Symmetry

What is symmetry? Symmetry is the principal feature of beauty. Beauty is subjective: “beauty is in the eye of the beholder” as the saying goes. On the contrary, symmetry can be identified and agreed upon. Symmetry is not subjective. Throughout the ages, scientists have searched for symmetry (as a proxy for beauty) and, often unexpectedly, have found profound truths: electricity and magnetism; energy and mass; light and radiation; space-time and gravity. Frontier physicists deal with super-symmetry. Therefore, it seems possible to infer that the modern scientific paradigm attempts to define symmetries (as proxies for beauty) with the hope of finding relevant truths.

Modern Scientific Paradigm

Beauty 美 Bellezza
↓ ↓
Symmetry 对称 Simmetria
↓ ↓
Truth 真理 Verità

Throughout the remainder of the course I will present various economic topics within symmetric contexts. It will be surprising how this simple mindset will produce unexpected results.
In terms of the Lagrange function we will deal, when possible and convenient, with a symmetric Lagrange function as opposed to the asymmetric Lagrange function.
The Monopsonist
(See also chapter 9 of the Symmetric Programming textbook)

A monopsony is a market condition similar to a monopoly except that a large buyer, not a seller, controls a large proportion of the market and drives prices down. A monopsony occurs when a single firm has exceptional market power in employing its factors of production. The word comes from the Greek opsōniā, *purchase of food*; in Economics it refers to the purchase of any commodity/factor of production.

There are several types of monopsonists. We will deal with two types:
1. The pure monopsonist
2. The perfectly discriminating monopsonist.

Symmetric Quadratic Programming (SQP)
I will introduce the monopsony topic in a rather unusual fashion. The idea is to build new knowledge on the strength of already acquired knowledge. In this case, we know almost everything about the monopolist. For example, we know that the $M$ matrix of the LCP associated with a pure monopolist has the following structure

$$
M = \begin{bmatrix}
2D & A' \\
-A & 0 \\
\end{bmatrix}, \quad A = [m \times n], m < n, \quad q = \begin{bmatrix}
-c \\
b
\end{bmatrix}
$$

The matrix $M$ contains a null matrix of dimensions $(m \times m)$. The $M$ matrix is analogous to a “book case” with an “empty shelf.” Hence, we can put something on the empty shelf. Let us put there the matrix $E$ of dimensions $(m \times m)$, symmetric and positive definite. Then, the original LCP is augmented as in

$$
M = \begin{bmatrix}
2D & A' \\
-A & E \\
\end{bmatrix}, \quad q = \begin{bmatrix}
-c \\
b
\end{bmatrix}
$$

### Questions:
What kind of quadratic programming problem (QP) does the LCP in (1) represent? Stated equivalently, what kind of primal and dual QP problems correspond to the LCP in (1)? And finally, what kind of economic problem does the LCP in (1) represent?

### Temporary Answer:
We do not know all those answers, as yet, but an intelligent strategy is to unpack the LCP in (1) and look at the resulting dual and primal constraints.

$$
Mz + q \geq 0
$$

The above LCP unpacks as

$$
\begin{bmatrix}
2D & A' \\
-A & E \\
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
+ \begin{bmatrix}
-c \\
b
\end{bmatrix}
\geq \begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

Note that we can rearrange the primal constraints as (according to the Economic Equilibrium)

$$
Ax \leq b + Ey
$$

Therefore, $b + Ey$ must represent the vector of direct supply functions ($s = b + Ey$) of limiting inputs (that is quantity, $s$, as a function of price). When we talk about input markets we are dealing with the possibility of a monopsonist. As stated above, a monopsonist is an economic agent who is the sole...
buyer of the limiting inputs. Examples of monopsonists: the National Football League (NFL), The National Basketball Association (NBA), government that buys state of the art weapon systems.

There remains to discuss the primal and dual objective functions. We know that the dual constraints belong to a pure monopolist since

\[ A'y \geq c - 2Dx \]

\[ MC \geq MR \]

of the pure monopolist

Hence, the dual constraints must be associated with a primal objective function that, at least in part, looks like

Primal:

\[
\max \ ? = c'x - x'Dx \ldots \\
\text{subject to} \\
Ax \leq b + Ey \\
D \leq S \quad \text{of limiting inputs}
\]

while the dual problem (as far as we remember) is

Dual:

\[
\min \ ? = b'y + x'Dx \ldots \\
\text{subject to} \\
A'y \geq c - 2Dx \\
MC \geq MR \quad \text{of the pure monopolist}
\]

What is missing in the two objective functions? Let us reason by analogy about \( x'Dx \).

