

# Provision of a public good with altruistic overlapping generations and many tribes\*

Larry Karp<sup>†</sup>

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## Abstract

I model the interaction and compare the relative importance of contemporaneous cooperation and intergenerational altruism in determining equilibrium climate policy. For this purpose, I imbed an OLG framework with intergenerational altruism into a differential game between nations. There is an isomorphism between pure and paternalistic altruism here. The game has a unique “limit equilibrium” (as the time horizon of a finite horizon game goes to infinity) and many other kinds of equilibria. In an analytic model of climate change, the limit equilibrium is more sensitive to contemporaneous cooperation than to intergenerational altruism. This comparison can be overturned in other equilibria.

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<sup>†</sup>Department of Agricultural and Resource Economics, University of California, Berkeley, and the Ragnar Frisch Center for Economic Research, email: karp@berkeley.edu

# 1 Introduction

Nations must be able to cooperate and people must value their descendants, to willingly invest in long-lived public goods such as climate protection. People's impatience for their own future utility, and their altruism toward their successors, are distinct categories. An overlapping generations model disentangles these two categories, making it possible to consider different levels of altruism, holding fixed the pure rate of time preference for own utility. The differential game captures contemporaneous rivalry amongst nations. By imbedding an OLG model in a differential game, I can examine how intergenerational altruism and international cooperation interact in determining the equilibrium provision of a global public good.

Two decades of negotiation show how hard it is for policymakers to cooperate on climate policy. A large game theoretic literature examines contemporaneous cooperation in the provision of public goods in general, and climate protection in particular. With few exceptions (e.g. Harstad 2012), this literature uses static models. This theory underpins suggestions for designing an effective climate agreement (Aldy and Stavins, 2007; Guesnerie and Tulkens, 2008). A distinct literature reveals fundamental disagreements about how discounting should be used to formulate climate policy (Stern, 2006; Nordhaus, 2007, Weitzman, 2007, Roemer, 2011). These, and many other authors, agree that discount rates are important to climate policy, but they disagree on whether the pure rate of time preference (PRTP) should reflect ethical concerns or market rates. They therefore disagree on the level of the discount rate appropriate for evaluating climate policy. One reason for this disagreement is that the infinitely lived agent (ILA) model conflates the PRTP and intergenerational altruism. It is difficult or impossible to choose a constant that reflects PRTPs consistent with market rates, and levels of altruism that people claim to have for future generations (Cropper, Ayded, and Portney 1994).

Intergenerational altruism and international cooperation are central in determining equilibrium policy. These are not policy instruments that can be easily dialed up or down, but real-world mutable characteristics can be mapped into these features of the model. I represent international cooperation using the parameter  $n$ , where

each political entity (a tribe) controls  $\frac{1}{n}$ 'th of the world; each entity internalizes  $\frac{1}{n}$ 'th of their emissions' effect on the global climate. European countries' delegating their climate policy to the European Union, or developing nations following the lead of the “BRIC” countries, both correspond to a decrease in  $n$ , i.e. an increase in international cooperation. Components of the Kyoto Agreement (e.g. the Clean Development Mechanism and Joint Implementation) and of the Copenhagen Accord (e.g. the global funding scheme to finance adaptation to climate change) can be construed as attempts to increase international cooperation, and thus correspond to a reduction in  $n$ . Intergenerational altruism is a characteristic of preferences, but the way that we *model* intergenerational altruism for climate policy has largely been a matter of convenience and custom. It is reasonable to ask to what extent replacing the ILA model with an (arguably) more descriptive model would change our conclusions. In addition, France and the UK recently started using lower discount rates to evaluate climate policy, and the US Environmental Protection Agency is considering doing the same (EPA, 2010). Changing views about the proper way to take into account the welfare of distant generations likely contributed to these policy changes.

Investment in a long-lived public good, such as the climate, involves transfers both across different people living at different times, and for the same person at different stages of their life. If people discount their own utility at a constant rate, it is sensible for a planner to use that rate in evaluating intra-personal intertemporal transfers. However, there is no reason that the planner should use that same rate to evaluate intertemporal transfers across different people. A planner who adopts constant, but different discount rates for these two types of transfers, and who assigns *the same weight to the preferences of all people currently living*, has time-inconsistent preferences (Calvo and Obstfeld, 1988; Schneider, Traeger and Winkler, 2012).

To assess the role of altruism and cooperation, in determining climate policy, I rely on a variation of the “linear-in-state” model in Golosov et al. 2013. I consider two types of equilibria. The “limit equilibrium” (as the length of a finite horizon model goes to infinity) is unique and is independent of the state variable. This equilibrium is also dominant: any agent's best response when all other agents use state-independent strategies does not depend on those strategies. The infinite horizon model also

has many other differentiable Markov perfect equilibria (MPE). The life of our planet is finite, but insisting on a finite horizon model (or its limit) implies that there is some generation that *knows* it is the last generation. Equilibria that rely on an infinite horizon can be motivated as  $\varepsilon$ -equilibria to a finite horizon game (Fudenberg and Levine 1983). These equilibria are used throughout economics: a Google Scholars search for “infinite horizon games” yields about 80,000 results. The “non-limit” MPE are functions of the state variable, and are not dominant; I call them “non-dominant MPEs”. The limit and the non-dominant equilibria have different properties, and different implications for the relative importance of, and interaction between, intergenerational altruism and international cooperation.

I use two measures of the relative importance of intergenerational altruism and international cooperation. The “local measure” compares elasticities of the equilibrium policy with respect to intergenerational altruism and international cooperation. For the limit equilibrium, this local measure implies that international cooperation has a much larger effect on the outcome than does intergenerational altruism. The “global measure” compares the effect of increasing either intergenerational altruism or international cooperation from an initial level to their maximal levels. In some cases, this global measure implies that altruism is more important in determining policy, compared to international cooperation.

The game has two dimensions; agents in the game have both a tribal and a time index. The tribal index accounts for the fact that each tribe internalizes only a fraction of the cost of its emissions. The time index is needed because whenever the measure of intergenerational altruism does not equal agents’ PRTP, agents have non-constant discount rates. The planners alive in the same period (one for each tribe) act simultaneously, and the planner of any tribe in any period acts before the future planners. The model nests three special cases, a single-agent problem of non-constant discounting (Strotz, 1956; Laibson, 1997), a differential game with constant discounting (Long, 2010; Haurie, Krawczyk and Zaccour, 2012), and a standard optimal control problem.<sup>1</sup>

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<sup>1</sup>Many papers use OLG models to study environmental and resource problems (Kemp and Long 1979; John, Pecchenino and Schhimmelfennig, 1995; Kosekla, Ollikainen and Puhakka, 2002), and

## 2 Discounting

The objective of this paper is to evaluate the interaction between, and the relative influence on investment in a public good, of agents' attitudes toward future generations and on the ability of different groups to cooperate at a point in time. Infinitely-lived-agent models conflate intra- and inter-personal intertemporal transfers. A two-parameter model of preferences, imbedded in an OLG structure, can distinguish between these transfers. There is no tribal index in this section, because I consider a representative tribe here. Agents' lifetime is exponentially distributed. The memoryless property of the exponential distribution means that any two currently living agents have the same probability distribution for their future lifetime, regardless of their current ages. I use the following assumption throughout this paper:

**Assumption 1** *(i) All agents have the same utility function, which depends only on the global public good and the agent's investment in the public good. (ii) In each period, agents in a tribe share equally the cost of investment in the public good. (iii) Agents might care about current and future members of their own tribe, but they do not care about the members of other tribes.*

Assumption 1.i means that there are no privately owned assets. The assumption of exponentially distributed lifetime, together with Assumption 1.i, means that any two currently living tribal members are identical, rendering Assumption 1.ii innocuous. Assumption 1.i&ii imply that all agents in a tribe have the same utility flow at a point in time. Assumption 1.iii makes it possible to consider discounting within a tribe independently of events in other tribes.

Agents' mortality rate is  $\theta$ ; the population is constant, so the birth rate is also  $\theta$ . Agents have the pure rate of time preference  $r$ , so their risk adjusted discount rate is  $\gamma = r + \theta$ . For a stream of utility  $\{u_\tau\}_{\tau=0}^\infty$ , the expected present discounted value of lifetime utility for an agent alive at time  $t$  is  $U(t) = \int_{\tau=t}^\infty e^{-\gamma(\tau-t)} u(\tau) d\tau$ ; this is the "selfish" component of the agent's welfare. An agent at  $t$  with paternalistic altruism

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a growing number use OLG models to study climate policy (Howarth, 1998; Gerlagh and van der Zwaan, 2001; Rasmussen, 2003; Laurent-Lucchetti and Leach, 2011 ). Those papers do not include the strategic elements that arise with non-constant discounting, which is central to my paper.

cares about the lifetime utility of her successors, all those born at  $s > t$ . But she does not take into account the fact that those born at  $s'$ , with  $s > s' > t$ , also care about the agents born at time  $s$ . In contrast, an agent with pure altruism does take into account the fact that her successors care about their own successors' welfare, not just their utility streams. For both types of agents, the altruism parameter is  $\lambda$ ; the agent with paternalistic altruism discounts her successors' utility streams using  $\lambda$ , and the agent with pure altruism discounts her successors' welfare using  $\lambda$ . Assumption 1, the exponential distribution, and the assumption that all agents have the same altruism parameter,  $\lambda$ , imply that all members of a tribe currently alive are identical; any currently living member can be chosen as the social planner who decides current (but not future) investment levels.

Given the utility stream  $\{u_\tau\}_{\tau=0}^\infty$ , the welfare of an agent with paternalistic altruism is

$$W(t) = U(t) + \theta \int_{\tau=t}^\infty e^{-\lambda(\tau-t)} U(\tau) d\tau = \int_{\tau=t}^\infty D(\tau-t) u(\tau) d\tau. \quad (1)$$

The first equality states that her welfare consists of two components: her own lifetime utility (the “selfish” component) and an altruistic component. Over the interval of time  $(\tau, \tau + d\tau)$ , approximately  $\theta d\tau$  new agents are born, and each of these has her own lifetime utility  $U(\tau)$ , which the agent at  $t$  discounts at rate  $\lambda$ . The second equality implicitly defines the discount factor,  $D(\tau-t)$ , under paternalistic altruism. Using the definition of  $U(\tau)$  in equation 1 and simplifying by changing the order of integration, produces Ekeland and Lazrak' (2010) discount factor under paternalistic altruism:

$$D(t) = \left( \frac{\lambda - r}{\lambda - \gamma} \right) e^{-\gamma t} - \frac{\theta}{\lambda - \gamma} e^{-\lambda t}. \quad (2)$$

Several papers use a convex combination of exponentials to represent non-constant discounting for a single infinitely lived agent (Li and Lofgren, 2000; Gollier and Weitzman, 2010; Zuber, 2010; and Jackson and Yariv, 2011). In Ekeland and Lazrak's OLG model, the discount factor is a weighted combination of exponentials;

it is a convex combination only if  $\lambda \leq r$ . Equation 2 shows the relation between the discount factor and the OLG structure. If  $\theta = \infty$ , a tribe consists of a succession of agents, each of whom lives for a single instant, implying a constant social discount rate  $\lambda$ . At the other extreme,  $\theta = 0$ , a tribe consists of an infinitely lived agent, with a constant discount rate  $r$ . For these two limiting cases, there is no time consistency problem.

Given the utility stream  $\{u_{t+\tau}\}_{\tau=0}^{\infty}$ , welfare at  $t$  for the agent with pure altruism,  $V(t)$ , satisfies the recursion

$$V(t) = \int_t^{\infty} e^{-\gamma(\tau-t)} u(\tau) d\tau + \theta \int_t^{\infty} e^{-\lambda(\tau-t)} V(\tau) d\tau = \int_t^{\infty} D(\tau-t) u(\tau) d\tau. \quad (3)$$

The first equality states that the agent's welfare consists of the discounted stream of her own utility, plus the stream of successors' welfare, discounted using the altruism parameter  $\lambda$ . The second equality implicitly defines the discount factor that aggregates the utility stream  $\{u_{t+\tau}\}_{\tau=0}^{\infty}$ . The discount factor under paternalistic altruism is obtained using equation 1 and then simplifying an integral, whereas the discount factor under pure altruism requires solving a recursion. Despite this difference in the mathematical structure of the two problems, the discount factors are related in a simple way:

**Proposition 1** *Suppose agents with pure altruism discount future agents' welfare at rate  $\lambda'$ , and agents with paternalistic altruism discount future agents' utility at rate  $\lambda$ . Both have mortality rate  $\theta$  and the pure rate of time preference  $r$ . (i) The two types of agents, and thus the planners who represent them, have the same preferences if and only if  $\lambda' = \lambda + \theta$ . (ii) Given the same mortality rate,  $\theta$ , and the same discounting preference parameters  $(r, \lambda)$ , the planner with paternalistic altruism discounts the future flow of utility more heavily than the planner with pure altruism.*

In view of the isomorphism described in Proposition 1.i, I hereafter consider only the case of paternalistic altruism. The discount rate,  $\eta(t)$ , corresponding to equation

2, is

$$\eta(t) \equiv -\frac{dD}{dt} \frac{1}{D} = \frac{-\gamma\lambda + \gamma r + \theta\lambda e^{-(\lambda-\gamma)t}}{-\lambda + r + \theta e^{-(\lambda-\gamma)t}}, \quad (4)$$

with

$$\begin{aligned} \text{sign} \frac{d\eta(t)}{dt} &= \text{sign} \lambda - r; \eta(0) = r \text{ for } \lambda < \infty; \\ \lim_{t \rightarrow \infty} \eta(t) &= \lambda \text{ for } \lambda \leq r; \text{ and } \lim_{t \rightarrow \infty} \eta(t) = \gamma \text{ for } \lambda > r. \end{aligned} \quad (5)$$

Different values of  $\lambda$  have obvious interpretations, but my results do not require choosing the “correct” value. Indeed, a primary motivation for this study is to determine the effect of different values of  $\lambda$  on equilibrium outcomes. For  $\lambda = \infty$ , currently living agents do not care about those born in the future. For  $\lambda = r$ , they make no distinction between a utility exchange from a person to her older self, and from a person to a different person born in the future. For  $\lambda = 0$ , agents put the same value on the lifetime welfare of all agents, regardless of their date of birth. Because the date of a person’s birth is a matter of chance, there is an obvious ethical justification for  $\lambda = 0$ . A reader who holds this view, might want to add a small positive constant to  $\lambda$ , to take into account the possibility that our species suddenly becomes extinct.

