phi^{-1} (\theta_j) \right),$$

(63)

where we use the fact that $E_t [M_{t,t+1}^*] = 1$ and $\theta_j \equiv \Phi \left( (q_t(\theta_j) - E_t [r_{t+1}]) / \sigma_t^r \right)$, which implies $q_t(\theta_j) = \Phi^{-1} (\theta_j) \sigma_t^r + E_t [r_{t+1}]$. Now, using $\Phi^{-1} (0.25) = -0.674$ and $\Phi^{-1} (0.75) = 0.674$, it follows

35
that SPOCQ for the three segments is

\[
SPOCQ_t(0, 25) = \Phi \left(-\frac{\sigma_{mr}}{\sigma^r_t} - 0.674 \right)
\]

\[
SPOCQ_t(25, 75) = \Phi \left(\frac{\sigma_{mr}}{\sigma^r_t} + 0.674 \right) - \Phi \left(\frac{\sigma_{mr}}{\sigma^r_t} - 0.674 \right)
\]

\[
SPOCQ_t(75, 100) = \Phi \left(\frac{\sigma_{mr}}{\sigma^r_t} - 0.674 \right)
\]

A second-order Taylor expansion of \(\Phi(z)\) around the point \(z = 0.674\) implies

\[
\Phi \left(\frac{\sigma_{mr}}{\sigma^r_t} + 0.674 \right) \approx \Phi (0.674) + \phi (0.674) \frac{\sigma_{mr}}{\sigma^r_t} - \frac{1}{2} (0.674) \phi (0.674) \left(\frac{\sigma_{mr}}{\sigma^r_t} \right)^2
\]

\[
= 0.75 + 0.318 \frac{\sigma_{mr}}{\sigma^r_t} - 0.107 \left(\frac{\sigma_{mr}}{\sigma^r_t} \right)^2
\]

It follows that

\[
SPOCQ_t(0, 25) - SPOCQ_t(75, 100) = \Phi \left(-\frac{\sigma_{mr}}{\sigma^r_t} - 0.674 \right) - \Phi \left(\frac{\sigma_{mr}}{\sigma^r_t} - 0.674 \right)
\]

\[
\approx 0.636 \frac{\sigma_{mr}}{\sigma^r_t}
\]

\[
SPOCQ_t(0, 25) + SPOCQ_t(75, 100) = \Phi \left(-\frac{\sigma_{mr}}{\sigma^r_t} - 0.674 \right) + \Phi \left(\frac{\sigma_{mr}}{\sigma^r_t} - 0.674 \right)
\]

\[
\approx 0.5 + 0.214 \left(\frac{\sigma_{mr}}{\sigma^r_t} \right)^2
\]
References


Wei, M., and J. Wright, 2010, Reverse regressions and long-horizon forecasting, Discussion paper, mimeo.