- \( x'Dx \) enters both objective functions
- \( x'Dx \) is symmetric positive definite quadratic form
- the derivative of \( x'Dx \) enters the dual constraints as \( 2Dx \)
- \( Ey \) enters the primal constraints and could be considered the derivative of \( \frac{1}{2}y' Ey \)
- In this case, \( \frac{1}{2}y' Ey \) should enter both objective functions, as \( x'Dx \) does
- Therefore, we guess that

Primal: \[ \max \ ? = c'x - x'Dx - \frac{1}{2}y'Ey \] (2)

subject to

\[ Ax \leq b + Ey \]

\[ D \leq S \]

Dual: \[ \min \ ? = b'y + x'Dx + \frac{1}{2}y'Ey \] (3)

subject to

\[ A'y \geq c - 2Dx \]

\[ MC \geq MR \]

**Is this reasoning correct?**

The only way to find out is to use KKT theory on the primal problem. If, by starting from the primal problem in (2) and using KKT theory, we arrive at a specification of the dual problem that is identical to the dual expressed in (3), the structure of the dual pair of problems is correct.

Lagrange Function: \[ L = c'x - x'Dx - \frac{1}{2}y'Ey + y'[b + Ey - Ax] \]
KKT conditions:

1. \[ \frac{\partial L}{\partial x} = c - 2Dx - A'y \leq 0 \quad \text{dual constraints} \]

2. \[ x'y' = x'c - 2x'Dx - x'A'y = 0 \quad \text{dual CSC} \]

3. \[ \frac{\partial L}{\partial y} = -Ey + b + 2Ey - Ax \geq 0 \quad \rightarrow \quad b + Ey - Ax \geq 0 \quad \text{primal constraints} \]

4. \[ y' \frac{\partial L}{\partial y} = y'b + y'Ey - y'Ax = 0 \quad \text{primal CSC} \]

So far, this Lagrange function has produced primal and dual constraints as desired. It remains to determine the dual objective function. Use the dual CSC in (2.) to simplify the Lagrange function, as usual:

\[
L = 2x'Dx + y'Ax - x'Dx - \frac{1}{2}y'Ey + y'b + y'Ey - y'Ax \\
= x'Dx + y'b + \frac{1}{2}y'Ey
\]

Therefore, the dual problem is precisely the dual structure appearing in (3), as desired.

It will turn out (in a while) that \( \frac{1}{2}y'Ey \) is the total cost of limiting inputs for a perfectly discriminating monopsonist and \((b'y + \frac{1}{2}y'Ey)\) is the total cost of market options for input supply functions. Therefore, we can restate the primal and dual problems of an economic agent that behaves as a pure monopolist on the output market and as a perfectly discriminating monopsonist of the input market in the following way

Primal:

\[
\text{max } \pi = TR - TC_{pp} \\
= (c'x - x'Dx) - \frac{1}{2}y'Ey \\
\text{subject to} \\
Ax \leq b + Ey \\
D \leq S
\]

Dual:

\[
\text{min } TCMO = TCMO_{inputs} + TCMO_{outputs} \\
= (b'y + \frac{1}{2}y'Ey) + x'Dx \\
\text{subject to} \\
A'y \geq c - 2Dx \\
MC \geq MR
\]

**Textbook Perfectly Discriminating Monopsonist – Pure Monopolist**

In almost every economics textbook, the monopsony model is presented using inverse supply functions for inputs (recall that in the monopsony discussion carried out in the previous section we dealt with direct supply functions of inputs). Let the vector of inverse supply functions of inputs be \( p_s = g + Gs \) where \( G \) is a SPD matrix. A perfectly discriminating monopsonist pays a different price for each unit of purchased input. For example, NFL and NBA pay different prices for each player.
Therefore, total cost of inputs (physical plant) is the integral under the vector of inverse input supply functions, that is

\[ TC_{pp} = \int_{0}^{s^*} (g + Gs)' ds = g's' + \frac{1}{2}(s^*)'Gs^* \]

Figure 1 illustrates this integral.

![Figure 1. Total cost of a perfectly discriminating monopsonist](image)

We now define the problem of an economic agent who behaves as
1. a pure monopolist on the output market and
2. a perfectly discriminating monopsonist (PDM) on the input market.

