State Prices of Conditional Quantiles:
New Evidence on Time Variation in the Pricing Kernel

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Abstract

We develop a set of statistics to represent the option-implied stochastic discount factor, and we apply them to S&P 500 returns between 1990 and 2012. Our statistics, which we call State Prices of Conditional Quantiles (SPOCQ), estimate the market’s willingness to pay for insurance against outcomes in various quantiles of the return distribution. By estimating state prices at conditional quantiles, we separate variation in the shape of the pricing kernel from variation in the probability of a particular event. Thus, without imposing strong assumptions about the distribution of returns, we obtain a novel view of pricing-kernel dynamics. We have two main empirical findings. First, consistent with several recent papers, our SPOCQ estimates imply a non-monotonic pricing kernel on average, although the shape of the pricing kernel varies significantly over time. Second, in contrast to recent studies, we find that the price of downside risk decreases when volatility increases. Under a simple asset pricing model, this result implies that most changes in volatility stem from fluctuations in idiosyncratic risk. Consistent with this interpretation, no known systematic risk factors such as consumer sentiment, liquidity, or macroeconomic risk can account for the negative relationship between the price of downside risk and volatility.

**JEL codes:** C5, G12, G13. **Keywords:** conditional quantile, non-monotonic, pricing kernel, state price, stochastic discount factor.

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1 Introduction

In a present value model, stock price changes stem either from changes in expected future dividends or changes in the discount rate. If the discount rate is constant, then price changes must reflect changes in expected future dividends. Shiller (1981) and LeRoy and Porter (1981) first showed that stock prices vary much more than can be explained by rational expectations about future dividends. This result suggests that discount rate variation is the dominant source of price volatility. Indeed, Cochrane (2011) writes that “discount rate variation is the central organizing question of current asset pricing research.”

We provide a new perspective in this line of research by developing a set of statistics to represent the option-implied stochastic discount factor. To generate these statistics, we first use a time-series model to estimate quantiles of the conditional distribution of returns. Then, taking these estimates to represent rational expectations about return quantiles, we infer from options prices the implied discount factor in ranges of the return distribution implied by the quantiles. Specifically, we estimate the prices of securities that pay one dollar in the event that the asset return falls in, for example, the bottom 10% of the conditional distribution and zero, otherwise. We call our statistics State Prices of Conditional Quantiles (SPOCQ) and apply them to S&P 500 monthly returns for the period 1990–2012.

We use quantiles for three reasons. First, we can obtain a parsimonious model of the important features of the conditional return distribution without imposing the assumptions required to fit the entire distribution. Second, quantiles often provide a more intuitive and useful characterization of tail behavior than moments. Value-at-risk (VaR) is one prominent example; a VaR statistic provides a clearer picture of tail risk than a skewness or kurtosis statistic. Third, and perhaps most important, time series variation in SPOCQ reveals information about discount rate variation that is obscured in other statistics.

For example, consider the event that the four-week return on the S&P 500 is less than $-4.4\%$, which occurs in 10% of the months in our sample. The state price of this event can be expressed as the event probability multiplied by the discount factor traders apply to the event. The state price increases when volatility increases because high volatility makes a return less than $-4.4\%$ more likely, even if the stochastic discount factor remains unchanged. A return in the bottom conditional decile, on the other hand, always has a 10% probability of occurring. As a result, variation in the associated SPOCQ comes only from changes in the discount applied to that event. By holding constant the event probability, we gain insight into the changes in discount rates over time.
We obtain two main empirical results. First, the SPOCQ for the top quartile typically exceeds its counterpart for the third quartile, implying that traders discount top-quartile returns more heavily than third-quartile returns. This finding is consistent with the early work by Jackwerth (2000) and the more recent findings by Bakshi, Madan, and Panayotov (2010), who show that the pricing kernel is non-monotonic when projected on S&P 500 returns. This stylized fact of non-monotonicity has been labeled the pricing kernel puzzle—it contradicts standard asset pricing models under risk aversion, which predict that state prices for lower-quantile events exceed those for higher-quantile events. Aside from the top quartile, the average SPOCQ decreases with returns as predicted by standard risk-aversion models.