Table I. Summary statistics for the S&P 500 returns

This table characterizes the S&P 500 returns data that we use to construct the quantiles underlying our SPOCQ statistics. Returns are defined as 100 times the log change in the S&P 500 index from the close on one Friday to the open on the following Friday (weekly) or the Friday four weeks hence (monthly). We report statistics for four overlapping monthly series, each beginning on a different Friday in January 1990. The asterisk (*) denotes significance at 5% based on a block bootstrap with block size 1 year.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Weekly</th>
<th>Monthly I</th>
<th>Monthly II</th>
<th>Monthly III</th>
<th>Monthly IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obs.</td>
<td>1,000</td>
<td>249</td>
<td>249</td>
<td>249</td>
<td>250</td>
</tr>
<tr>
<td>St. date</td>
<td>Jan-5-1990</td>
<td>Jan-12-1990</td>
<td>Jan-19-1990</td>
<td>Jan-26-1990</td>
<td>Jan-5-1990</td>
</tr>
<tr>
<td>Mean</td>
<td>0.068</td>
<td>0.394</td>
<td>0.288</td>
<td>0.291</td>
<td>0.405</td>
</tr>
<tr>
<td>Max</td>
<td>10.310</td>
<td>14.140</td>
<td>10.210</td>
<td>12.120</td>
<td>9.590</td>
</tr>
<tr>
<td>Median</td>
<td>0.255</td>
<td>1.070</td>
<td>0.860</td>
<td>0.700</td>
<td>0.930</td>
</tr>
<tr>
<td>Std. deviation</td>
<td>2.045 *</td>
<td>4.568 *</td>
<td>3.867 *</td>
<td>4.530 *</td>
<td>4.190 *</td>
</tr>
<tr>
<td>Skewness (mom.)</td>
<td>-1.421 *</td>
<td>-1.474</td>
<td>-0.613 *</td>
<td>-2.061 *</td>
<td>-1.630 *</td>
</tr>
<tr>
<td>Ex. Kurtosis (mom.)</td>
<td>12.198 *</td>
<td>8.630 *</td>
<td>1.040 *</td>
<td>12.103 *</td>
<td>8.675 *</td>
</tr>
<tr>
<td>Skewness (quant.)</td>
<td>-0.132 *</td>
<td>-0.227</td>
<td>-0.162 *</td>
<td>-0.052</td>
<td>-0.191</td>
</tr>
<tr>
<td>Ex. Kurtosis (quant.)</td>
<td>0.975 *</td>
<td>1.379 *</td>
<td>0.907 *</td>
<td>1.404 *</td>
<td>0.699 *</td>
</tr>
<tr>
<td>2.5% quantile</td>
<td>-4.345 *</td>
<td>-9.180 *</td>
<td>-8.550 *</td>
<td>-9.700 *</td>
<td>-8.710 *</td>
</tr>
<tr>
<td>25% quantile</td>
<td>-0.900 *</td>
<td>-1.500 *</td>
<td>-1.620 *</td>
<td>-1.410 *</td>
<td>-1.780 *</td>
</tr>
<tr>
<td>50% quantile</td>
<td>0.255 *</td>
<td>1.070 *</td>
<td>0.860 *</td>
<td>0.700 *</td>
<td>0.930 *</td>
</tr>
<tr>
<td>75% quantile</td>
<td>1.140 *</td>
<td>2.690 *</td>
<td>2.650 *</td>
<td>2.600 *</td>
<td>2.770 *</td>
</tr>
<tr>
<td>97.5% quantile</td>
<td>3.580 *</td>
<td>8.790 *</td>
<td>7.750 *</td>
<td>7.600 *</td>
<td>7.710 *</td>
</tr>
<tr>
<td>Ann. mean returns</td>
<td>3.536</td>
<td>5.122</td>
<td>3.744</td>
<td>3.783</td>
<td>5.265</td>
</tr>
<tr>
<td>Returns AR(1)</td>
<td>-0.095</td>
<td>0.055</td>
<td>0.080</td>
<td>-0.092</td>
<td>0.071</td>
</tr>
<tr>
<td>Sq. returns AR(1)</td>
<td>0.283</td>
<td>0.225</td>
<td>0.332</td>
<td>0.049</td>
<td>0.048</td>
</tr>
</tbody>
</table>
This table summarizes our options data, which consist of price quotes at the market close on Fridays that occur one week (weekly) or four weeks (monthly) before expiration. We have data for each contract expiration between January 1990 and November 2008. The columns under the heading “Strike Prices” show the minimum, mean, and maximum observed prices in our sample. The columns under “Average Prices” show mean bid and ask prices and the mean of the midpoint between the bid and ask prices.

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Postion</th>
<th>Number of contracts</th>
<th>Strike Prices</th>
<th>Average Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>min</td>
<td>mean</td>
</tr>
<tr>
<td>Weekly</td>
<td>Call</td>
<td>227</td>
<td>200</td>
<td>955</td>
</tr>
<tr>
<td></td>
<td>Put</td>
<td>227</td>
<td>275</td>
<td>1,104</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>227</td>
<td>200</td>
<td>1,001</td>
</tr>
<tr>
<td>Monthly</td>
<td>Call</td>
<td>227</td>
<td>200</td>
<td>966</td>
</tr>
<tr>
<td></td>
<td>Put</td>
<td>227</td>
<td>250</td>
<td>1,060</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>227</td>
<td>200</td>
<td>996</td>
</tr>
</tbody>
</table>
This table reports quasi-maximum likelihood parameter estimates for the MQ-CAViaR model applied to S&P 500 returns using data from January 1950-February 2009. Returns are defined as 100 times the log change in the S&P 500 index from the close on one Friday to the open on the following Friday (weekly) or the Friday four weeks hence (monthly). We report results for four overlapping monthly series, each beginning on a different Friday in January 1950. The DQ statistic allows a test of the null hypothesis of correct dynamic specification. Under the null, the DQ statistic is distributed as $\chi^2(4)$, which implies a 5% critical value of 9.487. The hit-rate statistic equals the fraction of returns that lie below the predicted conditional quantile from the model. Under correct specification, the entries in the “Hit Rate” column would match those in the “Quantile Estimation” column. The DQ and hit-rate statistics are computed over the period beginning in January 1990 to match the span of the options data.
### Table IV. Statistics for the CNLS fit of adjusted call and put option prices

This table summarizes the results of the curve-fitting exercises we perform to estimate the risk-neutral distribution on each of the dates for which we summarize prices in Table II. We show the minimum, median, and maximum estimates of the scale parameter for the logistic fit (single) and the mixture of logistics (mix). For the mixture, we report only the scale parameter for the highest-weighted component of the mixture (weights shown in the “Weight Mixture” column). The last two columns summarize the fit of the single and mix models using the standard $R^2$.