The primal problem maximizes profit

\[ \max \pi = TR - TC_{pp} \]

Primal:

\[ = (c'x - x'Dx) - (g's + \frac{1}{2}s'Gs) \]

subject to

\[ D \leq S \]
\[ Ax \leq s \]
\[ x \geq 0 \quad \text{output quantities} \]
\[ s \geq 0 \quad \text{input quantities} \]

The dual problem is derived, as usual, via KKT conditions.

Lagrange function:

\[ L = c'x - x'Dx - g's - \frac{1}{2}s'Gs + y'[s - Ax] \]

Relevant KKT conditions

1. \[ \frac{\partial L}{\partial x} = c - 2Dx - A'y \leq 0 \quad \text{dual constraints} \]
2. \[ x' \frac{\partial L}{\partial x} = x'c - 2x'Dx - x'A'y = 0 \quad \Rightarrow \quad c'x = 2x'Dx + x'A'y \]
3. \[ \frac{\partial L}{\partial s} = -g - Gs + y \leq 0 \quad \text{dual constraints} \]
4. \[ s' \frac{\partial L}{\partial s} = -s'g - s'Gs + s'y = 0 \quad \Rightarrow \quad s'y = s'g + s'Gs \]

Use (2.) and (4.) to simplify the Lagrange function
\[ L = c'x - x'Dx - g's - \frac{1}{2}s'Gs + y's - y'Ax \]
\[ = 2x'Dx + y'Ax - x'Dx - g's - \frac{1}{2}s'Gs + g's + s'Gs - y'Ax \]
\[ = x'Dx + \frac{1}{2}s'Gs \]

Hence, the dual problem is

\[ \text{min} TCMO = TCMO_{\text{outputs}} + TCMO_{\text{inputs}} \]

Dual:

\[ = x'Dx + \frac{1}{2}s'Gs \]

subject to

\[ A'y \geq c - 2Dx \quad \leftarrow \quad MC \geq MR \quad \text{of outputs} \]
\[ g + Gs \geq y \quad \leftarrow \quad MFC \geq MIV \quad \text{of inputs} \]

\( MIV \) stands for marginal input valuation (or marginal expenditure) and \( MFC \) stands for marginal factor cost. The dual economic agent who wishes to buy out the firm must minimize the cost of purchasing the demand functions for outputs, \( TCMO_{\text{outputs}} \), and the input supply functions for inputs, \( TCMO_{\text{inputs}} \).

He must also make sure that the prices he quotes are such that the marginal cost of producing outputs is not inferior to the corresponding marginal revenue because, in that case, he would not take advantage of all the profit opportunities. Finally, he cannot overvalue his inputs.

On the other hand, the profit of the primal economic agent is equal to \( x'Dx + \frac{1}{2}s'Gs \).

The corresponding LCP can easily arranged by organizing the dual and the primal constraints as follows

\[
\begin{bmatrix}
2D & A' \\
G & -I \\
-A & I
\end{bmatrix}
\begin{bmatrix}
x \\
s
y
\end{bmatrix}
\geq
\begin{bmatrix}
-c \\
g \\
0
\end{bmatrix}
\]

with

\[ M = \begin{bmatrix}
2D & A' \\
G & -I \\
-A & I
\end{bmatrix}, \quad q = \begin{bmatrix}
-c \\
g \\
0
\end{bmatrix}, \quad z = \begin{bmatrix}
x \\
s
y
\end{bmatrix} \]

The LCP system to solve includes \( (2m + n) \) equations. There are \( m \) more equations to solve than in the SQP problem.

There remains to establish the connection between the textbook Pure Monopolist–Perfectly Discriminating Monopsonist and the SQP model developed in the previous section.

**Connection Between the Textbook Monopsonist and the SQP Monopsonist**

Let us consider the dual constraint of the textbook PDM

\[ g + Gs \geq y. \]

Assuming that a small amount of inputs will be bought by the monopsonist agent, \( s > 0 \) and, therefore (by complementary slackness)

\[ p_s = g + Gs = y. \]