**Robustness** Here I compare the discount rate under the assumption of exponentially distributed lifetime and under a deterministic alternative where agents live for  $\Gamma$  periods. Setting  $\Gamma = \frac{1}{\theta}$ , the expected lifetime under the exponential distribution, makes the two comparable. When agents have deterministic lifetime, currently living agents have different incentives to invest in a long-lived public good, because some will die sooner than others. Other complications also become important in this setting, such as the possibility of transfers among currently living members of a tribe. For the exponentially distributed lifetime, where all tribal members are identical, there would be no reason for those transfers.

The discount factor for the deterministic case (with paternalistic altruism) can be calculated under Assumption 1 and the additional assumption that a utilitarian social planner aggregates the preferences of currently living tribal members, giving the same weight to each of these. The discrete time analog to this problem, in which

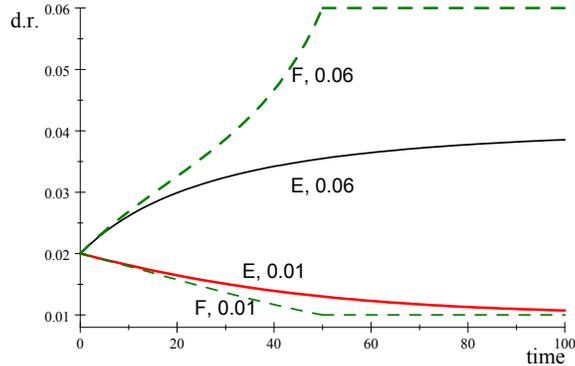


Figure 1: Discount rates (d.r.) for  $\theta = 0.02 = r = \frac{1}{\Gamma}$ . Solid curves (labelled E) correspond to exponentially distributed lifetime and dashed curves (labelled F) correspond to fixed lifetime. Numerical values in label show value of  $\lambda$ .

all people live for two periods, implies quasi-hyperbolic (or  $\beta, \delta$ ) discounting. Figure 1 shows the graphs of the discount rates under the exponential distribution and under finitely-lived agents, for  $\Gamma = \frac{1}{\theta}$ , with  $\theta = 0.02 = r$ , for  $\lambda = 0.01$  (the negatively sloped curves) and for  $\lambda = 0.06$  (the positively sloped curves). For  $\lambda < r$ , the two assumptions about lifetime lead to very similar discount profiles. For  $\lambda > r$ , the two profiles are similar for the first 15 - 20 years. However, for large discount rates, the future ceases to matter much after a few decades. An earlier version of this paper numerically compares equilibrium policy under the two assumptions about lifetime. Not surprisingly, the equilibrium properties were very similar for  $\lambda < r$ ; they were also similar for large  $\lambda$ . To satisfy space constraints, I hereafter consider only case of exponentially distributed lifetime, emphasizing the situation where  $\lambda \leq r$ . (Earlier numerical results show the equilibrium to be quite insensitive to  $\lambda$  above  $r$ .)

### 3 The game

This section relies on previous work to obtain the equilibrium conditions for the game with  $n$  tribes, where all agents have mortality rate  $\theta$ , PRTP  $r$ , and altruism parameter  $\lambda$ . Social planner  $i, t$  has a tribal index  $i$  and a time index  $t$ ; this planner

takes an action,  $x_{it}$ . The value of the state variable at  $t$ , common to all tribes, is  $S_t$ . A stationary symmetric MPE is a function  $\chi(S)$  with the Nash property: if planner  $i, t$  believes that all future planners (including those in her own tribe) and all other planners currently alive, will make their decision according to  $x = \chi(S)$  (where  $S$  is the value of the state when a particular decision is made) then it is optimal for planner  $i, t$  to set  $x_{it} = \chi(S_t)$ . (I use “believe” to mean “act as if she knows”.) Planner  $i, t$  chooses a single state-contingent action, rather than a (possibly infinite) sequence of actions, as in a control problem. A Nash equilibrium requires that her action maximize her current flow payoff plus a state-dependent continuation value.

Karp (2007), building on Harris and Laibson (2001), finds the formal limit, as the length of a period goes to 0, of the equilibrium conditions to the discrete time sequential game where  $n = 1$ . Ekeland and Lazrak (2010) independently obtain the same necessary conditions using a variational argument that begins with a continuous time model. The symmetry assumption makes it possible to adapt those necessary conditions for the game with  $n > 1$ . The succession of planners in tribe  $i$  who believe that all planners in all other tribes will use the policy rule  $\chi(S)$  face the problem considered in Karp (2007) and Ekeland and Lazrak (2010), apart from details regarding notation, described below. For the climate model, I establish existence constructively.

Denote  $\mathbf{x}_t \in \mathbb{R}^n$  as the vector of actions at time  $t$ , with  $i$ 'th element  $x_{it}$ , the action taken by planner  $i, t$ . The state variable  $S$  (possibly a vector) evolves according to  $\frac{dS}{dt} = f(S_t, \mathbf{x}_t; n)$ , with the initial condition (the current value of  $S$ ) given. For example, the elements of  $S_t$  may include greenhouse gas stocks in different sinks (e.g. the atmosphere or the deep ocean) and global average temperature as a deviation from pre-industrial levels;  $x_{it}$  equals tribe  $i$ 's emissions at time  $t$ . Consistent with Assumption 1, tribe  $i$ 's flow of utility at  $t$ ,  $u(S_t, x_{it}; n)$ , depends only on the state variable and  $i$ 's action (emissions, or investment in the public good).<sup>2</sup>

In order that the parameter  $n$  measure only the degree of international coopera-

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<sup>2</sup>At the cost of greater complexity, and dropping Assumption 1, I could include tribe  $j$ 's action in tribe  $i$ 's utility function. In the interest of parsimony, I use only changes in  $n$  to reflect different levels of concern for other groups.

tion, and not other things (e.g. population), this parameter must be introduced in the functions  $u(\cdot)$  and  $f(\cdot)$  so that it leaves unchanged the set of feasible utility. Fragmentation (a larger  $n$ ) has no intrinsic effect on aggregate (or per capita) utility or stock changes, but it alters the equilibrium decisions, thereby altering the equilibrium aggregate utility and stock changes:  $n$  has a strategic but not an intrinsic effect on outcomes.<sup>3</sup> Examples in Section 4 illustrate this procedure.

Denote  $\mathbf{i}_{n-1} \in \mathbb{R}^{n-1}$  as the vector consisting of 1's. Given the decision rule  $\chi$  that planners in tribe  $i$  expect planners in other tribes to use, denote  $F(S, x_i) \equiv f(S, \mathbf{i}_{n-1}\chi(S), x_i; n)$ . This function (with the same dimension as the state variable) is the time derivative of  $S(t)$  when the current value of the state variable is  $S$ , all other tribes use  $\chi(S)$ , and tribe  $i$  uses  $x_i$ . The payoff and constraint facing tribe  $i$  are

$$\int_{\tau}^{\infty} D(t - \tau)u(S_t, x_{it}; n)dt \quad \text{and} \quad \dot{S}_t \equiv \frac{dS_t}{dt} = F(S, x_i). \quad (6)$$

with discount factor,  $D$ , given by equation (2).

A larger value of  $n$  means that the time  $t$  social planner in tribe  $i$  internalizes a smaller fraction of the effect of her actions on other tribes. A larger value of  $\lambda$  means that this social planner internalizes a smaller fraction of the future effect of her actions. The point of this paper is to determine the interaction between, and the relative sensitivity of equilibrium investments in a public good, to changes in  $n$  and  $\lambda$ .

### 3.1 Equilibrium conditions

Given beliefs that all planners in other tribes use the decision rule  $\chi(S)$ , equation (6) contains the payoff and constraint facing tribe  $i$ . I use the conditions given in

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<sup>3</sup>For example, if the world consists of  $N$  countries, and each tribe controls  $m$  countries, then  $n = \frac{N}{m}$ . A tribal social planner internalizes the effect of her action on residents in all  $m$  of the countries that the tribe controls. If each tribe fragments into two equal tribes, then  $n = 2\frac{N}{m}$ . A larger  $n$  means not only that there are more tribes, but also that each is absolutely smaller. When EU members, for example, allow a central agent to determine their climate policy, that agent has the incentive to internalize a greater fraction of the effect of a single country's emissions, than would the country. Apart from the fact that countries in the real world are not identical, a coalition such as the EU is analagous to a smaller  $n$ .

Karp (2007) to obtain the necessary conditions to that game. The symmetric Nash condition requires that the equilibrium policy function equals  $\chi(S)$ . The validity of this procedure requires that the value function, defined below, and its first derivative exist – an assumption that can be checked given a particular equilibrium.

The limiting values, as  $t \rightarrow \infty$ , of the discount rate,  $\eta(t)$ , differ in the two cases corresponding to  $\lambda < r$  and  $\lambda > r$  (equation 5). The equilibrium conditions are therefore different in these two cases. I provide details for case  $0 < \lambda \leq r$  (where  $\lim_{t \rightarrow \infty} \eta(t) = \lambda$ ), relegating the other case to Appendix B.1.1. (Recall that for  $\lambda = r$  or  $\lambda = \infty$ , where the discount rate is constant, the tribes play a standard differential game, i.e. one without the strategic interactions within a tribe, across periods.)

Dropping the tribal index  $i$  (because of symmetry) Proposition 1 and Remark 1 of Karp (2007) imply that  $\chi(S)$  satisfies the necessary condition to the following auxiliary optimal control problem with constant discount rate  $\lambda$ :

$$J(S_0) = \max \int_0^\infty e^{-\lambda t} (u(S_t, x_t; n) - K(S_t)) dt \quad \text{subject to } \dot{S} = F(S, x), \quad (7)$$

with the side condition (a definition):

$$K(S_t) = (r - \lambda) \int_0^\infty e^{-\gamma \tau} u(S_{t+\tau}^*, \chi(S_{t+\tau}^*); n) d\tau. \quad (8)$$

The tribe's utility flow on the equilibrium path is  $u(S_\tau, \chi(S_\tau); n)$ , and  $S_\tau^*$  is the solution to the differential equation in (7) when all agents use the decision rule  $\chi(S)$ . The function  $K$  can be interpreted as an annuity, which if received in perpetuity and discounted at the rate  $r - \lambda$ , equals the present value of the stream of future utility, discounted at the rate  $\gamma = r + \theta$ .<sup>4</sup>

This model includes familiar special cases. For  $n > 1$ , the endogenous function  $F(S, x) = f(S, \mathbf{i}_{n-1}\chi(S), x; n)$  depends on the policies of the other  $n - 1$  agents.

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<sup>4</sup>The dynamic programming equation is  $\lambda J(S) = \max_x [u(S_t, x_t; n) - K(S_t) + J_S(S) F(S, x)]$ . For the models of Section 4,  $u$  is concave and  $F$  is linear in  $x$ , so a sufficient condition for equilibrium is that  $\chi(S) = \arg \max [u(S_t, x_t; n) + J_S(S) F(S, x)]$  and  $J$  solves the DPE. The sufficient conditions for a MPE are simpler in the game here, involving non-constant discount rates, than in the differential game with constant discount rates. Appendix B.1.2 discusses this issue.

Those agents do not exist if  $n = 1$ , in which case,  $F(S, x) = f(S, x; 1)$ , an exogenous function, and the model collapses to a sequential game with a single agent at each point in time. For  $\lambda = r$ ,  $K \equiv 0$  and the model collapses to a standard (constant discounting) differential game for  $n > 1$  or a control problem for  $n = 1$ .

### 3.2 Nonuniqueness

In general, the equilibrium to this game is not unique. Tsutsui and Mino (1990) note the existence of a continuum of stable steady states (an open interval) in the differential game with constant discounting when decision rules are differentiable. For each point in this interval there is an equilibrium policy function, defined at least in the neighborhood of that point. The economic explanation for this multiplicity in the differential game is that the decision whether to remain in a particular steady state depends on an agent's beliefs regarding the actions that rivals would take if a single agent were to drive the state away from that steady state. The MPE conditions do not pin down these beliefs. In a standard optimal control problem, the envelope theorem eliminates that kind of consideration, because the first order welfare effect of a deviation from the steady state is 0. This theorem is not applicable in the differential game, because rivals' actions do not maximize an agent's welfare. The same consideration applies for  $n = 1$  under non-constant discounting.

When  $n > 1$  and the discount rate is non-constant, there are two sources of multiplicity of steady states. (Under a special circumstance, studied in Ekeland, Karp, and Sumaila (2012), the equilibrium is unique, within the class that induce differentiable value functions.) The multiplicity of equilibria creates a coordination problem across tribes and generations.<sup>5</sup> Some MPE may Pareto dominate others.

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<sup>5</sup>Dropping the assumption that the value function is differentiable gives rise to many other MPE (Dutta and Sundaram 1993). Agents might "behave well" if the state variable is in a certain region, but follow a "bad" MPE if the state variable leaves that region. This kind of MPE has the flavor of trigger strategies in repeated games. There are many types of equilibria, apart from the MPE with differentiable value functions, studied here. Krusell and Smith (2003) and Vieille and Weibull (2009) discuss multiplicity in different settings.

### 3.3 The Green Golden Rule

For  $n = 1$ , there is no conflict amongst contemporaneous agents, and for  $\lambda = r$  or  $\lambda = \infty$  the time inconsistency problem also vanishes. These parameter values produce a standard optimization problem, with the usual normative interpretation. A third limiting case,  $\lambda = 0$ , implies non-constant discounting and therefore produces a game across generations. An equilibrium outcome to that game does not have normative properties, as is true in general for non-cooperative Nash equilibria.

For  $\lambda = 0$ , the discount rate converges to 0, so unless the flow payoff converges to 0 sufficiently rapidly, the payoff is unbounded. Nevertheless, the equilibrium policy function may be well defined even as  $\lambda \rightarrow 0$ , as in Section 4. For small positive  $\lambda$  and bounded  $u_{it}$ , the payoff is well defined and is approximately proportional to  $\frac{u_\infty}{\lambda}$ , the steady state utility flow divided by  $\lambda$ :

**Lemma 1** *For any bounded utility flow  $u(t)$  that converges to  $u_\infty \neq 0$ , and given the discount factor under paternalistic altruism where lifetime is exponentially distributed (equation (2)),  $\lim_{\lambda \rightarrow 0} \left( \frac{\lambda}{u_\infty} \int_0^\infty D(t)u(t)dt \right) = \frac{\theta}{\gamma}$ .*

For small  $\lambda$ , the payoff in the steady state determines the evaluation of welfare.