#### Panel A (Weekly)

<table>
<thead>
<tr>
<th>Scale Parameter</th>
<th>Weight</th>
<th>R-squared</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Single</td>
<td>Mixture</td>
</tr>
<tr>
<td>Min</td>
<td>2.587</td>
<td>1.107</td>
</tr>
<tr>
<td>Max</td>
<td>64.929</td>
<td>49.211</td>
</tr>
<tr>
<td>Median</td>
<td>11.298</td>
<td>9.373</td>
</tr>
</tbody>
</table>

#### Panel B (Monthly)

<table>
<thead>
<tr>
<th>Scale Parameter</th>
<th>Weight</th>
<th>R-squared</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Single</td>
<td>Mixture</td>
</tr>
<tr>
<td>Min</td>
<td>5.446</td>
<td>2.515</td>
</tr>
<tr>
<td>Max</td>
<td>101.263</td>
<td>62.520</td>
</tr>
<tr>
<td>Median</td>
<td>25.268</td>
<td>18.670</td>
</tr>
</tbody>
</table>
Table V. Summary statistics of variables in forecasting regressions

This table summarizes the variables that we use in the forecasting regressions. The relevant time period is from January 1990 to November 2008. We measure excess returns as the annualized log change in the S&P 500 index minus the three month T-Bill yield. Each return is measured over a period beginning on the date on which we calculate the monthly SPOCQ statistics (i.e., four weeks before each option expiration). The SPOCQ variables are each multiplied by the interquartile range and represent various quantiles as described in the text. The remaining predictors correspond to those in Bollerslev, Tauchen, and Zhou (2009); each observation corresponds to a calendar month. Specifically, IV is implied variance as measured by VIX, RV is realized variance, CAY is the consumption-wealth ratio of Lettau and Ludvigson (2001), DFSP is the spread between Moody’s BAA and AAA bond yield indices, PD is the log price-dividend ratio, PE is the log price-earnings ratio, RREL stochastically detrended 3-month T-bill rate, TBILL is the yield on 3-month T-Bills, and TMSP is the spread between the yields on 10-year T-Bonds and 3-month T-Bills.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. dev.</th>
<th>Skew. (mom.)</th>
<th>Ex. Kurt. (mom.)</th>
<th>Skew. (quant.)</th>
<th>Ex. Kurt. (quant.)</th>
<th>AR(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-month ex. return</td>
<td>2.280</td>
<td>53.087</td>
<td>-1.868</td>
<td>9.175</td>
<td>-0.127</td>
<td>2.216</td>
<td>0.016</td>
</tr>
<tr>
<td>3-month ex. return</td>
<td>0.058</td>
<td>29.808</td>
<td>-1.626</td>
<td>5.777</td>
<td>-0.096</td>
<td>0.817</td>
<td>0.660</td>
</tr>
<tr>
<td>6-month ex. return</td>
<td>0.400</td>
<td>22.721</td>
<td>-1.784</td>
<td>5.588</td>
<td>0.030</td>
<td>1.491</td>
<td>0.798</td>
</tr>
<tr>
<td>12-month ex. return</td>
<td>0.707</td>
<td>17.409</td>
<td>-1.305</td>
<td>1.904</td>
<td>-0.115</td>
<td>1.560</td>
<td>0.910</td>
</tr>
<tr>
<td>18-month ex. return</td>
<td>1.050</td>
<td>14.739</td>
<td>-1.133</td>
<td>1.033</td>
<td>-0.139</td>
<td>0.459</td>
<td>0.952</td>
</tr>
<tr>
<td>24-month ex. return</td>
<td>1.033</td>
<td>13.433</td>
<td>-0.977</td>
<td>0.362</td>
<td>-0.226</td>
<td>0.379</td>
<td>0.966</td>
</tr>
<tr>
<td>SPOCQ25</td>
<td>1.239</td>
<td>0.340</td>
<td>0.604</td>
<td>-0.203</td>
<td>0.046</td>
<td>-0.448</td>
<td>0.975</td>
</tr>
<tr>
<td>SPOCQ50</td>
<td>2.202</td>
<td>0.349</td>
<td>0.114</td>
<td>-0.387</td>
<td>0.124</td>
<td>-0.038</td>
<td>0.877</td>
</tr>
<tr>
<td>SPOCQ75</td>
<td>1.233</td>
<td>0.406</td>
<td>0.168</td>
<td>-1.177</td>
<td>0.169</td>
<td>-1.034</td>
<td>0.966</td>
</tr>
<tr>
<td>SPOCQD</td>
<td>-0.007</td>
<td>0.181</td>
<td>-0.422</td>
<td>-0.595</td>
<td>-0.187</td>
<td>-0.759</td>
<td>0.918</td>
</tr>
<tr>
<td>SPOCQS</td>
<td>2.474</td>
<td>0.680</td>
<td>0.399</td>
<td>-0.752</td>
<td>-0.070</td>
<td>-0.878</td>
<td>0.949</td>
</tr>
<tr>
<td>FSTAR ZERO</td>
<td>0.417</td>
<td>0.021</td>
<td>0.083</td>
<td>-0.740</td>
<td>0.014</td>
<td>-0.597</td>
<td>0.907</td>
</tr>
<tr>
<td>IV-RV</td>
<td>17.128</td>
<td>20.010</td>
<td>-3.329</td>
<td>43.450</td>
<td>0.396</td>
<td>0.759</td>
<td>0.293</td>
</tr>
<tr>
<td>IV</td>
<td>36.214</td>
<td>33.259</td>
<td>4.107</td>
<td>24.862</td>
<td>0.191</td>
<td>0.982</td>
<td>0.802</td>
</tr>
<tr>
<td>RV</td>
<td>19.085</td>
<td>38.124</td>
<td>8.849</td>
<td>96.706</td>
<td>0.283</td>
<td>2.002</td>
<td>0.622</td>
</tr>
<tr>
<td>CAY</td>
<td>0.402</td>
<td>2.118</td>
<td>-0.201</td>
<td>-1.291</td>
<td>-0.198</td>
<td>-1.219</td>
<td>0.977</td>
</tr>
<tr>
<td>DFSP</td>
<td>0.883</td>
<td>0.308</td>
<td>3.181</td>
<td>17.743</td>
<td>0.000</td>
<td>0.269</td>
<td>0.915</td>
</tr>
<tr>
<td>PD</td>
<td>3.926</td>
<td>0.327</td>
<td>-0.177</td>
<td>-1.012</td>
<td>-0.554</td>
<td>-0.777</td>
<td>0.991</td>
</tr>
<tr>
<td>PE</td>
<td>2.957</td>
<td>0.205</td>
<td>0.549</td>
<td>-0.655</td>
<td>0.259</td>
<td>-0.491</td>
<td>0.975</td>
</tr>
<tr>
<td>RREL</td>
<td>-0.180</td>
<td>0.875</td>
<td>-0.431</td>
<td>-0.195</td>
<td>-0.087</td>
<td>0.026</td>
<td>0.972</td>
</tr>
<tr>
<td>TBILL</td>
<td>4.086</td>
<td>1.784</td>
<td>-0.144</td>
<td>-0.584</td>
<td>-0.400</td>
<td>0.147</td>
<td>0.989</td>
</tr>
<tr>
<td>TMSP</td>
<td>1.615</td>
<td>1.200</td>
<td>0.061</td>
<td>-1.216</td>
<td>0.175</td>
<td>-1.014</td>
<td>0.970</td>
</tr>
</tbody>
</table>
Table VI. SPOCQ forecasting regressions over multiple horizons