Using the inverse matrix \( G^{-1} \), we obtain the direct input supply functions
that can be rearranged, conveniently, into
\[ s = b + Ey \]
direct input supply function used in SQP
where \( b = -G^{-1}g \), \( E = G^{-1} \). Now we use this version of the input supply function to transform the total cost of inputs stated in the textbook PDM into the total cost of input stated in the SQP PDM:
\[
TC_{pp} = g's + \frac{1}{2}s'Gs
\]
\[
= g'[b + Ey] + \frac{1}{2}[b' + y'E]E^{-1}[b + Ey]
\]
\[
= g'b + g'Ey + \frac{1}{2}b'E^{-1}b + b'y + \frac{1}{2}y'Ey
\]
\[
= -b'E^{-1}b - b'E^{-1}Ey + \frac{1}{2}b'E^{-1}b + b'y + \frac{1}{2}y'Ey
\]
\[
= -\frac{1}{2}b'E^{-1}b + \frac{1}{2}y'Ey
\]
The expression \( \frac{1}{2}b'E^{-1}b \) is a fixed (constant) known quantity that does not enter the optimization process. Similarly, the TCMO of the input supply functions can be restated as
\[
TCMO_{inputs} = \frac{1}{2}s'Gs = \frac{1}{2}[b' + y'E]G[b + Ey]
\]
\[
= \frac{1}{2}b'E^{-1}b + b'y + \frac{1}{2}y'Ey
\]
We have demonstrated that the SQP version of the PDM corresponds equivalently (identically) to the textbook PDM. The SQP model has the advantage of requiring a smaller number of equations to solve.

**Solution of Perfectly Discriminating Monopoly and Monopsony by LCP and Complementary Pivot Algorithm**

Consider the following SQP problem that represents the behavior of an economic agent who acts as a perfectly discriminating monopolist of the output market and as a perfectly discriminating monopsonist on the input market:

**Primal**
\[
\max \pi = c'x - \frac{1}{2}x'Dx - \frac{1}{2}y'Ey + \frac{1}{2}b'E^{-1}b
\]
subject to
\[
Ax \leq b + Ey, \quad x \geq 0, \ y \geq 0
\]
where the matrices \( D \) and \( E \) are assumed to be symmetric positive definite. The dual problem corresponds to the following specification:

**Dual**
\[
\min TCMO = b'y + \frac{1}{2}y'Ey + \frac{1}{2}b'E^{-1}b + \frac{1}{2}x'Dx
\]
subject to
\[
A'y \geq c - Dx, \quad x \geq 0, \ y \geq 0
\]
Note that the constant term \( \frac{1}{2}b'E^{-1}b \) does not enter into the optimal solution of the SQP problem. In other words, the term \( \frac{1}{2}b'E^{-1}b \) is not part of the KKT conditions. To have a cleaner specification of the primal and dual problems it could (and should) be left out.

In a numerical example, the various parameters take on the following values
\[
A = \begin{bmatrix}
1 & 0 & -2 \\
3 & 1 & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
3 \\
-2
\end{bmatrix}, \quad E = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}, \quad c = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad D = \begin{bmatrix}
13 & -4 & 3 \\
-4 & 5 & -2 \\
3 & -2 & 1
\end{bmatrix}
\]
The reformulation of the above numerical problem in the form of a LCP structure gives the following matrix $M$ and vector $\mathbf{q}$

$$
M = \begin{bmatrix}
D & A' \\
-A & E
\end{bmatrix} = \begin{bmatrix}
13 & -4 & 3 & 1 & 3 \\
-4 & 5 & -2 & 0 & 1 \\
3 & -2 & 1 & -2 & 0 \\
-1 & 0 & 2 & 2 & 1 \\
-3 & -2 & 0 & 1 & 2
\end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix}
-c \\
b
\end{bmatrix} = \begin{bmatrix}
0 \\
-1 \\
-2 \\
3 \\
-2
\end{bmatrix}
$$

The entire sequence of tableaux corresponding to the Lemke complementary pivot algorithm is presented in the next page. For reason of space, only the transformation vector is exhibited on the left of the tableaux. This SQP problem is identical to the problem presented on pages 97-103 of the Symmetric Programming textbook.
The structure of the LCP tableaux set up for solution using the Lemke Complementary Pivot Algorithm follows the specification

\[ w - Mz - sz_0 = q \]

<table>
<thead>
<tr>
<th>( T_0 )</th>
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<th>( w_2 )</th>
<th>( w_3 )</th>
<th>( w_4 )</th>
<th>( w_5 )</th>
<th>( z_1 )</th>
<th>( z_2 )</th>
<th>( z_3 )</th>
<th>( z_4 )</th>
<th>( z_5 )</th>
<th>( z_0 )</th>
<th>( q )</th>
<th>( BI )</th>
</tr>
</thead>
<tbody>
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<td>(-1)</td>
<td>1</td>
<td>-13</td>
<td>4</td>
<td>-3</td>
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<td>-3</td>
<td>-1</td>
<td>0</td>
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<td>1</td>
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<td>-1</td>
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<td>(-1)</td>
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<td>( w_2 )</td>
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\( \downarrow \) complement of \( w_3 \)

<table>
<thead>
<tr>
<th>( T_1 )</th>
<th>( \downarrow ) complement of ( w_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_2 )</td>
<td>( \downarrow ) complement of ( w_2 )</td>
</tr>
</tbody>
</table>

The mandatory initial step \( \downarrow \)
When the artificial variable \( z_0 \) is eliminated from the \( BI \) column we have a solution of the LCP.