Denote the steady state that maximizes the steady state utility flow as the “Green Golden Rule”, or GGR, the solution to  $\max_S u(S, x; 1)$  subject to  $f(S, x; 1) = 0$  (Chichilnisky, Heal, and Beltratti 1995). This is the steady state chosen by the infinitely patient planner ( $r = 0 = \lambda$ ) who controls the system ( $n = 1$ ). Suppose that this static optimization problem is concave, so that levels of the state variable closer to the GGR have higher utility levels. Consider the case where  $\lambda$  is small and where the MPE state trajectory does not approach the GGR. In this situation, Lemma 1 together with the concavity assumption imply that a deviation from the equilibrium that causes the state to move closer to the GGR, is a Pareto improvement over the MPE. Each generation prefers this deviation.

The following proposition states that for small  $\lambda$ , there is a MPE that supports a steady state arbitrarily close to the GGR.

**Proposition 2** *Consider the class of differentiable MPE policy rules. For  $n = 1$  and for arbitrarily small positive  $\varepsilon$ , it is possible to support a MPE steady state that leads to a utility flow within  $\varepsilon$  of the utility level at the GGR, provided that  $\lambda$  is sufficiently small (but positive).*

This proposition, together with Lemma 1 and the generic multiplicity of equilibria, implies that when  $\lambda$  is small but positive, there exists a MPE that maintains the state close to the GGR; moreover, that MPE Pareto dominates any MPE that maintains the state further from the GGR. The proof of the proposition uses the assumption that the policy function is differentiable. Section 3.2 notes that for this class of policy function, the set of MPE stable steady states is an open interval. Proposition 2 states that a boundary of that interval moves close to the GGR as  $\lambda$  becomes small.

## 4 Climate policy

I rely largely on an analytic model that is linear in the state variable. I first describe this model, and then study the unique equilibrium that is the limit, as the time horizon goes to  $\infty$ , of the finite horizon model (the “limit equilibrium”). Here I analytically compare the importance of altruism and fragmentation ( $\lambda$  and  $n$ ) in both a scalar and a higher dimensional model. I then consider “non-dominant equilibria”, that owe their existence to the infinite horizon assumption. As a robustness check I also report results for a model that is convex in the state variable.

### 4.1 The analytic model

The state variable is  $S = (s, G')'$ , where  $G \in \mathbb{R}^{m-1}$  is the vector of greenhouse gas stocks in different sinks, and possibly of temperatures in the different layers of the ocean and atmosphere, and  $s \in \mathbb{R}$  is a scalar, the only element of  $S$  that *directly* affects climate-related damage at a point in time. For example,  $s$  may be the average global temperature as a deviation from pre-industrial levels.;  $s = i_1' S$ , where  $i_1$  is the first unit vector. (Primes denote transpose.) Carbon emissions enter the atmosphere and disperse amongst the different sinks, eventually altering the variable

$s$ , and thus altering the level of damages. This linear model, with different sinks, can approximate the highly non-linear carbon cycle (Forster 2007, note *a*, page 213).

The equation of motion is

$$\dot{S} = BS + bX$$

$$\text{with } B = \begin{pmatrix} \varsigma & \varrho \\ 0 & v \end{pmatrix}, b = \begin{pmatrix} 0 \\ \tilde{b} \end{pmatrix} \text{ and } X = x + (n - 1) \chi. \quad (9)$$

The elements of  $B$  and  $b$  are constants.<sup>6</sup> Aggregate emissions,  $X$ , equal  $nx = n\chi(S)$ ; here, the scalar  $x$  is tribe  $i$ 's emissions, and  $\chi$  is the emission of any other tribe, all of which are equal in a symmetric equilibrium. The matrix of eigenvalues of  $B$  is  $\Lambda$  with  $i$ 'th diagonal element  $\Lambda_i$ , and the matrix of eigenvectors is  $P$ . I assume that  $\Lambda_i$  are non-positive real numbers and  $P$  is of full rank. If  $x$  and  $\chi$  are state-independent, but possibly time varying, then  $S_t = Pe^{\Lambda t}P^{-1}S_0 + H(t)$ ;  $H(t)$  depends on the profiles  $x(\tau), \chi(\tau)$  for  $\tau \in [0, t)$ ; the function  $H(t)$  plays no role in the analysis.

Where no confusion results, I suppress time subscripts, denoting  $X = nx$  as aggregate emissions in a symmetric equilibrium and  $S$  as the state variable at a point in time. In the linear-in-state model, aggregate utility flow is  $u(X, s, t; 1) = v(X, t; 1) - \kappa s$  for a concave function  $v$ ;  $\kappa$  is the damage parameter. The argument  $t$  in the function  $v$  allows for the possibility of exogenous changes, including those associated with changes in technology (e.g., carbon intensity) or capital stocks.<sup>7</sup> Increases in  $n$  represent greater fragmentation, not higher population. The utility flow of a particular tribe is  $u(x, s, t; n) = v(x, t; n) - \frac{\kappa}{n}s$  with  $v(x, t; n) = \frac{1}{n}v(nx, t; 1)$ . By construction, aggregate utility and the change in the stock depend on aggregate emissions and the stock, but not directly on  $n$ . To summarize: tribe  $i$ 's flow payoff is  $v(x, t; n) - \frac{\kappa}{n}s$ , it's discount factor is given by  $D(t)$  in equation 2, and the equation of motion for the state variable is equation 9.

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<sup>6</sup>The dimensions of the parameters in the equation of motion are:  $\varsigma$  is a scalar,  $\varrho$  is  $1 \times m - 1$ ,  $v$  is  $m - 1 \times m - 1$ ,  $\tilde{b}$  is  $m - 1 \times 1$ , and the elements of  $\tilde{b}$  sum to 1: emissions enter one of the sinks. The recursive structure of  $B$  implies that causation runs in one direction, from  $G$  to changes in  $s$ .

<sup>7</sup>For example, if  $\omega_t$  equals average carbon intensity of energy, and if all anthropogenic emissions were caused by energy consumption, then  $X_t = \omega_t \times \text{energy consumption}$ . This formulation provides one of many ways to link emissions to economic variables.

**Examples** One can take the function  $v(X, t; 1)$  as primitive; an alternative begins with the assumption that the aggregate utility flow,  $u(X, s, t; 1)$ , depends on consumption, a function of emissions,  $X$ , the damage stock,  $s$ , and the exogenously changing variables captured by the argument  $t$ . For particular combinations of damage function and utility function,  $u(X, s, t; 1)$  is linear in  $s$ . For example, if aggregate consumption equals  $C(X, t; 1)e^{-\kappa s}$ , and utility is logarithmic (as in Golosov et al (2013)), then  $u(X, s, t; 1) = \ln C(X, t; 1) - \kappa s$ ; defining  $v(X, t; 1) = \ln C(X, t; 1)$  gives the desired form. Two examples illustrate this formulation. In the first,  $C$  is Cobb Douglas in  $X$ , leading to a simplified version of Golosov et al (2013), the GeaS (“Golosov et al. Simplified”) model. In the second example,  $C$  is the exponential of a quadratic in  $X$ , causing  $v$  to be quadratic.

**Example 1** (*GeaS model*):  $C(X, t; 1) = A_t X_t^{\alpha_t} e^{-\kappa s_t}$ . Aggregate utility is  $u(X, s, t; 1) = \ln A_t + \alpha_t \ln X_t - \kappa s_t$ , i.e.  $v(X, t; 1) = \ln A_t + \alpha_t \ln X_t$ . Setting  $v(x, t; n) = \frac{\ln A_t}{n} + \frac{\alpha_t}{n} \ln(nx)$  means that in a symmetric equilibrium (where  $X = nx$ ) aggregate utility is  $n \left( \frac{\ln A_t}{n} + \frac{\alpha_t}{n} \ln(nx) - \frac{\kappa}{n} s \right) = u(X, s, t; 1)$ .

**Example 2** (*Quadratic model*):  $C(X, t; 1) = \exp \left( a_{0,t} + a_t X - \frac{d_t}{2} X^2 \right)$ , so  $v(X, t; 1) = a_{0,t} + a_t X - \frac{d_t}{2} X^2$ . Here, utility for a tribe emitting  $x$  is  $u(x, s, t; n) = v(x, t; n) - \frac{\kappa}{n} s$ , with  $v(x, t; n) = \frac{a_{0,t}}{n} + a_t x - n \frac{d_t}{2} x^2$ . Aggregate utility in a symmetric equilibrium is  $nu(x, s, t; n) = v(nx, t; 1) - \kappa s =$ , where  $v(nx, t; 1) = v(X, t; 1) = a_{0,t} + a_t X - \frac{d_t}{2} X^2$ .

**Representing equilibrium policies** It is convenient to present results in terms of a tax, measured in units of utility, instead of in terms of equilibrium emissions, or a tax measured (as is standard) in monetary units. The reason for this choice is that, for the “limiting equilibrium” considered below, the utility denominated tax depends only on the parameters of the discount function and the equation of motion and  $n$ ; in particular, that tax does not depend on the function  $C$ , and thus is unaffected by the time-dependence (e.g., via technology) of  $C$ . Both emissions and the tax measured in monetary units do depend on  $C$  even in the limiting equilibrium. The utility-denominated tax,  $\tau$ , that supports aggregate emissions  $X$ , in a

decentralized *aggregate* economy is

$$\tau(S, t) = \frac{1}{C(X, t; 1) e^{-\kappa s}} \frac{\partial C(X, t; 1)}{\partial X} e^{-\kappa s} = \frac{1}{C(X, t; 1)} \frac{\partial C(X, t; 1)}{\partial X},$$

which equals the marginal utility of consumption times the marginal increase in consumption due to an extra unit of emissions. This tax has units of utility/emissions. Dividing by the marginal utility of consumption (multiplying by  $C(X, t; 1) e^{-\kappa s}$ ) converts the tax from utility to monetary units.

**Discussion of this model** Integrated assessment models such as DICE and some of its successors treat capital as endogenous, although they typically treat other time-varying features such as technology as exogenous. My “stripped down” model treats everything except for the climate-related variables as exogenous.

Failure to treat capital as endogenous might not matter much. Golosov et al. (2013) use a discrete time model with endogenous investment, logarithmic utility, and Cobb Douglas production; capital depreciates 100% in a single period. In that setting (with  $n = 1$  and constant discounting), the optimal savings rate is a constant that is independent of climate parameters. Gerlagh and Liski (2012) and Iverson (2013) study that discrete time model under more general discounting (with  $n = 1$ ) and again find that the savings rate (in one equilibrium) is a constant, independent of climate parameters. These models decouple the investment and climate components.

In the continuous time setting, there is no analog to “100% depreciation in a period”, so the savings rate in the continuous time setting, extended to include endogenous capital, would not be constant. I avoid this complication by taking the capital stock, in addition to technology, as exogenous. Although the functional assumptions in Golosov et al. (2013) produce an exact form of decoupling between investment and the climate, in (some) other models the investment decision is insensitive to climate considerations. In those cases, studying the climate problem in isolation from the investment decision has little effect on climate policy. Hwang, Reynes, and Tol (2013) illustrate this point.

Technological progress and capital accumulation might make distant generations

so much richer than us, that climate-induced reductions in their consumption are unimportant. Reductions in future carbon intensity might make future abatement cheap. In these cases, we should not sacrifice much today to reduce our carbon emissions. These policy conclusions might be correct, but one might be concerned about the extent to which they are driven by strong assumptions about technology.

There are at least three reasons why we might want a model in which policy is not driven by the hypothesis that we will grow our way out of the climate problem. First, the familiar relation between high expected growth and a high discount rate arises in the standard model with time-additive expected utility; making growth uncertain leads to only a second-order correction. However, in a model that disentangles risk aversion from the elasticity of intertemporal substitution, Traeger (2014) shows that *stochastic* growth (compared to zero growth) might have little effect on the certainty equivalent discount rate. Second, the assumption, adopted by most integrated assessment models, that natural and man-made capital are highly substitutable, may be incorrect (Guesnerie 2004, Hoel and Sterner 2007, Traeger 2011). In that case, we may want to protect natural capital even if future generations have much more man-made capital than we do. Third, most integrated assessment models identify growth with increased GDP, leading to increased consumption. The limitations of GDP as the sole index of well-being are well understood; alternatives or supplements include the Genuine Progress Indicator (GPI), Human Development Index (HDI) and Ecological Footprint. Kubiszewski et al. (2013) discusses these, and notes that over the past 25 years GPI has been flat, while GDP has continued to grow: the indices might be only weakly correlated.

The modeling dilemma is that changes in technology likely are important, but we might want to avoid having today's climate policy driven by beliefs about future technological change. The linear-in-state model provides one solution to this dilemma. The parameters in the equation of motion,  $\dot{S}$ , are determined by natural processes, and thus independent of technology. The damage parameter,  $\kappa$ , could be altered by technology, but given the model's level of abstraction, treating  $\kappa$  as a constant parameter is defensible. With this assumption, only the function  $v$  depends explicitly on changing technology. For the limit equilibrium, the tax in utility units,

$\tau$ , is independent of  $v$ , and thus independent of technology. However, both emissions and the tax in monetary units do depend on  $v$ , and thus on current technology. They do not, however, depend on beliefs about *future* technology. The situation is slightly different under the other “non-dominant equilibria” introduced below.

## 4.2 The limit equilibrium

Here I assume that  $\lambda \in [0, r]$ , so I use the formulae in Section 3.1. In a finite horizon model, backward induction yields a unique equilibrium. The utility-denominated tax in the “limit equilibrium” (as the horizon goes to infinity) is independent of the state variable and the time argument,  $t$ . If other agents use state-independent decision rules, then the shadow value of the state, for an arbitrary tribe at an arbitrary point in time, depends on model parameters but not on the level of the state. Therefore, the agent’s optimal action is independent of the state. The independence with respect to time is then a consequence of the fact that the climate-related parameters (including  $\kappa$ ) do not depend on time.

**Proposition 3** *For the linear-in-state model, suppose that a planner of a tribe at a point in time believes that all other agents (future planners in her own tribe and all planners in all other tribes) will use state-independent (but possibly time- and tribe-dependent) emissions policies. (i) Her annuity function,  $K(S, t)$  and value function  $J(S, t)$  are linear in the state;  $J(S; t) = g_0t + g'S$  with:*

$$g' = \frac{\kappa}{n} (i'_1 - (r - \lambda) \tilde{q}') (B - \lambda I)^{-1}, \text{ with } \tilde{q}' = \int_0^\infty i'_1 P e^{-(\gamma I - \Lambda)\tau} P^{-1} d\tau, \quad (10)$$

where  $I$  is the  $m$  dimensional identity matrix.