This table shows the results of forecasting regressions over six horizons. We regress returns over each horizon on a set of SPOCQ variables measured at the beginning of the return interval. Panel A uses SPOCQ statistics that price the lower quartile, interquartile range, and the upper quartile of the conditional return distribution. Panel B uses the risk neutral distribution evaluated at zero, rather than at a conditional quantile. Panel C re-specifieds the model in Panel A using SPOCQD=SPOCQ25-SPOCQ75 and SPOCQS=SPOCQ25+SPOCQ75. Each SPOCQ variable is multiplied by the interquartile range before entering the regression, as implied by equation (26). Hodrick (1992) standard errors are in parentheses below the coefficient estimates, and (*) denotes significance at the 5% level. \( R^2_{full} \) denotes the conventional adjusted \( R^2 \) over the full sample and \( R^2_{out} \) denotes the \( R^2 \) from a pseudo-out-of-sample experiment in which we predict returns using an expanding estimation sample that ends just prior to the forecast interval. The row denoted std.dev(E[R]) reports the standard deviation of the fitted values from the full-sample regression.

### Panel A

<table>
<thead>
<tr>
<th>Model</th>
<th>Forecasting horizon in months</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>SPOCQ25</td>
<td>-0.207</td>
</tr>
<tr>
<td></td>
<td>(0.293)</td>
</tr>
<tr>
<td>SPOCQ50</td>
<td>0.119</td>
</tr>
<tr>
<td></td>
<td>(0.096)</td>
</tr>
<tr>
<td>SPOCQ75</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>(0.170)</td>
</tr>
<tr>
<td>( R^2_{full} )</td>
<td>0.00</td>
</tr>
<tr>
<td>( R^2_{out} )</td>
<td>-0.01</td>
</tr>
<tr>
<td>std.dev(E[R])</td>
<td>5.44%</td>
</tr>
</tbody>
</table>

