Online Appendix to “Grouped Coefficients to Reduce Bias in Heterogeneous Dynamic Panel Models with Small $T$”

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A1 Bias Formulas for Large $T$

The heterogeneous dynamic panel that we consider is written as follows:

\[ y_{it} = \gamma_i y_{i,t-1} + \beta_i x_{it} + \alpha_i + \varepsilon_{it}, \quad (A1) \]

\[ x_{it} = \mu_i (1 - \rho) + \rho x_{i,t-1} + \eta_{it}. \]

We assume that the random coefficients $\alpha_i, \beta_i, \gamma_i,$ and $\mu_i$ are independent of each other and $\varepsilon_{it}$ and $\eta_{it}$. We also assume that $x_{it}$ is strictly exogenous.

We use formulas derived by Pesaran and Smith (1995) for the asymptotic bias of pooled fixed effects for the case where $N \to \infty$ and $T \to \infty$ to show how the bias from coefficient heterogeneity depends on various parameters. Bias formulas are cumbersome even when $T \to \infty$, so to simplify the exposition we consider two cases of the model in equation (A1).

Case 1. In the case of $N \to \infty$ and $T \to \infty$, with a heterogeneous coefficient on an autocorrelated regressor ($\sigma^2_\gamma = 0, \sigma^2_\beta > 0, \rho > 0$), the asymptotic bias of pooled fixed effects is as follows:

\[ \left[ \begin{array}{c} \hat{\gamma}^{FE} - \gamma \\ \hat{\beta}^{FE} - \beta \end{array} \right] \overset{p}{\rightarrow} \frac{1}{\Psi_1} \left[ \begin{array}{c} \rho (1 - \rho \gamma) (1 - \gamma^2) \sigma^2_\beta \\ -\beta \rho^2 (1 - \gamma^2) \sigma^2_\beta \end{array} \right], \quad (A2) \]

where

\[ \Psi_1 = \frac{\sigma^2_\varepsilon}{\sigma^2_x} (1 - \rho^2) (1 - \gamma \rho)^2 + (1 - \gamma^2 \rho^2) \sigma^2_\beta + (1 - \rho^2) \beta^2. \quad (A3) \]

The asymptotic bias formulas in equations (A2)-(A3) are the same as equations (2.7)-(2.9) in Pesaran and Smith (1995), except that we think they have a minor typo.\(^1\) The estimate of $\gamma$ is biased up if $\rho$ is positive and biased down if $\rho$ is negative. Intuitively, ignoring the coefficient heterogeneity causes the coefficient on the lagged dependent variable to capture the autocorrelation in the independent variable in addition to the dynamics of the model. The estimate of $\beta$ is biased towards zero, regardless of the sign of $\rho$.

\(^1\)For the last term in $\Psi_1$, they have $(1 - \rho^2) \beta$, whereas we think this should be $(1 - \rho^2) \beta^2$. 
The magnitude of the asymptotic bias in case 1 depends on the following parameters: \( \gamma \), \( \beta \), \( \rho \), \( \sigma^2_{\beta} \), and \( \sigma^2_{\varepsilon}/\sigma^2_{\gamma} \). As expected, the bias is increasing in \( \sigma^2_{\beta} \). The bias is also increasing in the signal-to-noise ratio \( \sigma^2_{\varepsilon}/\sigma^2_{\gamma} \)—that is, if \( x_{it} \) explains a relatively large proportion of the variation in \( y_{it} \), then bias from ignoring the heterogeneity of \( \beta_i \) increases.

**Case 2.** In the case of \( N \to \infty \) and \( T \to \infty \), with a heterogeneous coefficient the lagged dependent variable and an autocorrelated regressor \( (\sigma^2_{\gamma} > 0, \sigma^2_{\beta} = 0, \rho > 0) \), the asymptotic bias of pooled fixed effects is as follows:

\[
\begin{bmatrix}
\hat{\gamma}^{FE} - \gamma \\
\hat{\beta}^{FE} - \beta
\end{bmatrix}
\xrightarrow{p} \frac{1}{\Psi_2} \begin{bmatrix}
E[(\gamma_i - \gamma) \Lambda] - \rho \beta \\
\rho \beta E \left[ \frac{\gamma_i - \gamma}{1 - \rho \gamma_i} \right] E \Lambda - \rho \beta E \left[ \frac{1}{1 - \rho \gamma_i} \right] E [(\gamma_i - \gamma) \Lambda]
\end{bmatrix},
\tag{A4}
\]

where

\[
\Psi_2 = E \Lambda - \rho^2 \beta^2 E \left[ \frac{1}{1 - \rho \gamma_i} \right]^2,
\tag{A5}
\]

and

\[
\Lambda = \frac{\beta^2 (1 + \rho \gamma_i)}{(1 - \rho \gamma_i) (1 - \gamma_i^2)} + \frac{\sigma^2_{\gamma} (1 - \rho^2)}{\sigma^2_{\varepsilon} (1 - \gamma_i^2)}.
\tag{A6}
\]

The bias formulas in equations (A4)-(A6) are obtained from the derivations in the appendix of Pesaran and Smith (1995). If \( x_{it} \) is not autocorrelated \( (\rho = 0) \), then the fixed effects estimate of the coefficient on the independent variable is not biased and the sign of the bias of \( \hat{\gamma}^{FE} \) corresponds to the sign of \( E \left[ \frac{\gamma_i - \gamma}{1 - \gamma_i^2} \right] \).