(ii) *Her optimal emission level is independent of the state, and is a dominant strategy: it does not depend on her beliefs about the state-independent emissions of any future planner, or about the actions of other current planners. Equilibrium emissions are also independent of beliefs about future technology. Within the class*

of state-independent policies, the unique equilibrium is given by

$$\chi = \arg \max_x v(x, t; n) + g' C x. \quad (11)$$

Iverson (2013) shows, for the discrete-time log-linear model with investment, 100% depreciation in a period, and  $n = 1$ , that the first period action of a planner who can commit does not depend on her beliefs about future state-independent actions; this first period action equals the policy in a state-independent MPE. Phelps and Pollack (1968) obtain this result for a simpler model. This result is consistent with the dominance result in Proposition 3.ii, which holds for arbitrary  $n$  and concave function  $v$ . Dominance is a consequence of the model's linearity in the state variable.

The utility denominated tax depends on climate-related parameters (including  $\kappa$ )  $n$ , and discounting parameters, but not on the function  $v(\cdot)$ ; the tax is also independent of both the state and  $t$ :

**Corollary 1** *The utility-denominated tax that in the aggregate economy supports the equilibrium level of emissions, defined as  $\tau = v'(X(n), t; 1)$ , is*

$$\tau = -\frac{\kappa}{n} (i'_1 - (r - \lambda) \tilde{q}') (B - \lambda I)^{-1} C, \quad (12)$$

and thus independent of the state variable, time, and the payoff function  $v(\cdot)$ . The absolute value of the elasticity of this tax with respect to  $n$  is 1, and the elasticity of the tax with respect to  $\lambda$  is

$$\varepsilon \equiv -\frac{d\tau}{d\lambda} \frac{\lambda}{\tau} = -\lambda \frac{[\tilde{q}' + (i'_1 - (r - \lambda) \tilde{q}') (B - \lambda I)^{-1}] (B - \lambda I)^{-1} C}{(i'_1 - (r - \lambda) \tilde{q}') (B - \lambda I)^{-1} C}. \quad (13)$$

From equation (11), equilibrium emissions (unlike the tax) does depend on  $v$ , and thus on current technology (via  $t$ ); emissions do not depend on beliefs about future technology. Examples 1 and 2 illustrate the relation between  $v(\cdot)$  and equilibrium emissions. For the GeaS model, if each of  $n$  tribes fragments into two, aggregate equilibrium emissions double, despite no fundamental (i.e., non-strategic) change in the economy. The Quadratic model does not have this extreme feature:

**Corollary 2** *In GeaS model, equilibrium emissions per tribe are independent of  $n$ , so aggregate emissions are proportional to  $n$ . In the Quadratic model, equilibrium emissions per tribe fall with  $n$ , and aggregate emissions are a strictly concave increasing function of  $n$ .*

**The climate application** Climate-related damages are related to temperature change, and temperature adjusts to GHG stocks with a lag. Consequently, damages caused by current emissions are likely non-monotonic with respect to time. A two-dimensional ( $m = 2$ , i.e.  $G$  is a scalar) version of equation 9 is adequate to capture this feature, and still leads to explicit solutions. In this two-dimensional model,  $\varsigma$ ,  $\varrho$  and  $v$  are scalars, and the scalar  $\tilde{b} = 1$ . I choose units of  $s$  so that  $\kappa = -1$ , so I need three assumptions to calibrate the model. For this purpose, I define  $G$  as the difference between actual and pre-industrial atmospheric CO<sub>2</sub> measured in Teratons (Tt CO<sub>2</sub>). The emissions flow,  $X$ , increases this stock, which decays at a constant rate,  $v$ . The damage stock, a proxy for temperature change, is  $s \leq 0$ , and responds to increased  $G$  with a lag.

Rezai (2010) reports estimates of dissipation rates for CO<sub>2</sub> that imply half-lives between 126 and 276 years, for a midpoint of 200 years, the value that I use. With multiplicative exponential damages, the fractional consumption loss at  $t$  due to climate change is  $1 - e^{s(t)}$ . Gerlagh and Liski (2012) calibrate a higher dimensional linear model, based primarily on DICE. I adopt their assumptions that: (i) doubling atmospheric stocks (relative to preindustrial levels) reduces output (in my setting, consumption) by 2.6%, once  $s$  has adjusted, and (ii) following a pulse increase in atmospheric CO<sub>2</sub> at time 0, the loss rises (from 0) during the first 60 years, and then falls slowly. These three assumptions imply the “baseline parameter values”  $v = -3.47 \times 10^{-3}$ ,  $\varsigma = -4.685 \times 10^{-2}$ , and  $\varrho = -5.66 \times 10^{-4}$ . (The baseline also includes  $r = 0.02 = \theta$ .)<sup>8</sup>

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<sup>8</sup>It is well-understood that no single half-life provides a close approximation of the carbon cycle (Forster 2007, note *a*, page 213). A higher dimensional system can provide a better approximation of the carbon cycle and can describe additional aspects of the relation between emissions and damages. Those higher dimensional models require expressing results in terms of eigenvalues of  $B$  (as in equation 12) rather than primitives. It is noteworthy that even the two-dimensional model

Using the two-dimensional analog of equation 9 and the formula in equation 12, the equilibrium tax is

$$\tau = -\frac{\rho}{n} \frac{1}{\lambda - v} M_1' M_2 \quad \text{with} \quad (14)$$

$$M_1 = \begin{pmatrix} (\varsigma - \gamma)(r - \lambda) + 1 \\ \frac{r - \lambda}{(\varsigma - v)(\gamma - v)} ((\gamma - v)(\varsigma - \gamma) + 1) \end{pmatrix} \quad M_2 = \begin{pmatrix} \frac{-1}{(\varsigma - \lambda)} \\ 1 \end{pmatrix}.$$

Figure 2 shows the graphs of the equilibrium tax (at  $n = 1$ ) and the elasticity of the tax with respect to  $\lambda$  under the baseline parameters. The elasticity is non-monotonic in  $\lambda$ , but is always less than 0.6. As noted above, the elasticity of the tax with respect to  $n$  equals 1. For this calibration, the equilibrium policy is much less sensitive to the altruism parameter than it is to the fragmentation parameter. The tax increases by a factor of 4.9 as  $\lambda$  falls from  $r = 0.02$  to its lower bound 0.

To convert the tax in utility units,  $\tau$ , to dollars, I multiply  $\tau$  by annual gross world consumption. At an estimated gross world output of \$63 trillion, annual consumption corresponding to a 25% investment rate is \$47.25 trillions. By the choice of calibration method, CO<sub>2</sub> units are Teratons, so the units of  $\tau \times$  consumption (in trillions of \$) are dollars/ton of CO<sub>2</sub>. My baseline calibration implies, for  $n = 1$ , taxes (\$/ton) ranging from 17 (for  $\lambda = 0.02 = r$ ) to 83 (for  $\lambda = 0$ ).

Due to the functional forms assumed here, there is observational equivalence between the model with discount parameters  $r, \lambda$  and a second model with a *constant* discount rate  $R(\lambda; r)$ ; these two models yield the same equilibrium tax. For the baseline parameters in the two-dimensional model, with  $n = 1$ ,  $R(\lambda; r)$  increases from 0.003 (at  $\lambda = 0$ ) to 0.02 (at  $\lambda = r = 0.02$ ).

The tax and elasticity formulae are even simpler if  $S$  is a scalar. The scalar model is less descriptive, because it implies that there is no delay between emissions and changes in damages. However, my interest lies in comparing the relative sensitivity of equilibrium policy to  $n$  and  $\lambda$ . The noteworthy result is that the scalar and two-

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linear captures the most important dynamics contained in earlier models. Appendix B.1.3 provides details of the calculations and also discusses the relation between my two-dimensional model and Gerlagh and Liski (2012) and Nordhaus (2008).

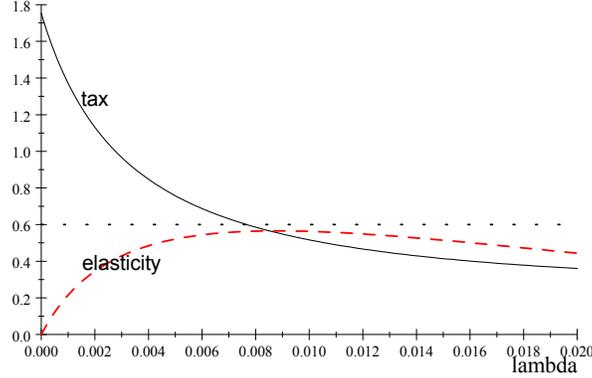


Figure 2: The elasticity of the tax with respect to  $\lambda$ , and the tax (for  $n = 1$ ) under baseline parameters, with  $r = \theta = 0.02$

dimensional models yield the same qualitative result: the tax is much less sensitive to the altruism parameter than it is to the fragmentation parameter. This result is important because in the next section I rely on a scalar model.

In the scalar model (so  $S = s$  and  $B$  is a scalar), with  $\dot{s} = Bs + X$ ,  $\Lambda = B < 0$  and  $\tilde{q} = \frac{1}{\gamma - B}$ . Specializing Corollary 1 implies:

**Corollary 3** *For the case where  $S$  is a scalar,*

$$\tau = \frac{\kappa(\theta - B + \lambda)}{n(\lambda - B)(\gamma - B)}, \text{ and } \varepsilon = \frac{\lambda\theta}{(\lambda - B)(\theta + \lambda - B)};$$

$\varepsilon$  is independent of  $r$  and increases with  $B$ , reaching its upper bound at  $B = 0$ , where  $\varepsilon = \frac{\theta}{\theta + \lambda} < 1$ : as the pollutant becomes more persistent ( $B$  increases toward 0), the tax becomes more sensitive to  $\lambda$ . As  $B$  varies over its domain  $(-\infty, 0)$ , the tax varies monotonically over its range,  $\left(0, \frac{\kappa(\theta + \lambda)}{n(r + \theta)\lambda}\right)$ .

The elasticity  $\varepsilon$  is maximized at  $\lambda = \sqrt{B^2 - B\theta}$ . A 2%/year mortality implies  $\varepsilon < 0.33$  for a 100 year half-life of the stock, and  $\varepsilon < 0.51$  for a 300 year half-life. Figure 3 shows the elasticity contours as  $B$  takes values corresponding to a range of half-lives between 80 and 1000 years, and  $0 < \lambda < 0.02$ .

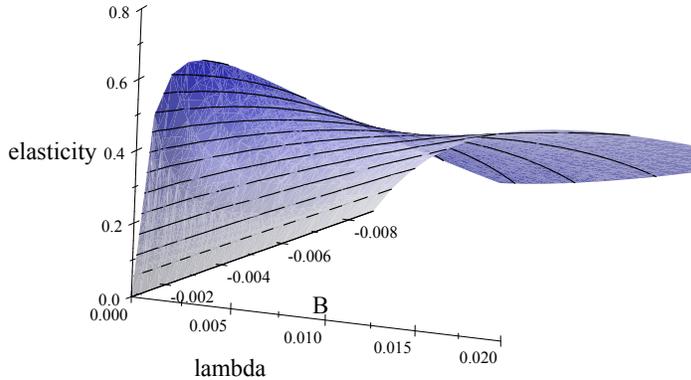


Figure 3: Elasticity of the tax, with respect to  $\lambda$ , for the baseline parameters, for  $\lambda \in [0, 0.02]$  and half-lives ranging from 80 to 1000 years.

### 4.3 Non-dominant MPE

The limit equilibrium is dominant and state-independent, conditional on other agents using state-independent policies. There are many other, qualitatively different, differentiable Markov Perfect equilibria in the infinite horizon setting. The procedure for obtaining the necessary conditions for such equilibria in the *scalar* model is straightforward.<sup>9</sup> This section uses exclusively the scalar model, where  $\dot{s} = Bs + X$  and  $B < 0$ . As noted above, in the limiting equilibrium, the scalar and higher dimensional models yield qualitatively similar conclusions regarding the relative importance of intergenerational altruism and international cooperation. I use the GeaS model, where consumption is  $C(X, t; 1) = A_t X_t^{\alpha_t} e^{-\kappa s t}$ . I set  $\alpha_t = \alpha$ , a constant, in order to consider stationary equilibria;  $A_t$  can depend on time. For non-dominant equilibria, the equilibrium depends on the flow payoff,  $v(\cdot)$ .

The function  $\Phi(s) = \frac{dX(s)}{ds} \Big|_{s=s_\infty}$  is the derivative of aggregate equilibrium emissions, evaluated at the steady state. Because of the mapping between the value

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<sup>9</sup>Numerical methods using function iteration can find a differentiable MPE for a non-scalar model. My experience with these methods suggests that they identify only the “limit equilibrium” (Fujii and Karp 2008). Other papers that consider multiplicity of equilibria under non-constant discounting (with  $n = 1$ ) in the scalar case include Karp, 2005 and Karp and Tsur, 2011.

of this function and the value of the equilibrium steady state stock of GHGs, the function  $\Phi(s)$  provides a simple means of comparing different equilibria. For the limit equilibrium, where emissions are independent of the state variable everywhere, trivially  $\Phi = 0$ . Local asymptotic stability (“stability”) of any equilibrium requires  $B + \Phi < 0$ . Thus,  $\Phi = -B$  is the supremum of permissible values of  $\Phi$ , i.e. those values that correspond to a stable equilibrium. This supremum defines the infimum of the set of feasible steady state stocks, as the next proposition shows. I state the proposition in terms of the utility denominated tax,  $\tau$ , instead of emissions,  $X$ , by using the one-one mapping between the tax and emissions. This transformation makes the results here and in the previous section more easily comparable. The proposition uses the definitions

$$\begin{aligned} Q &= \frac{1}{Bn}\alpha(n-1), \quad L = \left(-\frac{1}{n}\kappa\right)s + \alpha\frac{(1-n)\lambda + B(2n-1) + (\gamma(1-n) - r)}{Bn} \\ h &= \left(-\frac{1}{n}\kappa(B-\gamma) - \frac{1}{n}\kappa(r-\lambda)\right)s + \frac{1}{B}\alpha(B-\lambda)(B-\gamma). \end{aligned} \quad (15)$$

**Proposition 4** *A differentiable MPE,  $\tau(s)$ , and the corresponding annuity function,  $K(s)$ , in the GeaS model solve the pair of ODEs*

$$\frac{d\tau}{ds} = -\frac{\left(\frac{\kappa}{n} + K'(s)\right)\tau - (\lambda - B)\tau^2}{(Bs\tau + \frac{\alpha}{n})} \quad \text{and} \quad \frac{dK}{ds} = \frac{\gamma K(s) - (r - \lambda)\left(\frac{\alpha}{n}\ln\left(\frac{\alpha}{\tau}\right) - \frac{\kappa}{n}s\right)}{Bs + \frac{\alpha}{\tau}} \quad (16)$$

with boundary condition

$$(s_\infty, \tau_\infty, K_\infty) = \left(s_\infty, -\frac{\alpha}{Bs_\infty}, \frac{(r-\lambda)}{\gamma n} [\alpha \ln(-Bs_\infty) - \kappa s_\infty]\right) \quad (17)$$

for  $s_\infty \in \Delta$ , where

$$\Delta = \{s \mid \exists \Phi \text{ for which } B + \Phi < 0 \wedge Q\Phi^2 + L\Phi + h = 0\}. \quad (18)$$

As consistency checks, note that for  $n = 1$  and  $\lambda = r$ , where the game collapses to a standard control problem,  $\Delta$  is a singleton, with  $s = \frac{\alpha(B-r)}{B\kappa}$ . As required, this is the steady state in the control problem ( $n = 1$  and constant PRTP,  $r$ ). The GGR

steady state ( $s = \arg \max_s \alpha \ln X - \kappa s$ , subject to  $0 = s + BX$ ) equals the limit of this value as  $r \rightarrow 0$ ; here the GGR steady state is  $\frac{\alpha}{\kappa}$ .