### Panel B

<table>
<thead>
<tr>
<th>Model</th>
<th>Forecasting horizon in months</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>FSTAR ZERO</td>
<td>-0.057</td>
</tr>
<tr>
<td></td>
<td>(1.501)</td>
</tr>
<tr>
<td>constant</td>
<td>0.047</td>
</tr>
<tr>
<td></td>
<td>(0.625)</td>
</tr>
<tr>
<td>( R^2_{full} )</td>
<td>0.00</td>
</tr>
<tr>
<td>( R^2_{out} )</td>
<td>-0.01</td>
</tr>
<tr>
<td>std.dev(E[R])</td>
<td>0.12%</td>
</tr>
</tbody>
</table>

### Panel C

<table>
<thead>
<tr>
<th>Model</th>
<th>Forecasting horizon in months</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>SPOCQD</td>
<td>-0.040</td>
</tr>
<tr>
<td></td>
<td>(0.190)</td>
</tr>
<tr>
<td>SPOCQS</td>
<td>-0.062</td>
</tr>
<tr>
<td></td>
<td>(0.069)</td>
</tr>
<tr>
<td>constant</td>
<td>0.176</td>
</tr>
<tr>
<td></td>
<td>(0.153)</td>
</tr>
<tr>
<td>( R^2_{full} )</td>
<td>0.00</td>
</tr>
<tr>
<td>( R^2_{out} )</td>
<td>-0.01</td>
</tr>
<tr>
<td>std.dev(E[R])</td>
<td>3.97%</td>
</tr>
</tbody>
</table>
Table VII. Univariate Forecasting Regressions: 12 months

This table shows the results of forecasting regressions at the 12-month horizon. Each row reports the results from a regression of 12-month returns on a single variable. The predictor variables are as summarized in Table V and the regression results are displayed as in Table VI.

<table>
<thead>
<tr>
<th>Model</th>
<th>Slope</th>
<th>Constant</th>
<th>$R^2_{\text{full}}$</th>
<th>$R^2_{\text{out}}$</th>
<th>std.dev(E[R])</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPOCQD</td>
<td>0.301 *</td>
<td>0.009</td>
<td>0.09</td>
<td>0.07</td>
<td>5.45%</td>
</tr>
<tr>
<td></td>
<td>(0.144)</td>
<td>(0.034)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CAY</td>
<td>0.034 *</td>
<td>-0.007</td>
<td>0.16</td>
<td>0.01</td>
<td>7.14%</td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.038)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DFSP</td>
<td>-0.123</td>
<td>0.116</td>
<td>0.04</td>
<td>0.00</td>
<td>3.80%</td>
</tr>
<tr>
<td></td>
<td>(0.108)</td>
<td>(0.087)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>0.000</td>
<td>0.008</td>
<td>0.00</td>
<td>-0.04</td>
<td>0.09%</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.030)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV-RV</td>
<td>0.001</td>
<td>-0.001</td>
<td>0.00</td>
<td>-0.01</td>
<td>0.93%</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.037)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PD</td>
<td>-0.153</td>
<td>0.607</td>
<td>0.08</td>
<td>-0.12</td>
<td>4.99%</td>
</tr>
<tr>
<td></td>
<td>0.101</td>
<td>0.381</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PE</td>
<td>-0.225</td>
<td>0.673</td>
<td>0.07</td>
<td>-0.01</td>
<td>4.61%</td>
</tr>
<tr>
<td></td>
<td>(0.175)</td>
<td>(0.509)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RREL</td>
<td>0.080</td>
<td>0.021</td>
<td>0.16</td>
<td>0.04</td>
<td>6.98%</td>
</tr>
<tr>
<td></td>
<td>(0.044)</td>
<td>(0.031)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RV</td>
<td>0.000</td>
<td>0.010</td>
<td>0.00</td>
<td>-0.22</td>
<td>0.57%</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.031)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TBILL</td>
<td>0.013</td>
<td>-0.046</td>
<td>0.01</td>
<td>-0.17</td>
<td>2.33%</td>
</tr>
<tr>
<td></td>
<td>(0.018)</td>
<td>(0.089)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TMSP</td>
<td>0.013</td>
<td>-0.014</td>
<td>0.00</td>
<td>-0.13</td>
<td>1.59%</td>
</tr>
<tr>
<td></td>
<td>(0.024)</td>
<td>(0.057)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table VIII. Univariate Forecasting Regressions: 24 months

This table shows the results of forecasting regressions at the 24-month horizon. Each row reports the results from a regression of 24-month returns on a single variable. The predictor variables are as summarized in Table V and the regression results are displayed as in Table VI.