The solution of the LCP can be read as follows:

\[
\begin{align*}
z_1 &= x_1 = 0 & w_1 &= y_{s1} = \frac{55}{9} \\
z_2 &= x_2 = \frac{8}{3} & w_2 &= y_{s2} = 0 \\
z_3 &= x_3 = \frac{22}{3} & w_3 &= y_{s3} = 0 \\
z_4 &= y_1 = 0 & w_4 &= x_{s1} = 20 \\
z_5 &= y_2 = \frac{7}{3} & w_5 &= x_{s2} = 0
\end{align*}
\]

With this information we can evaluate the various economic targets of this entrepreneur.

\[
TR = c' \mathbf{x} - \frac{1}{2} \mathbf{x}' \mathbf{D} \mathbf{x}
\]

\[
= \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{8}{3} & \frac{22}{3} \\ \frac{8}{3} & \frac{22}{3} \\ \frac{8}{3} & \frac{22}{3} \end{bmatrix} \begin{bmatrix} 13 & -4 & 3 \\ -4 & 5 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{8}{3} \\ \frac{22}{3} \end{bmatrix}
\]

\[
= \frac{106}{9}
\]

\[
TC_{pp} = \frac{1}{2} \mathbf{y}' \mathbf{E} \mathbf{y} = \frac{1}{2} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & \frac{7}{3} \\ \frac{7}{3} & \frac{7}{3} \\ \frac{7}{3} & \frac{7}{3} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{7}{3} \\ \frac{7}{3} \end{bmatrix} = \frac{40}{9}
\]

\[
\pi = TR - TC_{pp} = \frac{106}{9} - \frac{40}{9} = \frac{57}{9}
\]

\[
TCMO_{outputs} = \frac{1}{2} \mathbf{x}' \mathbf{D} \mathbf{x} = \frac{1}{2} \begin{bmatrix} 0 & \frac{8}{3} & \frac{22}{3} \\ \frac{8}{3} & \frac{22}{3} \\ \frac{8}{3} & \frac{22}{3} \end{bmatrix} \begin{bmatrix} 13 & -4 & 3 \\ -4 & 5 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{8}{3} \\ \frac{22}{3} \end{bmatrix} = \frac{40}{9}
\]

\[
TCMO_{inputs} = \mathbf{b}' \mathbf{y} + \frac{1}{2} \mathbf{y}' \mathbf{E} \mathbf{y} = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{7}{3} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & \frac{7}{3} \\ \frac{7}{3} & \frac{7}{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{7}{3} \end{bmatrix} = \frac{7}{9}
\]

\[
TCMO = TCMO_{outputs} + TCMO_{inputs} = \frac{50}{9} + \frac{7}{9} = \frac{57}{9}
\]

**Pure Monopolist and Pure Monopsonist**

The treatment of this model will begin with the textbook version followed by the SQP version.

**The Textbook Pure Monopolist and Pure Monopsonist Model**

11
The required initial information regards the vector of inverse demand functions for monopoly outputs \( \mathbf{p} = \mathbf{c} - \mathbf{Dx} \) and the vector of inverse supply functions for limiting inputs \( \mathbf{p_s} = \mathbf{g} + \mathbf{Gs} \), where the matrices \( \mathbf{D} \) and \( \mathbf{G} \) are SPD. The economic agent in charge uses a linear technology matrix \( \mathbf{A} \) of dimension \((m \times n)\), \( m < n \) to transform inputs into outputs. The firm’s goal is to maximize profit. Hence the primal problems can be stated as

**Primal:**

\[
\max \pi = TR - TC_{pp} = \mathbf{p'}x - \mathbf{p'}s = [\mathbf{c - Dx}]'x - [\mathbf{g + Gs}]'s
\]

subject to

\[
\mathbf{D} \leq \mathbf{S} \\
\mathbf{A}x \leq \mathbf{s}, \quad x \geq \mathbf{0}, s \geq \mathbf{0}
\]