I obtain the steady state GHG stock corresponding to a value of  $\Phi < -B$  by solving  $Q\Phi^2 + L\Phi + h = 0$  (which is linear in  $s$ ) for  $s$ . The steady state depends on the parameters  $\alpha$  (the elasticity of consumption with respect to emissions) and  $\kappa$  (the damage parameter). I divide the steady state by the GGR steady state, and denote the resulting ratio as  $\Upsilon(\Phi)$ ; this ratio equals the factor by which a particular steady state (indexed by  $\Phi$ ) exceeds the GGR, and it is independent of  $\alpha$  and  $\kappa$ :

$$\Upsilon(\Phi) = -\frac{(n(B - \lambda + \frac{1}{n}\Phi(n-1))(B + \Phi - \gamma) - \Phi(r - \lambda))}{B(\lambda - r - \Phi - B + \gamma)}.$$

Recalling that  $\Phi = 0$  corresponds to the limit equilibria, and  $\Phi = -B$  is the supremum of permissible values of  $\Phi$ , the ratios  $\Upsilon(0)$  and at  $\Upsilon(-B)$  both have simple interpretations:  $\Upsilon(0)$  is the factor by which the steady state in the limit equilibrium exceeds the GGR, and  $\Upsilon(-B)$  is the factor by which the infimum of stable steady states exceeds the GGR. These two ratios are<sup>10</sup>

$$\Upsilon(0) = \frac{n}{B}(B - \lambda)\frac{B - \gamma}{B + r - \lambda - \gamma} \text{ and } \Upsilon(-B) = 1 - n\lambda\frac{\gamma}{B(\gamma - r + \lambda)}.$$

The fact that  $\lim_{\lambda \rightarrow 0} \Upsilon(-B) = 1$  (together with continuity) implies

**Corollary 4** *For all values of  $n$ , there exists a MPE steady state arbitrarily close to the GGR for  $\lambda$  close to 0. This steady state is supported by a policy function corresponding to  $\Phi$  close to its upper bound ( $\Phi = -B$ ).*

Corollary 4 is in the same spirit as Proposition 2, which considers general functional forms with  $n = 1$ . Corollary 4 applies to the GeaS model, for arbitrary  $n$ .

**The climate application** I define the “local criterion” as the ratio of the elasticities of the steady states with respect to  $\lambda$  and to  $n$ . For example, if this ratio is 0.4,

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<sup>10</sup>Negative values of  $\Phi$  are feasible, but these yield higher, i.e. less cooperative, steady state stocks. I do not consider these, because it seems that agents should be able to coordinate on an equilibrium at least as good as the unique limit equilibrium, corresponding to  $\Phi = 0$ .

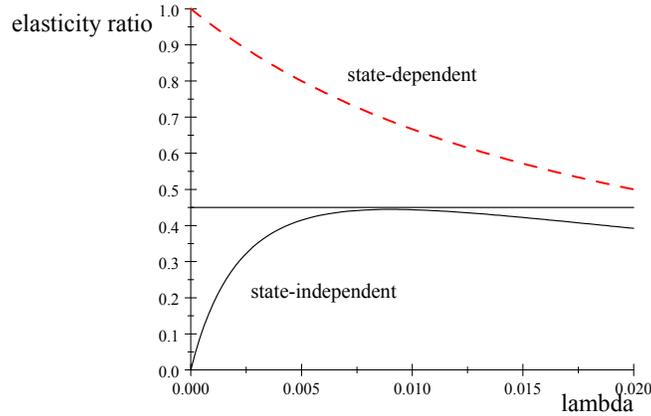


Figure 4: The local criterion: the elasticity of the steady state with respect to  $\lambda$  divided by the elasticity of the steady state with respect to  $n$ . Solid curve: the criterion corresponding to the limit equilibrium ( $\Phi = 0$ ); dashed curve: the criterion for the infimum of feasible steady states ( $\Phi = -B$ ). Half-life of stock = 200 years.

the elasticity of the steady state with respect to  $\lambda$  is 40% as large as the elasticity of the steady state with respect to  $n$ . The solid curve in Figure 4 is the graph of the local criterion for the limit equilibrium, where  $\Phi = 0$ ; the dashed curve is the graph of the criterion for the infimum of permissible steady states, where  $\Phi = -B$ .<sup>11</sup> For both  $\Phi = 0$  and  $\Phi = -B$ , the local criterion is less than 1: the equilibrium steady state is more sensitive to  $n$  than to  $\lambda$ . Using this criterion, the limit equilibrium is more than twice as sensitive to  $n$  as to  $\lambda$ ; the state-dependent equilibrium corresponding to  $\Phi = -B$  is less than twice as sensitive to  $n$  as to  $\lambda$ , and for small  $\lambda$  the equilibrium is approximately as sensitive  $\lambda$  as to  $n$ . This comparison suggests that exclusive attention on the state-independent (limit) equilibrium overstates the extent to which equilibria are more sensitive to  $n$  than to  $\lambda$ .

The local criterion is a natural way to compare the sensitivity of equilibria to the fragmentation and altruism parameter, if we are interested in small changes in these parameters. A “global criterion” provides an alternative, if we are interested in the effect of moving from a given level of altruism and cooperation, to their maximum

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<sup>11</sup>For  $\Phi \neq 0$ , the elasticity of the steady state with respect to  $n$  does not equal 1.

levels, corresponding to  $\lambda = 0$  and  $n = 1$ . Figure 5 provides this information, by showing the graphs of  $\Upsilon(-B)$  (solid curves) and of  $\Upsilon(0)$  (dashed curves) for  $\lambda \in (0, r)$  at three values of  $n$ . The figure reveals two insights.

First, judged by the global criterion, the altruism parameter might be more important than the fragmentation parameter. For example,  $\lambda = r$ , and  $n = 5$ , correspond to low or modest levels of altruism and cooperation. Decreasing  $n$  from 5 to 1, holding  $\lambda$  constant, achieves maximal cooperation. This greater cooperation decreases the steady state stock of atmospheric CO<sub>2</sub> by a factor of about 5, depending on the value of  $\Phi$  one selects. Decreasing  $\lambda$  from  $r$  to close to 0, holding  $n$  fixed, achieves near-maximal altruism. For  $\Phi \approx -B$ , this change decreases the steady state by a factor of about 25. The system can get close to the GGR steady state if altruism is high, even if cooperation is low; it cannot get close to the GGR if altruism is low, even if cooperation is high.

Second, Figure 5 provides information about the circumstances where it is important to consider multiple equilibria. The equilibrium steady state monotonically decreases in  $\Phi$ , so the distance between the solid graph and its dashed counterpart measures the range of stable steady states (for  $\Phi \geq 0$ ). Where this range is small, the equilibrium policies corresponding to the different steady states are similar, at least in the neighborhood of their respective steady states. Thus, small distances between the graphs suggest that multiplicity is unimportant, and large distances suggest the reverse. By this criterion, the multiplicity of equilibria is unimportant for  $n = 1$ , but is important at  $n = 5$ , especially at low values of  $\lambda$ .<sup>12</sup> Gerlagh and Liski (2012) and Iverson (2013) study the limit equilibrium to the Golosov et al. model under non-constant discounting with  $n = 1$ . My result suggests that there may be little loss in generality in focusing exclusively on this equilibrium for  $n = 1$ .<sup>13</sup>

The relation  $\tau(s) = \frac{\alpha}{X(s)}$  in the GeaS model implies  $\text{sign}\left(\frac{d\tau}{ds}\right) = \text{sign}\left(-\frac{dX}{ds}\right)$ . For  $0 < \Phi$  ( $= \left.\frac{dX(s)}{ds}\right|_{s=s_\infty}$ ), the equilibrium tax is decreasing in the state in the neighbor-

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<sup>12</sup>The vertical distance is exactly 0 at  $\lambda = r$  and  $n = 1$ , where, as noted above,  $\Delta$  is a singleton.

<sup>13</sup> Section 3.2 notes that non-uniqueness occurs in constant-discounting differential games and in the one-tribe problem with non-constant discounting. For  $n > 1$  and  $\lambda \neq r$  there are two sources of multiplicity; one of these sources vanishes at  $n = 1$  and is unimportant for small  $n$ ; for large  $n$ , the two sources reinforce each other.

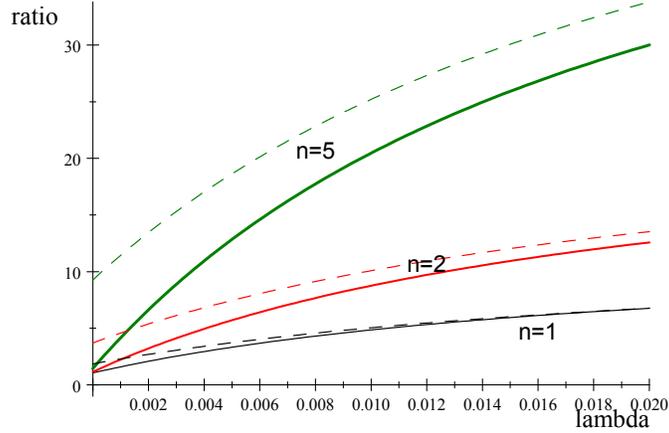


Figure 5: Solid curves:  $\Upsilon(-B)$ , the infimum of stable steady states (as  $\Phi \rightarrow -B$ ) as a ratio of the GGR steady state. Dashed curves:  $\Upsilon(0)$ , the steady state in the dominant equilibrium ( $\Phi = 0$ ) as a ratio of the GGR steady state. Half life of stock = 200 years,  $r = \theta = 0.02$ .

hood of the steady state;  $\frac{ds}{ds} < 0$  implies that taxes are strategic complements. If a tribe reduces its current tax below the equilibrium level, the future stock is above the equilibrium level, causing future taxes to be lower than the equilibrium level: future taxes respond in the same direction as a current deviation. “Better equilibria” (lower steady state stocks) correspond to strategic complementarity. When taxes are strategic complements, tribes have a built-in incentive to use high taxes, in order to keep the stock low, thereby encouraging future decision-makers, both in their own and in other tribes, to use high taxes and also maintain a low stock. This incentive is absent in the limit equilibrium with constant taxes, and the steady state stock is consequently higher there.

Four insights, none of which are self-evident, emerge from this analysis.

1. The multiplicity of differentiable MPE is unimportant in this OLG setting if there is a single tribe, but can be important if there are several (e.g. 5) tribes.
2. The local criterion suggests that the equilibrium is more sensitive to  $n$  than to  $\lambda$ , although consideration of only the state-independent (limit) equilibrium

overstates this comparison.

3. The global criterion illustrated by Figure 5 shows that large changes in altruism or cooperation can overturn this comparison.
4. The equilibria associated with lower steady stocks involve actions that are strategic complements, at least in the neighborhood of the steady state.

**Robustness** An earlier version of this paper considered a model that is non-linear in the state. That model is equivalent to the Quadratic model in Example 2, if one replaces the damage function  $e^{-\kappa s}$  by  $e^{-\kappa s^2}$ , so that consumption is  $A_t \exp(a_t X - \frac{d_t}{2} X^2 - \kappa s^2)$  and utility is  $\ln A_t + a_t X - \frac{d_t}{2} X^2 - \kappa s^2$ , resulting in the familiar linear-quadratic payoff. The greater complexity of this model requires numerical analysis. The qualitative results obtained above also hold in the numerical results under the stationary linear-quadratic specification.

The tractability of the linear-in-state model is appealing, especially for a research question that seeks general insights. However, that model implies that the climate-related loss is linear in the state when measured in utility units, and is concave when measured in output units. The linear-quadratic model implies that the climate-related loss in utility units is convex in  $s$ ; the loss in output units is convex at low  $s$  and concave for large  $s$ . Many environmental economics models assume convex damages. The limit equilibrium to the linear-in-state model implies that strategies are dominant; the limit equilibrium in the linear-quadratic model does not involve dominant strategies. There are strategic interactions in the linear-quadratic model that are absent from the linear-in-state model (for the limit equilibrium). In short, both the linear-in-state and the linear-quadratic models have distinct advantages.

## 5 Discussion

The provision of a long-lived public good, such as a stable climate, depends on the ability of contemporaneous agents to cooperate, and on their degree of altruism

towards future generations. A differential game/overlapping generations model illustrates the sensitivity of climate policy to these two features. A local criterion considers the equilibrium effect of small changes in parameter values, and a global criterion considers the effect of large changes. Analytic results for the climate application rely on a linear-in-state model. Based on the local criterion, the limit equilibrium is much more sensitive to international cooperation than to intergenerational altruism; the same ranking holds for other equilibria, although less strongly. The global criterion can reverse these conclusions. At initial levels of cooperation and altruism, a move to full cooperation increases the provision of the public good, but is unlikely to get the steady state close to the “Green Golden Rule” level. In contrast, a move close to maximal altruism can get the state close to this level, even at low levels of contemporaneous cooperation. There are thus circumstances where altruism is more important than international cooperation, in determining equilibrium provision of the public good.

The multiplicity of equilibria opens the possibility that actions may be strategic complements, rather than strategic substitutes (or dominant, as in the limit equilibrium). The logic of Nash’s noncooperative equilibrium does not doom us to bad outcomes, even if we exclude trigger or other punishment strategies. (Consideration of such strategies increases the equilibrium set, leading to the possibility of still better outcomes.) This conclusion, although not specific to this paper, is nevertheless worth stating, because many non-cooperative models of climate policy build in strategic substitutability, implying that agents have an incentive to undertake less public investment, partly to induce their successors to invest more. This built-in free riding causes the models to be quite pessimistic about the chance of a meaningful climate agreement amongst sovereign nations. Recognition of the possibility of strategic complementarity, where agents have an incentive to increase their current investment partly to induce higher future investment, moderates this pessimism. International negotiations on climate policy are important, even if they do not result in enforceable agreements. Negotiations make coordination on a good equilibrium easier to achieve.