<table>
<thead>
<tr>
<th>Model</th>
<th>Slope</th>
<th>Constant</th>
<th>$R^2_{\text{full}}$</th>
<th>$R^2_{\text{out}}$</th>
<th>std.dev(E[R])</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPOCQD</td>
<td>0.388 *</td>
<td>0.013</td>
<td>0.27</td>
<td>0.17</td>
<td>7.01%</td>
</tr>
<tr>
<td></td>
<td>(0.143)</td>
<td>(0.034)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CAY</td>
<td>0.036 *</td>
<td>-0.004</td>
<td>0.32</td>
<td>0.09</td>
<td>7.58%</td>
</tr>
<tr>
<td></td>
<td>(0.016)</td>
<td>(0.038)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DFSP</td>
<td>-0.043</td>
<td>0.048</td>
<td>0.01</td>
<td>-0.14</td>
<td>1.32%</td>
</tr>
<tr>
<td></td>
<td>(0.067)</td>
<td>(0.064)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>0.000</td>
<td>0.011</td>
<td>0.00</td>
<td>-0.17</td>
<td>0.04%</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
<td>(0.033)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV-RV</td>
<td>0.000</td>
<td>0.008</td>
<td>0.00</td>
<td>-0.03</td>
<td>0.33%</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.037)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PD</td>
<td>-0.175</td>
<td>0.699</td>
<td>0.18</td>
<td>-0.56</td>
<td>5.73%</td>
</tr>
<tr>
<td></td>
<td>0.098</td>
<td>0.366</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PE</td>
<td>-0.198</td>
<td>0.595</td>
<td>0.09</td>
<td>-0.02</td>
<td>4.05%</td>
</tr>
<tr>
<td></td>
<td>(0.174)</td>
<td>(0.506)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RREL</td>
<td>0.027</td>
<td>0.015</td>
<td>0.03</td>
<td>0.03</td>
<td>2.37%</td>
</tr>
<tr>
<td></td>
<td>(0.026)</td>
<td>(0.033)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RV</td>
<td>0.000</td>
<td>0.011</td>
<td>0.00</td>
<td>-0.55</td>
<td>0.21%</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
<td>(0.033)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TBILL</td>
<td>-0.007</td>
<td>0.038</td>
<td>0.00</td>
<td>-0.11</td>
<td>1.21%</td>
</tr>
<tr>
<td></td>
<td>(0.012)</td>
<td>(0.059)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TMSP</td>
<td>0.041</td>
<td>-0.057</td>
<td>0.13</td>
<td>-0.14</td>
<td>4.97%</td>
</tr>
<tr>
<td></td>
<td>(0.022)</td>
<td>(0.062)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table IX. Multivariate Forecasting Regressions: 12 months

This table shows the results of forecasting regressions at the 12-month horizon. Each row reports the results from a regression of 12-month returns on a set of variables. The predictor variables are as summarized in Table V and the regression results are displayed as in Table VI.

<table>
<thead>
<tr>
<th>Model</th>
<th>CAY</th>
<th>PD</th>
<th>PE</th>
<th>RREL</th>
<th>TMSP</th>
<th>SPOCQD</th>
<th>Constant</th>
<th>$R^2_{full}$</th>
<th>$R^2_{out}$</th>
<th>std.dev(E[R])</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPOCQD</td>
<td>0.301 *</td>
<td>0.009</td>
<td>0.09</td>
<td>0.07</td>
<td>5.45%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.144)</td>
<td>(0.034)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD CAY</td>
<td>0.028</td>
<td>-0.108</td>
<td>0.234</td>
<td>0.433</td>
<td>0.13</td>
<td>-0.09</td>
<td>6.37%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.101)</td>
<td>(0.142)</td>
<td>(0.384)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD PD</td>
<td>-0.108</td>
<td>0.234</td>
<td>0.253</td>
<td>0.508</td>
<td>0.13</td>
<td>-0.03</td>
<td>6.39%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.101)</td>
<td>(0.141)</td>
<td>(0.176)</td>
<td>(0.514)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD PE</td>
<td>-0.169</td>
<td>0.253</td>
<td>0.265</td>
<td>0.022</td>
<td>0.23</td>
<td>0.05</td>
<td>8.46%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.176)</td>
<td>(0.141)</td>
<td>(0.176)</td>
<td>(0.514)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD RREL</td>
<td>0.074</td>
<td>0.265</td>
<td>0.286</td>
<td>0.022</td>
<td>0.23</td>
<td>0.05</td>
<td>8.46%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.044)</td>
<td>(0.140)</td>
<td>(0.176)</td>
<td>(0.514)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD TMSP</td>
<td>-0.015</td>
<td>0.354 *</td>
<td>0.034</td>
<td>0.10</td>
<td>0.00</td>
<td>5.66%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.027)</td>
<td>(0.161)</td>
<td>(0.176)</td>
<td>(0.514)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD CAY PD PE</td>
<td>0.043</td>
<td>0.199</td>
<td>0.167</td>
<td>0.150</td>
<td>0.25</td>
<td>-0.15</td>
<td>8.89%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.026)</td>
<td>(0.185)</td>
<td>(0.180)</td>
<td>(0.180)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD CAY REL TMSP</td>
<td>0.034 *</td>
<td>0.093</td>
<td>0.089</td>
<td>-0.005</td>
<td>0.38</td>
<td>0.05</td>
<td>10.91%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.017)</td>
<td>(0.051)</td>
<td>(0.030)</td>
<td>(0.156)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD CAY PD PE REL TMSP</td>
<td>0.042</td>
<td>0.121</td>
<td>0.088</td>
<td>0.023</td>
<td>0.037</td>
<td>0.051</td>
<td>0.39</td>
<td>-0.35</td>
<td>11.08%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.025)</td>
<td>(0.195)</td>
<td>(0.216)</td>
<td>(0.054)</td>
<td>(0.032)</td>
<td>(0.137)</td>
<td>(0.698)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table X. Multivariate Forecasting Regressions: 24 months