Second, we find that high volatility in stock returns is associated with lower state prices for bottom-quartile returns and higher state prices for top-quartile returns. Figure 1 illustrates this finding by plotting the bottom- and top-quartile SPOCQ statistics against the log of volatility. The negative correlation in the bottom quartile contradicts recent models that specify the pricing kernel as a linear function of squared returns—see Christoffersen, Heston, and Jacobs (2013). In those models, state prices in both tails of the return distribution increase with volatility.

Our results support the hypothesis that changes in volatility during our sample were predominantly idiosyncratic to the S&P 500 and thus unrelated to the pricing kernel. By definition, an increase in idiosyncratic volatility does not affect expected returns. Moreover, an increase in idiosyncratic volatility decreases the signal-to-noise ratio in S&P 500 returns. If returns contain relatively more noise, then they convey less information about systematic risk. We show that, as a result, the left-tail SPOCQ decreases when idiosyncratic volatility increases. In contrast, an increase in systematic volatility increases the left-tail SPOCQ because it causes traders to apply a larger discount to left-tail returns.

The return-volatility literature is consistent with our results. Since French, Schwert, and Stambaugh (1987), numerous papers have found weak and inconsistent association between volatility and expected returns (see Yu and Yuan (2011) for a summary). If increases in volatility tend to reflect greater idiosyncratic rather than systematic variation, then volatility should only weakly predict returns. Most of the papers in this literature cover time periods prior to our sample, which suggests that our results are not specific to the period since 1990. The recent financial crisis is a notable exception to our finding that most variation in volatility is idiosyncratic to the S&P 500. During the months of the crisis that exhibited the highest volatility (Sept 2008–April 2009), the left-tail SPOCQs were about their average values.
for the 1990–2012 sample. Thus, volatility had a higher systematic component during the financial crisis than in other high volatility periods. Given the prolonged slump that followed the crisis, it is unsurprising that returns in this period had a larger systematic component than other high-volatility periods.

The remainder of the paper is organized as follows. In Section 2, we provide a framework for SPOCQ and how to interpret its variation over time. We then describe the details regarding the estimation of the SPOCQ components in Section 3, and we discuss our estimation results in Section 4. An analysis of the salient features of the SPOCQ time variation and its implications for the pricing kernel follow in Section 5, and in Section 6 we present regression results to relate SPOCQ to known risk factors. Section 7 concludes. The tables and figures follow the main body of the text, and the appendices provide further details on our data and methods.

2 State Prices of Conditional Quantiles

2.1 Conceptual Framework

In dynamic equilibrium models, the price of an asset equals the expected value of discounted future payoffs on the asset (e.g., Cochrane (2001)). Let \( S_t \) denote the price of the asset at time \( t \), \( S_T \) the payoff on that asset at time \( T > t \), and \( M_{t,T} \) denote the stochastic discount factor (SDF) between \( t \) and \( T \). Suppose the state of the economy at \( t \) can be described by a vector \( W_t \). In equilibrium, the asset price is given by

\[
S_t = E_t [M_{t,T} S_T],
\]

where \( E_t [\cdot] \) denotes the expectation conditional on \( W_t \). Equivalently to (1), we can express \( S_t \) as the expected future payoff under the risk-neutral probability measure (Ross (1976)) discounted at the risk-free rate. Based on the fundamental theorem of asset pricing, the risk-neutral measure always exists if the market is arbitrage-free (Harrison and Kreps (1979)) and is unique if the market is complete (Harrison and Pilska (1981)) with respect to the state space defined by \( W_t \).

At time \( t \), the researcher observes \( S_t \) and the prices of any derivatives defined by payoffs on the asset, but not \( W_t \). Thus, we focus on the risk-neutral distribution implied by the

\[^{1}\text{See Hansen, Heaton, Lee, and Roussanov (2007), Hansen and Renault (2009), and Section 5 on Scaled Consumption-Based Models in Ludvigson (2012) for up-to-date discussion of the pricing kernel.}\]
observed asset and derivative prices, which is the risk-neutral distribution of returns on the asset after integrating out the unobserved component of the state space. Using the law of iterated expectations, the expression in (1) becomes

$$1 = E_t [M_{t,T} R_{t,T}] = E_t [E_t [M_{t,T} R_{t,T}] R_{t,T}] = \int E_t [M_{t,T} | R_{t,T}] R_{t,T} dF_t (R_{t,T}),$$