The derivation of the dual problem follows the familiar procedure beginning with the Lagrange function

\[
L = \mathbf{c'}x - \mathbf{x'Dx} - \mathbf{g's} - \mathbf{s'Gs} + \mathbf{y}'[\mathbf{s} - \mathbf{Ax}]
\]

and the statement of the relevant KKT conditions

1. \( \frac{\partial L}{\partial \mathbf{x}} = \mathbf{c} - 2\mathbf{Dx} - \mathbf{A'y} \leq \mathbf{0} \quad \text{dual constraints} \)
2. \( \mathbf{x'} \frac{\partial L}{\partial \mathbf{x}} = \mathbf{x'c} - 2\mathbf{x'Dx} - \mathbf{x'A'y} = \mathbf{0} \quad \rightarrow \quad \mathbf{c'} = 2\mathbf{x'Dx} + \mathbf{x'A'y} \)
3. \( \frac{\partial L}{\partial \mathbf{s}} = -\mathbf{g - 2Gs} + \mathbf{y} \leq \mathbf{0} \quad \text{dual constraints} \)
4. \( \mathbf{s'} \frac{\partial L}{\partial \mathbf{s}} = -\mathbf{s'g - 2s'Gs} + \mathbf{y} = \mathbf{0} \quad \rightarrow \quad \mathbf{g'} = -2\mathbf{s'Gs} + \mathbf{y's} \)

Using KKT conditions (2.) and (4.) (the complementary slackness conditions) it is possible to simplify the Lagrange function that will turn into the dual objective function:

\[
\begin{align*}
L &= \mathbf{c'}x - \mathbf{x'Dx} - \mathbf{g's} - \mathbf{s'Gs} + \mathbf{y's} - \mathbf{y'Ax} \\
&= 2\mathbf{x'Dx} + \mathbf{y'Ax} - \mathbf{x'Dx} + 2\mathbf{s'Gs} - \mathbf{y's} - \mathbf{s'Gs} + \mathbf{y's} - \mathbf{y'Ax} \\
&= \mathbf{x'Dx} + \mathbf{s'Gs}
\end{align*}
\]

Therefore, the dual model becomes:

**Dual**

\[
\min TCMO = TCMO_{outputs} + TCMO_{inputs} = \mathbf{x'Dx} + \mathbf{s'Gs}
\]

subject to

\[
\begin{align*}
\mathbf{A'y} &\geq \mathbf{c - 2Dx} = (\mathbf{c - Dx}) - \mathbf{Dx} = \mathbf{p} - \text{monopolist market power} \\
\mathbf{MC} &\geq \mathbf{MR} \\
\text{monopsonist market power} + \mathbf{p_s} &= \mathbf{Gs} + (\mathbf{Gs} + \mathbf{g}) = \mathbf{g} + 2\mathbf{Gs} \geq \mathbf{y} \\
\mathbf{MFC} &\geq \mathbf{MIV}
\end{align*}
\]
where \( MFC \) stands for marginal factor cost and \( MIV \) stands for marginal input valuation. Figure 1 illustrates diagrammatically the behavior of this economic agent.

![Figure 1. Textbook Pure Monopolist and Pure Monopsonist](image)

**SQP version of Pure Monopolist and Pure Monopsonist**

The symmetric quadratic programming specification of this problem requires the statement of the vector of direct input supply functions, as in the case of the perfectly discriminating monopsonist on page 3. This discussion is found also on pages 181-182 of the Symmetric Programming textbook. An analogous discussion to the procedure presented in page 6 leads to the definition of the following parameters

\[
\bar{b} = -\frac{1}{2}G^{-1}g \quad \text{and} \quad \bar{E} = \frac{1}{2}G^{-1}.
\]

Then, the symmetric quadratic programming version of the pure monopolist and pure monopsonist can be stated as

Primal
\[
\begin{align*}
\max \pi &= TR - TC_{pp} \\
&= (c'x - x'Dx) - \frac{1}{2} y'Ey \\
\text{subject to} & \\
D &\leq S \\
A x &\leq \bar{b} + \bar{E}y \\
x &\geq 0, y \geq 0 \\

\text{Dual} & \\
\min TCMO = TCMO_{outputs} + TCMO_{inputs} \\
&= x'Dx + \{b'y + \frac{1}{2} y'Ey\} \\
\text{subject to} & \\
MC &\geq MR \\
A'y &\geq c - 2 Dx
\end{align*}
\]

with the usual economic interpretation for each component of the dual pair of problems.