This table shows the results of forecasting regressions at the 24-month horizon. Each row reports the results from a regression of 24-month returns on a set of variables. The predictor variables are as summarized in Table V and the regression results are displayed as in Table VI.

<table>
<thead>
<tr>
<th>Model</th>
<th>CAY</th>
<th>PD</th>
<th>PE</th>
<th>RREL</th>
<th>TMSP</th>
<th>SPOCQD</th>
<th>Constant</th>
<th>$R^2_{full}$</th>
<th>$R^2_{out}$</th>
<th>std.dev(E[R])</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPOCQD</td>
<td>0.388*</td>
<td>0.013</td>
<td>0.27</td>
<td>0.17</td>
<td>7.01%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.143)</td>
<td>(0.034)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD CAY</td>
<td>0.028</td>
<td>0.279*</td>
<td>0.001</td>
<td>0.44</td>
<td>0.14</td>
<td>8.94%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.131)</td>
<td>(0.037)</td>
<td>(0.037)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD PD</td>
<td>-0.115</td>
<td>0.316*</td>
<td>0.463</td>
<td>0.33</td>
<td>-0.32</td>
<td>7.84%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.090)</td>
<td>(0.119)</td>
<td>(0.337)</td>
<td>(0.337)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD PE</td>
<td>-0.119</td>
<td>0.354*</td>
<td>0.365</td>
<td>0.30</td>
<td>0.06</td>
<td>7.39%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.169)</td>
<td>(0.132)</td>
<td>(0.496)</td>
<td>(0.496)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD RREL</td>
<td>0.019</td>
<td>0.378*</td>
<td>0.016</td>
<td>0.28</td>
<td>0.11</td>
<td>7.20%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.025)</td>
<td>(0.142)</td>
<td>(0.033)</td>
<td>(0.033)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD TMSP</td>
<td>0.015</td>
<td>0.336*</td>
<td>-0.011</td>
<td>0.28</td>
<td>0.01</td>
<td>7.16%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.022)</td>
<td>(0.140)</td>
<td>(0.058)</td>
<td>(0.058)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD CAY PD PE</td>
<td>0.035</td>
<td>0.101</td>
<td>-0.178</td>
<td>0.263*</td>
<td>0.129</td>
<td>0.46</td>
<td>-0.17</td>
<td>9.24%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.024)</td>
<td>(0.147)</td>
<td>(0.145)</td>
<td>(0.145)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD CAY RREL TMSP</td>
<td>0.029</td>
<td>0.042</td>
<td>0.021</td>
<td>0.179</td>
<td>-0.027</td>
<td>0.49</td>
<td>-0.08</td>
<td>9.52%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.016)</td>
<td>(0.034)</td>
<td>(0.025)</td>
<td>(0.147)</td>
<td>(0.065)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SPOCQD CAY PD PE RREL TMSP</td>
<td>0.040</td>
<td>0.175</td>
<td>-0.269</td>
<td>0.035</td>
<td>0.040</td>
<td>0.107</td>
<td>0.047</td>
<td>0.53</td>
<td>-0.40</td>
<td>9.91%</td>
</tr>
<tr>
<td></td>
<td>(0.024)</td>
<td>(0.163)</td>
<td>(0.177)</td>
<td>(0.036)</td>
<td>(0.027)</td>
<td>(0.106)</td>
<td>(0.683)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 1. SPX 500 SPOCQ and subsequent excess returns. We calculate SPOCQD once each month from January 1990 to November 2008 on the Friday four weeks before the next option expiration. SPOCQD equals the conditional interquartile range multiplied by a SPOCQ term. The SPOCQ term equals the difference between the price of an Arrow-Debreu security that pays one dollar in the event that the return falls in the bottom quartile of the return distribution, SPOCQ(0,25), and the price of an Arrow-Debreu security that pays one dollar in the event that the return falls in the upper quartile, SPOCQ(75,100). The subsequent excess returns series is the annualized log change in the S&P 500 from the date on which SPOCQD is measured until the date 96 weeks hence minus the 3-month T-Bill yield.
Figure 2. Interpretation of SPOCQs. We illustrate how we obtain SPOCQ at a particular percentile \((\theta_j)\) by first inverting the physical return distribution to find the quantile \(q_t(\theta_j)\) and then evaluating the risk-neutral distribution at \(q_t(\theta_j)\).
Figure 3. Daily S&P 500 closing values for 1990-2008.
Figure 4. Weekly MQ-CAViaR conditional quantiles. Constructed from predicted values of the weekly CAViaR model for which we report results in Table III.
Figure 5. Monthly MQ-CAViaR conditional quantiles. Constructed from predicted values of the monthly I CAViaR model for which we report results in Table III.