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# A Proofs

The supplementary material provides more details of some of these proofs.

**Proof.** (Proposition 1) Discovering the isomorphism given in part (i) of the proposition requires some work. I began with a discrete stage approximation to the continuous time model and solved the discrete time recursion corresponding to pure altruism to find the discount factor for the discrete time problem. I then took the limit of that discount factor as the length of each stage goes to 0, to obtain the continuous time discount factor. I treat this as the trial solution for the continuous time recursion. With lengthy but straightforward calculations, involving simplification of integrals, I confirm that this trial solution solves the continuous time recursion defined in equation 3. Comparing this trial solution to the discount factor in equation 2 gives part (i) of the proposition. Part (ii) uses part (i) and the fact that for  $D(t)$  given by equation (2),  $\frac{dD}{d\lambda} < 0$  for  $t > 0$ . ■

The proof of Lemma 1 involves straightforward calculation.

**Proof.** (Proposition 2) I use a Taylor expansion to evaluate  $K'(S_\infty)$ . This information, together with the necessary conditions for the auxiliary control problem, evaluated at the steady state, and the requirement that the steady state is locally asymptotically stable, produces a set of  $S$ , each element of which can be supported as a locally stable steady state in a MPE. I then show that for  $n = 1$ , the infimum of this set approaches the GGR as  $\lambda \rightarrow 0$ . ■

**Proof.** (Proposition 3) The argument proceeds by showing that if all other tribes, and if future planners in one's own tribe, use state-independent decision rules, then the solution to the auxiliary control problem is linear in the state, with value function and control rule given in the proposition. ■

The subsequent three corollaries involve straightforward calculation.

**Proof.** (Proposition 4) This proof parallels the method that Tsutsui and Mino (1990) use to find non-linear equilibria in a linear quadratic differential game with constant discounting. However, the endogenous function  $K(S)$  in the game with non-constant discounting creates an additional dimension, ruling out Tsutsui and Mino's use of two-dimensional phase portrait analysis. Extending their approach

to the non-constant discounting setting, one manipulates the necessary conditions to the auxiliary control problem (equation 7) to obtain a two-dimensional system of ordinary differential equations that a stationary differentiable MPE must solve. There is some latitude in choosing which two variables to use for this system, but a natural choice consists of emissions (or the tax) and the annuity function  $K$ , both expressed as a function of the state variable. The condition that the steady state of the resulting equilibrium state trajectory be locally asymptotically stable identifies an open interval; the steady state to a stable MPE lies in this interval. Values of state variables in this interval, and the corresponding steady state values of the control variable, are boundary conditions for the system of ODEs. ■

## B Supplementary material

This material contains supplementary material that can be made available for online publication. I begin by providing more details of the proofs that are only sketched above, and then elaborate on three issues mentioned in the text. I omit a more detailed proof of Proposition 1, because that proof consists merely of several pages of calculation, verifying that the trial solution indeed satisfies the recursion.

**Proof.** (Lemma 1) For small  $\varepsilon > 0$  define the  $\tau$  as the smallest time beyond which  $\left| \frac{U(t) - U_\infty}{U_\infty} \right| \leq \varepsilon$ . That is

$$\tau = \inf_t \left\{ t : \left| \frac{U(t) - U_\infty}{U_\infty} \right| \leq \varepsilon \forall \tau \geq t \right\}.$$

Note that  $\tau < \infty$ . Use ■

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \left( \frac{\int_0^\infty D(t)U(t)dt}{U_\infty} \right) \\ &= \lim_{\lambda \rightarrow 0} \frac{\lambda \left[ \left( \int_0^\tau D(t)U(t)dt \right) + \left( U_\infty \int_\tau^\infty D(t)dt \right) + \left( \int_\tau^\infty U(t)D(t)dt - U_\infty \int_\tau^\infty D(t)dt \right) \right]}{U_\infty}. \end{aligned} \quad (19)$$

Consider each of the three terms on the right side of this equation. The fact that  $\tau < \infty$  implies that

$$\lim_{\lambda \rightarrow 0} \lambda \int_0^\tau D(t)U(t)dt = 0$$

A calculation confirms that

$$\int_\tau^\infty \left( \left( \frac{\lambda - r}{\lambda - \gamma} \right) e^{-\gamma t} - \frac{\theta}{\lambda - \gamma} e^{-\lambda t} \right) dt = \frac{-e^{-\gamma \tau} \lambda^2 + e^{-\gamma \tau} \lambda r + \theta e^{-\lambda \tau} \gamma}{(-\lambda + \gamma) \gamma \lambda}.$$

Taking the limit as  $\lambda \rightarrow 0$  of this expression, implies that the second term on the second line of equation (19) equals  $\frac{\theta}{\gamma}$ . By definition of  $\tau$ ,

$$\left| \frac{\int_\tau^\infty U(t)D(t)dt - U_\infty \int_\tau^\infty D(t)dt}{U_\infty} \right| < \varepsilon \int_\tau^\infty D(t)dt.$$

The limit as  $\lambda \rightarrow 0$  of the last expression is  $\varepsilon \frac{\theta}{\gamma}$ .

**Proof.** (Proposition 2) I first derive the necessary and sufficient condition, for general  $n$ , that must be satisfied at a stable steady state in a differentiable MPE. I then specialize to  $n = 1$  and show that the boundary of the open interval of states that satisfies this condition is arbitrarily close to the GGR for  $\lambda$  close to 0. Because I am interested in the case where  $\lambda$  is small, I assume throughout that  $\lambda < r$ .

Denote agent  $i$ 's policy function as  $\chi(S)$  and the aggregate decision as  $\Psi \equiv n\chi$ , so  $\Psi' = n\chi'$ . Define

$$z = (f_S + f_\Psi \Psi')|_\infty,$$

where the subscript  $\infty$  denotes that the function is evaluated at a steady state. Stability requires  $z < 0$ . For  $S_t \approx S_\infty$ , a first order approximation gives

$$S_{t+\tau} = e^{z\tau} S_t + S_\infty (1 - e^{z\tau}) + o(S_t - S_\infty) \implies \frac{dS_{t+\tau}}{dS_t} \approx e^{z\tau} \quad (20)$$

for  $\tau \geq 0$ . Equation (8) implies

$$K'(S_t) = (r - \lambda) \int_0^\infty e^{-\gamma\tau} (u_S(S_{t+\tau}, \chi(S_{t+\tau})) + u_x(S_{t+\tau}, \chi(S_{t+\tau}))\chi'(S_{t+\tau})) \frac{dS_{t+\tau}}{dS_t} d\tau. \quad (21)$$

Using equation (20) and evaluating equation (21) at  $S_t = S_\infty$  gives

$$\begin{aligned} K'(S_\infty) &= (r - \lambda) (u_S + u_x \chi')|_\infty \int_0^\infty e^{-\gamma t} e^{z t} dt \\ &= \frac{(r - \lambda)(u_S + u_x \chi')|_\infty}{\gamma - z} = \frac{(r - \lambda)(u_S + u_x \frac{\Psi'}{n})|_\infty}{\gamma - z}. \end{aligned} \quad (22)$$

The Hamiltonian corresponding to the fictitious optimal control problem in equation (7) is

$$H = u(S, x) - K(S) + \mu f\left(S, x + \frac{n-1}{n}\Psi(S)\right),$$

where  $\mu$  is the current value costate variable. The necessary conditions for optimality are

$$u_x + \mu f_x = 0 \implies \mu = -\frac{u_x}{f_x} \text{ and } \dot{\mu} = \lambda\mu - \left(u_S - K' + \mu \left(f_S + \frac{n-1}{n}f_x \Psi'\right)\right).$$

Using the first necessary condition and evaluating the costate equation at a steady state (setting  $\dot{\mu} = 0$ ) gives the condition

$$\begin{aligned} & \left[ -u_S + K' + \frac{u_x}{f_x} \left( f_S + \frac{n-1}{n} f_x \Psi' - \lambda \right) \right]_{|\infty} = \\ & \left[ -u_S + \frac{(r-\lambda) \left( u_S + u_x \frac{\Psi'}{n} \right)}{\gamma-z} + \frac{u_x}{f_x} \left( f_S + \frac{n-1}{n} f_x \Psi' - \lambda \right) \right]_{|\infty} = 0, \end{aligned} \quad (23)$$

where the first equality uses equation (22). Using the definition of  $z$  and rearranging the second line of equation (23) implies that  $\Psi' = \Psi'(S_\infty)$  is a solution to the quadratic equation

$$Q \times (\Psi')^2 + L \times \Psi' + C = 0 \quad (24)$$

with<sup>14</sup>

$$\begin{aligned} Q & \equiv u_x \frac{n-1}{n} f_x \\ L & \equiv \left( (r-\lambda) \frac{1}{n} + \frac{n-1}{n} (\gamma - f_S) - \left( (f_S - \lambda) - \frac{u_S}{u_x} f_x \right) \right) u_x \\ C & \equiv (-\lambda - \theta + f_S) u_S + \frac{u_x}{f_x} (f_S - \lambda) (\gamma - f_S) \end{aligned}$$

Hereafter I set  $n = 1$ , so

$$\begin{aligned} \Psi'_{|\infty} & = \frac{(\lambda + \theta - f_S) \frac{u_S}{u_x} - \frac{1}{f_x} (f_S - \lambda) (\gamma - f_S)}{r - f_S + \frac{u_S}{u_x} f_x} \implies \\ z & = f_S + f_x \frac{(\lambda + \theta - f_S) \frac{u_S}{u_x} - \frac{1}{f_x} (f_S - \lambda) (\gamma - f_S)}{r - f_S + \frac{u_S}{u_x} f_x} = \\ & \frac{\theta \left( \frac{u_S}{u_x} - \frac{f_S}{f_x} \right) f_x + \lambda \left( \frac{u_S}{u_x} - \frac{f_S}{f_x} + \frac{\gamma}{f_x} \right) f_x}{r + \left( \frac{u_S}{u_x} - \frac{f_S}{f_x} \right) f_x}. \end{aligned} \quad (25)$$

The GGR is a solution to

$$\left( \frac{u_S}{u_x} - \frac{f_S}{f_x} \right) u_x = 0.$$

I define the state and the control variables so that  $u_S < 0$  and  $u_x < 0$ . For example, in the climate model,  $S$  is the stock of atmospheric carbon and  $x$  is the level of abatement, so the flow of utility is decreasing in both variables. These

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<sup>14</sup>The function  $C$  defined here is unrelated to consumption, used in the section on climate policy.

definitions (the state variable is a “bad” and the action is costly) mean that the model is sensible if and only if  $f_x < 0$  (so that incurring a cost reduces the public bad). Given the concavity of the static optimization problem (which determines the GGR), a stock level slightly greater than the GGR satisfies

$$\left(\frac{u_S}{u_x} - \frac{f_S}{f_x}\right) u_x = \varepsilon < 0 \quad \text{or} \quad \left(\frac{u_S}{u_x} - \frac{f_S}{f_x}\right) = \frac{\varepsilon}{u_x} > 0, \quad (26)$$

for  $\varepsilon$  small in absolute value. Such a stock level yields approximately the maximum steady state level of utility. (Given that the costly action  $x$  reduces the stock, it would never be part of an equilibrium to drive the stock below the optimal static level.)

Using equation (26) in (25) gives

$$z = \frac{\theta \frac{\varepsilon}{u_x} f_x + \lambda \left(\frac{\varepsilon}{u_x} + \frac{\gamma}{f_x}\right) f_x}{r + \frac{\varepsilon}{u_x} f_x}.$$

The denominator is positive for small  $\varepsilon$ . For  $\varepsilon$  small in absolute value (so that  $\frac{\varepsilon}{u_x} f_x + \gamma > 0$ ), the numerator is negative if and only if

$$\frac{-\theta \frac{\varepsilon}{u_x} f_x}{\left(\frac{\varepsilon}{u_x} f_x + \gamma\right)} > \lambda,$$

i.e. if and only if  $\lambda$  is sufficiently small, as was to be shown. ■

**Proof.** (Proposition 3) Because  $P$  is of full rank,  $B = P\Lambda P^{-1}$ . In a symmetric MPE, i.e. where all tribes emit  $x(t) = \chi(t)$ ,  $\dot{S}(t) = BS(t) + bn\chi(t)$ . Here, the equilibrium value of the state  $t$  periods in the future, given the current value  $S_0$  is:  $S_t = Pe^{\Lambda t} P^{-1} S_0 + H(t)$ , where  $H(t)$  depends on the trajectory of controls from time 0 to  $t$ . If  $\chi$  is a constant, then  $H(t) = P\Omega(t) Pn^{-1}(bn\chi)$ . Because all eigenvalues are negative,  $\Omega(t)$  is a diagonal matrix with element  $\frac{\exp(\Lambda_i t - 1)}{\Lambda_i}$  in the  $i$ 'th diagonal position.

Under the assumption that the policy maker in tribe  $i$  at time  $t$  expects future

emissions to be independent of the state, and using the flow payoff  $v(x, t; n) - \frac{\kappa s}{n} = \frac{1}{n} [v(nx, t; 1) - \kappa i'_1 S]$  and equation (8), the annuity function is :

$$\begin{aligned} K(S_t, t) &= \frac{(r-\lambda)}{n} \int_0^\infty e^{-\gamma\tau} [(v(n\chi(t+\tau), t+\tau; 1) - \kappa i'_1 S_{t+\tau})] d\tau \\ &= \frac{(r-\lambda)}{n} \int_0^\infty e^{-\gamma\tau} [(v(n\chi(t+\tau), t+\tau; 1) - \kappa i'_1 (Pe^{\Lambda\tau} P^{-1} S_t + H(t+\tau)))] d\tau. \end{aligned}$$

From this formula, it is apparent that  $K$  is linear in  $S$ ,  $K(S_t, t) = q_{0,t} + q' S_t$ , with the gradient  $q'$  given by the first equation in (10). If future policies are constant, and  $v$  does not depend on time, then  $q_{0,t}$  is a constant,  $q_0$ .