The magnitude of the bias in case 2 depends on the following parameters: \( \gamma \), \( \beta \), \( \rho \), \( \sigma^2_{\gamma} \), and \( \sigma^2_{\varepsilon}/\sigma^2_{\gamma} \). Although \( \sigma^2_{\gamma} \) does not appear directly in equations (A4)-(A6), terms that include some form of the expected value of \( \frac{1}{1 - \gamma_i} \) depend on \( \sigma^2_{\gamma} \).

### A2 Additional Monte Carlo Results

Figures A1–A6 display additional results from Monte Carlo simulations. In the main paper, we only show Monte Carlo results with \( \sigma_x = 0.5 \) and \( T = 6 \). In the following figures we
show other combinations of the parameters including $\sigma_x = 2$ and $T = 10$. Table A1 shows the probability of rejecting the null hypothesis of no autocorrelation of the errors for the additional combinations of parameter values.
Figure A1: Monte Carlo Results with $\gamma = 0.5$, $\sigma_x=2$, and $T = 6$

**Coefficient on Lagged Dependent Variable**

- $\sigma=0.25, \sigma_\beta=0.5$
- $\sigma=0.25, \sigma_\beta=1.0$
- $\sigma=0.5, \sigma_\beta=0.5$
- $\sigma=0.5, \sigma_\beta=1.0$

**Coefficient on Independent Variable**

- $\sigma=0.25, \sigma_\beta=0.5$
- $\sigma=0.25, \sigma_\beta=1.0$
- $\sigma=0.5, \sigma_\beta=0.5$
- $\sigma=0.5, \sigma_\beta=1.0$
Figure A2: Monte Carlo Results with $\gamma = 0.8$, $\sigma_x=2$, and $T = 6$
Figure A3: Monte Carlo Results with $\gamma = 0.5$, $\sigma_x = 0.5$, and $T = 10$

Coefficient on Lagged Dependent Variable

Coefficient on Independent Variable

A6
Figure A4: Monte Carlo Results with $\gamma = 0.8$, $\sigma_x = 0.5$, and $T = 10$

**Coefficient on Lagged Dependent Variable**

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<thead>
<tr>
<th>$\sigma_x = 0.25, \sigma_\gamma = 0.5$</th>
<th>$\sigma_x = 0.25, \sigma_\gamma = 1.0$</th>
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<tr>
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<tbody>
<tr>
<td><img src="image3.png" alt="Graph" /></td>
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Legend:
- True
- OLS
- FE
- IC-OLS
- GC-OLS
- GC-BB

**Coefficient on Independent Variable**

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<thead>
<tr>
<th>$\sigma_x = 0.25, \sigma_\gamma = 0.5$</th>
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<tbody>
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<td><img src="image8.png" alt="Graph" /></td>
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Legend:
- True
- OLS
- FE
- IC-OLS
- GC-OLS
- GC-BB
Figure A5: Monte Carlo Results with $\gamma = 0.5$, $\sigma_x=2$, and $T = 10$

Coefficient on Lagged Dependent Variable

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Coefficient on Independent Variable

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Figure A6: Monte Carlo Results with $\gamma = 0.8$, $\sigma_x=2$, and $T = 10$

**Coefficient on Lagged Dependent Variable**

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<tr>
<th>$\sigma = 0.5$, $\sigma_{\gamma} = 0.5$</th>
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**Coefficient on Independent Variable**

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<th>$\sigma = 0.5$, $\sigma_{\gamma} = 1.0$</th>
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<tbody>
<tr>
<td><img src="image7" alt="Graph" /></td>
<td><img src="image8" alt="Graph" /></td>
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A9
Table A1: Probability of Rejecting Null Hypothesis of No Autocorrelation of Error for Additional Parameter Values

<table>
<thead>
<tr>
<th>$\sigma_\gamma$</th>
<th>$\sigma_\beta$</th>
<th>Probability of Rejecting Null</th>
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</thead>
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<td>0.096</td>
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<td>$\gamma = 0.8$, $\sigma_x = 2$, $T = 6$</td>
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A3 Additional Details on Application to Labor Demand

Table A2: Observations in each year of the original data from Arellano and Bond (1991)

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References