Figure 6. Weekly Constrained-NLS fit of adjusted put and call options prices. The top panel shows the put option prices on the contracts traded on November 14, 2008 for expiration on November 22, 2008. Each put price is adjusted (discounted) by the risk-free rate, for which we use LIBOR. The bottom panel shows parity-adjusted call prices (i.e., the discounted put option prices implied by observed call options). On each panel, we plot using a dark line the fit achieved by the logistic mixture distribution.
Figure 7. Monthly Constrained-NLS fit of adjusted put and call options prices. The top panel shows the put option prices on the contracts traded on October 24, 2008 for expiration on November 22, 2008. Each put price is adjusted (discounted) by the risk-free rate, for which we use LIBOR. The bottom panel shows parity-adjusted call prices (i.e., the discounted put option prices implied by observed call options). On each panel, we plot using a dark line the fit achieved by the logistic mixture distribution.
Figure 8. Weekly SPOCQs. The two SPOCQ series represent estimated price of an Arrow-Debreu security that pays one dollar in the event that the one-week return falls in the bottom quartile of the return distribution, SPOCQ(0.25), and in the event that the return falls below the upper quartile, SPOCQ(0.75). We generate SPOCQ(0.25) and SPOCQ(0.75) on a single day each month, the Friday one week before that month’s option expiration date. We smooth each series with a trailing four-month moving average. Shaded gray areas indicate NBER-defined recessions. Vertical lines indicate notable volatility events – including events highlighted in Bloom (2009), Alan Greenspan’s famous “irrational exuberance” speech, the peak of dotcom boom, and the peak of the U.S. housing market.
Figure 9. Monthly SPOCQs. The two SPOCQ series represent estimated price of an Arrow-Debreu security that pays one dollar in the event that the four-week return falls in the bottom quartile of the return distribution, SPOCQ(0.25), and in the event that the return falls below the upper quartile, SPOCQ(0.75). We generate SPOCQ(0.25) and SPOCQ(0.75) on a single day each month, the Friday four weeks before the next month’s option expiration date. We smooth each series with a trailing four-month moving average. Shaded gray areas indicate NBER-defined recessions. Vertical lines indicate notable volatility events – including events highlighted in Bloom (2009), Alan Greenspan’s famous “irrational exuberance” speech, the peak of dotcom boom, and the peak of the U.S. housing market.
Figure 10. Monthly SPOCQs and volatility premium from Bollerslev et al. (2009). The SPOCQ series represents the estimated price of an Arrow-Debreu security that pays one dollar in the event that the four-week return falls in the bottom or top quartile of the return distribution, i.e., $\text{SPOCQ}(0.25) + \text{SPOCQ}(0.75)$. We generate $\text{SPOCQ}(0.25)$ and $\text{SPOCQ}(0.75)$ on a single day each month, the Friday four weeks before the next month’s option expiration date. The dark line is taken directly from Bollerslev, Tauchen, and Zhou (2009) and equals the square root of implied variance minus the square root of realized variance over the calendar month. We smooth both series with a trailing four-month moving average. Shaded gray areas indicate NBER-defined recessions. Vertical lines indicate notable volatility events— including events highlighted in Bloom (2009), Alan Greenspan’s famous “irrational exuberance” speech, the peak of dotcom boom, and the peak of the U.S. housing market.
Figure 11. **Temperature Plot: Returns, GDP Growth, SPOCQ and other predictors.** Dark (light) colors indicate the negative (positive) values of a variable, with the shading becoming increasingly darker (lighter) as the values become more negative (positive). All variables with the exception of the binary variable that tracks the NBER recessions are standardized to have mean zero and variance one, and the shading is assigned such that any values less than -1 are indicated by black and any values that exceed 1 are indicated by white. The binary variable that tracks the NBER recessions takes the value -1 for the recessionary quarters and 1 otherwise. We plot the negative of the log PD and PE ratios because PD and PE are thought to be negatively related to expected returns. Although the series for the returns and our predictors are recorded on a monthly basis, the temperature plot tracks their values in the first month of each quarter.