Using  $K = q_{0,t} + q' S$  and the utility flow  $v(x, t; n) - \frac{\kappa s}{n}$  in equation (7), produces the dynamic programming equation (DPE)

$$\begin{aligned} \lambda J(S, t) &= \max_x \{v(x, t; n) - \frac{\kappa}{n} i'_1 S \\ &- (q_{0,t} + q' S) + J'_S(S, t) [BS + b(x + (n-1)\chi_t)]\}. \end{aligned}$$

Because this problem is linear in the state, the obvious trial solution is a linear function,  $J(S, t) = g_{0,t} + g'_t S$ . Using this trial solution, the DPE becomes

$$\begin{aligned} \lambda (g_{0,t} + g'_t S) &= \max_x \{v(x, t; n) + g'_t b x\} \\ &- \frac{\kappa}{n} i'_1 S - (g_{0,t} + q' S) + g'_t [BS + b(n-1)\chi]. \end{aligned}$$

The first order condition (which is sufficient due to concavity of  $v$ ) is

$$\frac{\partial v(x, t; n)}{\partial x} + g'_t b = 0. \quad (27)$$

The solution,  $x^*$ , possibly depends on time, but is independent of the state. Substituting the optimal flow payoff into the DPE gives the maximized DPE

$$\begin{aligned} \lambda (g_{0,t} + g'_t S) &= v(x^*(t), t; n) + g'_t b x^* \\ &- \frac{\kappa}{n} i'_1 S - (g_{0,t} + q' S) + g'_t [BS + b(n-1)\chi]. \end{aligned}$$

Equating coefficients of  $S$  gives

$$\lambda g'_t = -\frac{\kappa}{n} i'_1 - q' + g'_t B.$$

Because  $B$  and  $q'$  are constants,  $g'_t$  is also a constant,  $g'$ , given by the last equation in (10). If  $v$  is independent of  $t$ , then equilibrium emissions are also constant, in which case,  $q_0$ , and  $g_0$  are also constants.

In summary, regardless of planner  $i$ ,  $t$ 's beliefs about other planners' state-independent policies, planner  $i$ ,  $t$ 's optimal policy is the constant given by  $x = \arg \max_x \{v(x, t; n) + g'bx\}$ . Because  $g$  is independent of other planners' policies, the equilibrium policy is dominant both respect to actions by future planners in one's own tribe, and by all current and future planners in other tribes. ■

**Proof.** (Corollary 1) The first order condition for the problem in equation (11) is  $\frac{dv(x;n)}{dx} = -g(n)'b$ , where I make the dependence of  $g$  on  $n$  explicit for emphasis, and I drop the argument  $t$  in  $v$  to simplify notation. By concavity of  $v$ , this first order condition is sufficient. Using the definition  $v(x; n) = \frac{1}{n}v(nx; 1)$ , and the chain rule, I have  $\frac{dv(x;n)}{dx} = \frac{1}{n} \frac{dv(nx;1)}{d(nx)} \frac{d(nx)}{dx} = \frac{dv(X(n);1)}{dX}$ . Using this relation in the first order condition gives  $\frac{dv(X(n);1)}{dX} = -g(n)'b$ . Using the definition of  $g(n)'$  from equation (10) in this first order condition gives equation (12).

By inspection, the absolute value of the elasticity of this tax with respect to  $n$  is 1. In order to obtain equation (13), use

$$\begin{aligned} \frac{d\tau}{d\lambda} &= -\frac{\kappa}{n} \frac{d[(i'_1 - (r - \lambda)\tilde{q}')^{(B - \lambda I)^{-1}b}]}{d\lambda} = \\ &= -\frac{\kappa}{n} [\tilde{q}'(B - \lambda I)^{-1}b + (i'_1 - (r - \lambda)\tilde{q}')^{(B - \lambda I)^{-2}b}] = \\ &= -\frac{\kappa}{n} [\tilde{q}' + (i'_1 - (r - \lambda)\tilde{q}')^{(B - \lambda I)^{-1}}] (B - \lambda I)^{-1}b. \end{aligned}$$

Multiplying this expression by  $-\frac{\lambda}{\tau}$  to convert to an absolute value elasticity, gives equation (13). ■

**Proof.** (Corollary 2) For the GeaS model, where  $v(x; n) = \frac{\alpha}{n} \ln(nx)$ , the first order condition for the problem in (11) is  $\frac{\alpha}{nx} + g'b = 0 \implies x = -\frac{\alpha}{ng'b} \implies nx = X = -\frac{\alpha}{g'b} = \frac{-\alpha n}{\kappa(i'_1 - (r - \lambda)\tilde{q}')^{(B - \lambda I)^{-1}b}}$ . Emissions are positive ( $g'b < 0$ ), so aggregate emissions are

an increasing linear function of  $n$ .

For the quadratic model,  $v(x, t; n) = a_t x - \frac{d_t n}{2} x^2$ . Suppressing the time subscripts, the first order condition for the problem in (11) is  $a - dnx + g'b = 0 \implies x = \frac{a+g'b}{dn} \implies nx = X = \frac{a+g'b}{d}$ . Using  $\frac{d(g'b)}{dn} = -\frac{\kappa}{n^2} (i'_1 - (r - \lambda) \tilde{q}') (B - \lambda I)^{-1} b > 0$ , aggregate emissions is an increasing strictly concave function of  $n$ . ■

**Proof.** (Corollary 3) For the scalar case,  $\tilde{q} = \frac{1}{\gamma - B}$ . Using this result in equation (12) produces the formula for  $\tau^*$  in the scalar case. Straightforward calculation establishes the other claims in the corollary. ■

**Proof.** (Proposition 4) I use the necessary conditions for a differentiable MPE to find two ordinary differential equations (ODEs) in  $S$ , with dependent variables  $X$  and  $K$  (emissions and the annuity function). For ease of interpretation, I then translate the ODEs in  $X$  and  $K$  into equivalent ODEs in  $\tau$  (the tax in units of utility) and  $K$ . I then use the stability condition to find the set of stable steady states in a MPE. ■

*Construct the ODEs in  $X$  and  $K$ .* For a given symmetric decision rule,  $\chi(S)$  define aggregate emissions as  $\Psi(S) = n\chi(S)$ . In the scalar linear model,  $\dot{S} = BS + \sum_i x_i$ . With  $\sum_{j \neq i} x_j = \frac{n-1}{n} \Psi(S)$ , a representative tribe (so I drop the tribal index) faces the equation of motion

$$\dot{S} = F(S, x) = BS + \frac{n-1}{n} \Psi(S) + x.$$

The GeaS model allows non-stationarity, entering via the time-dependent term  $A_t$ . Thus, the annuity function,  $K$ , also has time as an argument. This fact requires a slight change in notation, but because the non-stationarity enters additively, it does not complicate the derivations. In the GeaS model,  $v(x; n) = \frac{\ln A_t}{n} + \frac{\alpha}{n} \ln(nx)$ . To incorporate the non-stationarity, I replace the definition of the annuity, equation 8, by

$$\begin{aligned} \tilde{K}(S_t, t) &= (r - \lambda) \int_0^\infty e^{-\gamma\tau} \left( v(x_{t+\tau}; n) - \frac{\kappa}{n} S_{t+\tau} \right) d\tau \\ &= (r - \lambda) \int_0^\infty e^{-\gamma\tau} \left( \frac{\ln A_{t+\tau}}{n} + \frac{\alpha}{n} \ln(nx_{t+\tau}) - \frac{\kappa}{n} S_{t+\tau} \right) d\tau \\ &= \bar{A}(t) + K(S_t), \end{aligned}$$

which uses the definitions

$$\begin{aligned}\bar{A}(t) &= (r - \lambda) \int_0^\infty e^{-\gamma\tau} \left( \frac{\ln A_{t+\tau}}{n} \right) d\tau \\ K(S_t) &= (r - \lambda) \int_0^\infty e^{-\gamma\tau} \left( \frac{\alpha}{n} \ln(nx_{t+\tau}) - \frac{\kappa}{n} S_{t+\tau} \right) d\tau.\end{aligned}$$

In a symmetric equilibrium, the function  $K$  satisfies the ODE

$$\begin{aligned}\gamma K(S) &= (r - \lambda) \left( \frac{\alpha}{n} \ln(X) - \frac{\kappa}{n} S \right) + K'(S) (BS + X) \Rightarrow \\ K'(S) &= \frac{\gamma K(S) - (r - \lambda) \left( \frac{\alpha}{n} \ln(X) - \frac{\kappa}{n} S \right)}{BS + X}.\end{aligned}\tag{28}$$

The second line of equation 28 is the ODE for  $K$ . The numerator and denominator are both 0, evaluated at a steady state. I therefore use a Taylor expansion (below) to find the value of  $K'$  at a steady state.

The auxiliary control problem for the GeaS model (see equation 7) is

$$\begin{aligned}\tilde{J}(S_t, t) &= \max \int_0^\infty e^{-\lambda\tau} \left( \frac{\ln A_{t+\tau}}{n} + \frac{\alpha}{n} \ln(nx_{t+\tau}) - \frac{\kappa}{n} S_{t+\tau} - \tilde{K}(S_{t+\tau}) \right) d\tau \\ &\quad \text{subject to } \dot{S} = F(S, x) \\ &= \int_0^\infty e^{-\lambda\tau} \left( \frac{\ln A_{t+\tau}}{n} - \bar{A}_{t+\tau} \right) d\tau + \max \int_0^\infty e^{-\lambda\tau} \left( \frac{\alpha}{n} \ln(nx_{t+\tau}) - \frac{\kappa}{n} S_{t+\tau} - K(S_{t+\tau}) \right) d\tau \\ &\quad \text{subject to } \dot{S} = F(S, x).\end{aligned}$$

Defining

$$J(S_t) = \tilde{J}(S_t, t) - \int_0^\infty e^{-\lambda t} \left( \frac{\ln A_t}{n} - \bar{A} \right) dt,$$

produces the stationary auxiliary control problem

$$J(S) = \max \int_0^\infty e^{-\lambda t} \left( \frac{\alpha}{n} \ln(nx) - \frac{\kappa}{n} S - K(S) \right) dt \quad \text{subject to } \dot{S} = F(S, x).$$

Defining  $\mu$  as the costate variable for  $S$ , the Hamiltonian and necessary conditions

to this problem (taking the function  $K$  and  $\Psi$  as given for the time being) are

$$\begin{aligned}
H &= \max_x \left[ \frac{\alpha}{n} \ln(nx) - \frac{\kappa}{n} S - K(S) + \mu \left( BS + \frac{n-1}{n} \Psi(S) + x \right) \right] \\
\frac{\alpha}{n} \frac{1}{nx} n + \mu &= 0 \\
\dot{\mu} &= \lambda \mu + \frac{\kappa}{n} + K'(S) - \mu \left( B + \frac{n-1}{n} \Psi'(S) \right) \\
&= \frac{\kappa}{n} + K'(S) + \mu \left( \lambda - B - \frac{n-1}{n} \Psi'(S) \right).
\end{aligned} \tag{29}$$

Evaluating the necessary conditions at a symmetric equilibrium (replacing  $\Psi'$  with  $X'$ ) gives

$$\begin{aligned}
\mu &= -\frac{\alpha}{X} \\
\dot{\mu} &= \frac{\kappa}{n} + K'(S) + \mu \left( \lambda - B - \frac{n-1}{n} X'(S) \right) \\
&= \frac{\kappa}{n} + K'(S) - \frac{\alpha}{X} \left( \lambda - B - \frac{n-1}{n} X'(S) \right)
\end{aligned} \tag{30}$$

Differentiating the first equation with respect to time, using the second, gives

$$\frac{\alpha}{X^2} \dot{X} = \frac{\kappa}{n} + K'(S) - \frac{\alpha}{X} \left( \lambda - B - \frac{n-1}{n} X'(S) \right)$$

Dividing this equation by  $\dot{S} = BS + X$ , using  $\frac{\dot{X}}{\dot{S}} = \frac{dX}{dS} = X'(S)$  gives

$$\begin{aligned}
\frac{\frac{\alpha}{X^2} \dot{X}}{\dot{S}} &= \frac{\frac{\kappa}{n} + K'(S) - \frac{\alpha}{X} \left( \lambda - B - \frac{n-1}{n} X'(S) \right)}{BS + X} \\
\frac{\alpha}{X^2} \frac{\dot{X}}{\dot{S}} &= \frac{\alpha}{X^2} X' = \frac{\left[ \frac{\kappa}{n} + K'(S) - \frac{\alpha}{X} \left( \lambda - B - \frac{n-1}{n} X'(S) \right) \right]}{BS + X}
\end{aligned}$$

Solving for  $X'$  gives

$$X' = \frac{\left[ \frac{\kappa}{n} + K'(S) - \frac{\alpha}{X} (\lambda - B) \right]}{\alpha \left( BS + \frac{1}{n} X \right)} X^2 \tag{31}$$

A differentiable MPE must solve the ODEs 28 and 31.

*Translation from  $X$  to  $\tau$ .* In the GeaS model,  $\tau = \frac{\alpha}{X}$ , or  $X = \frac{\alpha}{\tau}$ , so  $\frac{dX}{dS} = -\frac{\alpha}{\tau^2} \frac{d\tau}{dS}$ . Using this fact in equations 28 and 31 produces equation 16.

*Find the set of boundary conditions for these ODEs.* A steady state, a triple

$S_\infty$ ,  $X_\infty$  and  $K_\infty$ , is a feasible boundary condition for the ODEs if it is locally asymptotically stable. Denote  $\Phi$  as the value of  $X'$  at a steady state, i.e. where  $0 = BS + X$ , and define  $z = B + \Phi$ . Stability requires  $z < 0$ . Equation 22, repeated here, is

$$K'(S_\infty) = \frac{(r - \lambda) \left( u_s + u_x \frac{\Psi'}{n} \right) \Big|_\infty}{\gamma - z}. \quad (32)$$

In the GeaS model,  $u = v(x; n) - \frac{\kappa}{n} s = \frac{\alpha}{n} \ln(nx) - \frac{\kappa}{n} s$ , so  $u_s = -\frac{\kappa}{n}$  and  $u_x = \frac{\alpha}{n} \frac{1}{nx} n = \frac{\alpha}{X}$ . Using these results and equation 32 gives

$$\begin{aligned} K'(S_\infty) &= \left( \frac{(r-\lambda) \left( -\frac{\kappa}{n} + \frac{\alpha}{X} \frac{X'}{n} \right)}{\gamma - B - X'} \right) \Big|_\infty \\ &= \left( \frac{(r-\lambda) \left( -\frac{\kappa}{n} + \frac{\alpha}{-BS} \frac{\Phi}{n} \right)}{\gamma - B - \Phi} \right) \Big|_\infty, \end{aligned} \quad (33)$$

where the second equality uses the fact that  $X = -BS$  at a steady state, and the notation  $\Phi = X' \Big|_\infty$ .

Using equation 30 and setting  $\dot{\mu} = 0$  gives

$$0 = \frac{\kappa}{n} + K'(S) - \frac{\alpha}{X} \left( \lambda - B - \frac{n-1}{n} X'(S) \right).$$

Using this equation and the second line of equation 33, and  $X = -BS$  at a steady state, I obtain

$$\begin{aligned} 0 &= \frac{\kappa}{n} + K'(S) - \frac{\alpha}{-BS} \left( \lambda - B - \frac{n-1}{n} \Phi \right) \\ &= \frac{\kappa}{n} + \left[ \frac{(r-\lambda) \left( -\frac{\kappa}{n} + \frac{\alpha}{-BS} \frac{\Phi}{n} \right)}{\gamma - B - \Phi} \right] - \frac{\alpha}{-BS} \left( \lambda - B - \frac{n-1}{n} \Phi \right). \end{aligned}$$

Rearranging this equation gives

$$0 = Q\Phi^2 + L\Phi + h,$$

using the definitions in equation 15. This equation, together with the stability requirement  $B + \Phi < 0$ , establish the condition in equation 18. The steady state

values in equation 17 follow from the fact that  $X = -BS$  at a steady state, and from evaluating the definition of  $K$  at a steady state.

## B.1 Additional supporting material

I provide the necessary conditions for the case not considered in the text, then discuss the issue of sufficiency, and then discuss calibration of the two dimensional GeaS model.

### B.1.1 Equilibrium conditions

There are two cases under the exponentially distributed lifetime, because  $\lim_{t \rightarrow \infty} \eta(t)$  depends on whether  $\lambda < r$  or  $\lambda > r$ . For the exponential case with  $0 < \lambda \leq r$ , and using the differentiability of  $J(S)$  (already assumed in deriving the problem comprised of (7) and (8), a necessary condition for the MPE is that

$$x_t = \chi(S_t) \equiv \arg \max (u(S_t, x_t) - K(S_t) + J_S(S) F(S, x)), \quad (34)$$

and that  $J(S)$  satisfy the dynamic programming equation<sup>15</sup>

$$\lambda J(S) = (u(S, \chi(S)) - K(S) + J_S(S) F(S, \chi(S))). \quad (35)$$

With  $\lambda > r$ , where  $\lim_{t \rightarrow \infty} \eta(t) = \gamma$ , the fictitious control problem is

$$J(S) = \max \int_0^\infty e^{-\gamma t} (u(S_t, x_t) - K(S_t)) d\tau \quad \text{subject to } \dot{S} = F(S, x), \quad (36)$$

with the side condition (definition):

$$K(S_t) \equiv \int_0^\infty D(\tau) (\eta(\tau) - \gamma) u(S_\tau^*, \chi(S_\tau^*)) d\tau. \quad (37)$$

Equation (2) and the first line of equation (4) imply  $D(t) (\eta(t) - \gamma) = -\theta e^{-\lambda t}$  so

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<sup>15</sup>Appendix B.1.2 explains why these necessary conditions, together with the definition in equation (8), are also sufficient for a MPE.

equation (37) simplifies to

$$K(S_t) = -\theta \int_0^\infty e^{-\lambda\tau} u(S_{t+\tau}^*, \chi(S_{t+\tau}^*)) d\tau. \quad (38)$$

The integral in equation (38) is the present discounted value of the equilibrium future flow of payoff, computed using the discount rate  $\lambda$ . Thus,  $-K(S_t)$  is an annuity, which if received in perpetuity and discounted at  $\theta$  (the constant birth = death rate), equals the value of this future stream of payoff. The flow payoff in the fictitious control problem equals the flow payoff in the original model, plus this annuity. A necessary condition for the MPE is that

$$x_t = \chi(S_t) \equiv \arg \max (u(S_t, x_t) - K(S_t) + J_S(S) F(S, x)), \quad (39)$$

and that  $J(S)$  satisfy the dynamic programming equation

$$\gamma J(S) = (u(S, \chi(S)) - K(S) + J_S(S) F(S, \chi(S))). \quad (40)$$

### B.1.2 Sufficiency

The discussion of sufficiency in Karp (2007) is misleading, and I take this opportunity to clarify it. The endogeneity of the function  $K(S)$ , and the resulting difficulty in determining its curvature, makes it difficult to apply standard sufficiency conditions for optimal control problems, to the auxiliary control problem defined by equations 36 and 37. However, the auxiliary control problem is merely a device for describing the equilibrium to the sequential game induced by non-constant discounting; for that purpose, we use only the necessary conditions to the auxiliary problem. The maximization problem in equation (21) of Karp (2007) is a statement of the problem for the planner in a particular period in a discrete time setting, under the assumption of Markov perfection. Equation (5) of that paper (equivalently, equation 40 above) is the limiting form of the discrete time condition, as the length of a period of commitment goes to 0. Therefore, provided that we are willing to restrict attention to the limiting game (as the length of a period goes to zero in the discrete time

game), and provided that the value function is differentiable, a sufficient condition for the MPE is that the control rule satisfy equation 39 above, and that the value function satisfy the dynamic programming equation 40.

The primitive functions of some interesting optimal control problems do not have the curvature need to satisfy familiar sufficient conditions. Sufficiency in optimal control problems is therefore sometimes a difficult issue, and the analysis sometimes proceeds without reference to sufficiency. The difficulty arises because sufficiency is a global property in optimal control problems. In contrast, sufficiency is a much simpler issue in the type of sequential game induced by non-constant discounting and the requirement of Markov perfection. In this game, each of the succession of social planers chooses a single action; given her beliefs about successors' policy function, each policy maker thus solves a static optimization problem. Because each of the policymakers treats the functions  $J(S)$  and  $K(S)$  as predetermined (although they are endogenous to the game), sufficiency requires (in the limit as  $\varepsilon \rightarrow 0$ ) only that  $x = \chi(S)$  maximizes  $(U(S_t, x_t) + J_S(S) f(S, x))$ .

For the climate model,  $U$  is concave in the control and  $f$  is linear, so the necessary condition to  $\max_x (U(S_t, x_t) + J_S(S) f(S, x))$  is also sufficient.

### B.1.3 The two-dimensional GeaS model

The parameters in equation ?? imply

$$B = \begin{pmatrix} \varsigma & \varrho \\ 0 & v \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix of eigenvalues and eigenvectors corresponding to  $B$  are

$$\Lambda = \begin{pmatrix} v & 0 \\ 0 & \varsigma \end{pmatrix} \text{ and } P = \begin{pmatrix} -\frac{\varrho}{\varsigma-v} & 1 \\ 1 & 0 \end{pmatrix}. \quad (41)$$

Solving the pair of differential equations gives

$$\begin{pmatrix} s(t) \\ G(t) \end{pmatrix} = P e^{\Lambda t} P^{-1} \begin{pmatrix} s(0) \\ G(0) \end{pmatrix} + P \begin{pmatrix} e^{vt} \int_0^t e^{-vy} X(y) dy \\ \frac{\rho}{\zeta - v} e^{\zeta t} \int_0^t e^{-\zeta y} X(y) dy \end{pmatrix}.$$

Using this equation and

$$P e^{\Lambda t} P^{-1} = \begin{pmatrix} e^{t\zeta} & \frac{\rho e^{t\zeta} - \rho e^{tv}}{\zeta - v} \\ 0 & e^{tv} \end{pmatrix}$$

implies that a one unit increase in  $G(0)$  causes an  $E(t) = \frac{\rho e^{t\zeta} - \rho e^{tv}}{\zeta - v}$  increase in  $s(t)$  (“ $E(t)$ ” for “effect”).

A half-life of  $h$  years implies  $v = \frac{\ln(0.5)}{h}$ . Thus,  $h = 200$  implies  $v = \frac{\ln(0.5)}{200} = -3.47 \times 10^{-3}$ , while  $h = 400$  implies  $v = \frac{\ln(0.5)}{400} = -1.733 \times 10^{-3}$ . Doubling atmospheric stocks relative to pre-industrial levels implies an increase of 280 parts per million by volume (ppm) of  $\text{CO}_2$ . One ppm corresponds to 2.13 Gigatons of carbon, or 2.13 (3.66) Gigatons of  $\text{CO}_2$  or  $\frac{2.13(3.66)}{1000}$  Teratons of  $\text{CO}_2$  (Tt  $\text{CO}_2$ ). Therefore, an increase of 280 ppm represents an increase of  $\frac{2.13(3.66)}{1000} 280 \approx 2.18$  Tt  $\text{CO}_2$ . The steady state level of  $s_\infty$  equals  $-\frac{\rho}{\zeta} G_\infty$ . I assume that doubling the steady state of  $G$  (relative to preindustrial levels), leads to a 2.6% reduction in output (and consumption). This assumption implies  $\left(1 - e^{-\frac{\rho}{\zeta} G_\infty}\right) = \left(1 - e^{-\frac{\rho}{\zeta} 2.18}\right) = 0.026$  or  $-\frac{\ln 0.974}{2.18} = \frac{\rho}{\zeta}$ .

An instantaneous one unit increase in the stock at time 0 leads to a change in  $s(t)$  of  $E(t) = \frac{\rho e^{t\zeta} - \rho e^{tv}}{\zeta - v}$  units. The increase in percent reduction in output after  $t$  years, due to this time-zero increase in  $G$ , is  $100 \left( (1 - e^{s(t)+E(t)}) - (1 - e^{s(t)}) \right) = 100 e^{s(t)} (1 - e^{E(t)})$ , where  $s(t)$  is the equilibrium value of  $s$  absent the initial increase in stock  $G$ . The time profile of  $100 e^{s(t)} (1 - e^{E(t)})$  depends on  $s(t)$ . For purpose of calibration, I consider the case where  $s(t)$  is a constant, as in a steady state. In this case, the increase in loss is maximized where

$$\frac{d(1 - e^{E(t)})}{dt} = -e^{E(t)} \frac{dE}{dt} = -e^{E(t)} \rho \frac{\zeta e^{t\zeta} - v e^{tv}}{\zeta - v} = 0.$$

Following Gerlagh and Liski (2012), I assume that the increase in loss is maximized at  $t = 60$  years, giving the calibration equation

$$\frac{\zeta e^{60\zeta} - \nu e^{60\nu}}{\zeta - \nu} = 0.$$

Solving the last equation, using the calibration assumption  $\nu = -3.47 \times 10^{-3}$ , corresponding to a half-life of 200 years, gives  $\zeta = -4.685 \times 10^{-2}$ . Using this result and  $-\frac{\ln 0.974}{2.18} = \frac{\rho}{\zeta}$  implies  $\rho = -5.66 \times 10^{-4}$ .

With these values, the additional consumption loss, due to a unit increase in  $G$ , beginning at a steady state, increases for the first 60 years and then slowly falls. The increased loss after  $t$  years, as a fraction of the (maximal) increased loss after 60 years, is

$$\frac{\left(1 - \exp\left(\frac{\rho e^{t\zeta} - \rho e^{t\nu}}{\zeta - \nu}\right)\right)}{\left(1 - \exp\left(\frac{\rho e^{60\zeta} - \rho e^{60\nu}}{\zeta - \nu}\right)\right)}. \quad (42)$$

Figure 6 shows the graph of this ratio assuming a half-life of 200 (solid curve) or 400 (dashed curve) years. Given the other two calibration assumptions, the graph is insensitive to the half-life for  $t < 60$ . Beyond that time, a longer half-life causes the additional stock, and thus the additional damage, to decay more slowly, causing the curve to rotate up.

Comparing Figure 6 with Gerlagh and Liski's (2012) Figure 1 shows that for a half-life of 200 years, the damage trajectory in my two-dimensional model has the same shape as their representation of the DICE results. For a 400 year half-life, the trajectory has the same shape as in their model for the first 600 years or so. The profiles in my two-dimensional model and in their four dimensional model differ in the very long run. Their climate system is closed, whereas for  $\nu < 0$ , emissions and thus damages eventually dissipate in my model. Therefore,  $\lim_{t \rightarrow \infty} E(t) = 0$  in my setting, whereas it approaches a positive constant in theirs. Their closed climate system would present a problem in the GeaS setting, where equilibrium emissions are constant; with constant emissions, damages in the closed climate system become unbounded. In contrast, in my setting, steady state damages for constant  $X$  equals

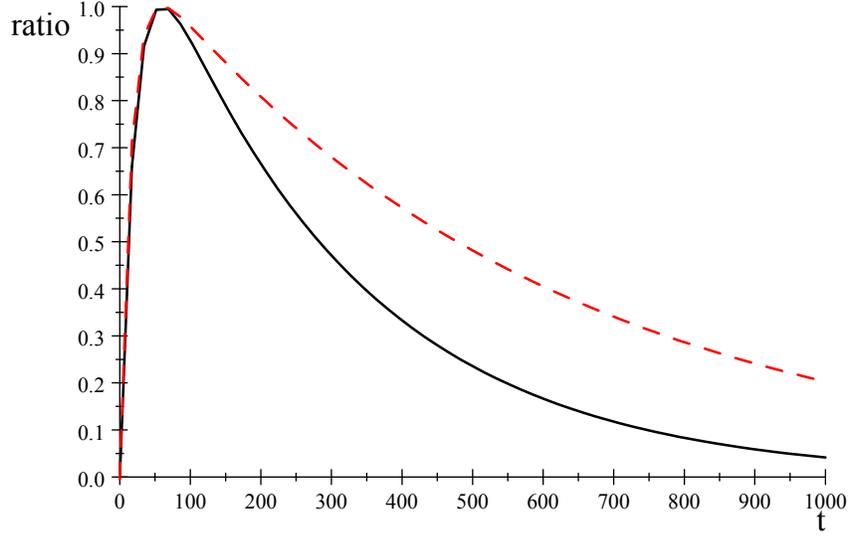


Figure 6: The ratio given in expression 42 for a half life of 200 years (solid curve) and a half life of 400 years (dashed curve). Other two calibration assumption as in the text.

$$\frac{\varrho X}{\varsigma v}.$$

I use the expressions for  $\Lambda$  and  $P$  in equation 41 to calculate

$$\begin{aligned} \tilde{q}' &= \int_0^\infty i'_1 P e^{-(\gamma I - \Lambda)\tau} P^{-1} d\tau = i'_1 P \int_0^\infty e^{-(\gamma I - \Lambda)\tau} d\tau P^{-1} = \\ &\left( \gamma - \varsigma, \quad -\frac{\varrho}{(\varsigma - v)(\gamma - v)} (\varsigma\gamma - \gamma^2 - \varsigma v + \gamma v + 1) \right) \end{aligned}$$

and

$$B - \lambda I = \begin{pmatrix} \varsigma - \lambda & \varrho \\ 0 & v - \lambda \end{pmatrix} \Rightarrow (B - \lambda I)^{-1} = \begin{pmatrix} \frac{1}{\varsigma - \lambda} & -\frac{\varrho}{\lambda^2 - \varsigma\lambda + \varsigma v - \lambda v} \\ 0 & -\frac{1}{\lambda - v} \end{pmatrix}.$$

Using these results in the formula for the tax (with  $\kappa = -1$ ), equation 12, and simplifying the result, gives equation 14.