(2)

where $R_{t,T} \equiv S_T / S_t$ denotes the absolute return, and $E_t [\cdot | R_{t,T}]$ is the expectation conditional on $\{W_t, R_{t,T}\}$. Multiplying and dividing the last expression by $E_t [M_{t,T}]$ produces

$$1 = E_t [M_{t,T}] \int \frac{E_t [M_{t,T} | R_{t,T}]}{E_t [M_{t,T}]} R_{t,T} dF_t (R_{t,T}) = E_t [M_{t,T}] \int R_{t,T} dF_t^* (R_{t,T}).$$

(3)

Thus, the risk-neutral conditional distribution of the asset return is given by

$$F_t^* (R) \equiv \int_{-\infty}^{R} \frac{E_t [M_{t,T} | R_{t,T}]}{E_t [M_{t,T}]} dF_t (R_{t,T}) \equiv E \left[ M_t^* | R_{t,T} \leq R \right] F_t (R),$$

(4)

where we define $M_t^*$ as

$$M_t^* \equiv M_t^* (R_{t,T}) \equiv \frac{E_t [M_{t,T} | R_{t,T}]}{E_t [M_{t,T}]}.$$

(5)

At time $t$, that is, taking the current state of the world $W_t$ as given, $M_t^* (R)$ is the discount applied to the return outcome $R$. The development in (2)–(5) is similar to the projection of the pricing kernel onto the payoffs of a tradable asset used in Engle and Rosenberg (2002). Therefore, $M_t^* (R_{t,T})$ can be labeled the projected pricing kernel.\(^2\)

Equation (4) decomposes the risk-neutral conditional distribution into two components and thereby reveals two sources of time variation. The first source of variation is changes in the future return distribution, $F_t (R)$. The second source of variation is changes in the price of risk, $M_t^* (R)$. We focus on the second source, which we isolate by evaluating $F_t^* (R)$ at conditional quantiles of the asset returns. Specifically, we define the conditional quantile $q_t (\theta_j)$ such that $F_t (q_t (\theta_j)) = \theta_j$. The state price of the event $R_{t,T} \leq q_t (\theta_j)$, which occurs with fixed probability $\theta_j$, is then given by

$$F_t^* (q_t (\theta_j)) = E \left[ M_t^* | R_{t,T} \leq q_t (\theta_j) \right] \theta_j.$$

(6)

\(^2\)See Chabi-Yo (2012) for a more recent application of the projected pricing kernel. The pricing implications of a projected pricing kernel for an asset are the same with the pricing implications of the original kernel—e.g., see Section 4.1 in Cochrane (2001). The former depends on the realization of the asset payoff, while the latter depends on the realization of the state variable. Similar to our derivation here, Engle and Rosenberg derive their projected pricing kernel through the law of iterated expectations; see Equation (4) in their paper.
Equation (6) is an expression for a state price reflecting the market’s willingness to pay for insurance against a state with a fixed probability.

We now provide the statistic that is our main focus in this paper, the state price of conditional quantiles (SPOCQ):

\[
SPOCQ_{t,T}(\theta_{j-1}, \theta_j) = F^*_t(q_t(\theta_j)) - F^*_t(q_t(\theta_{j-1})).
\]  

Figure 2 illustrates how we obtain SPOCQ following a two-step approach. In the first step, we invert the physical distribution of returns to find the quantiles \(q_t(\theta_{j-1})\) and \(q_t(\theta_j)\). In the second step, we evaluate the risk-neutral distribution at these quantiles.

Equivalently, using (6), we can write SPOCQ as

\[
SPOCQ_{t,T}(\theta_{j-1}, \theta_j) = \int_{q_t(\theta_{j-1})}^{q_t(\theta_j)} M^*_{t,T} dF_t(r_{t,T}) = (\theta_j - \theta_{j-1}) E_t[M^*_{t,T}|\mathcal{R}_{t,T}^{j-1,j}],
\]  

where \(\theta_j > \theta_{j-1}\) and \(\mathcal{R}_{t,T}^{j-1,j}\) denotes the states of the world at time \(T\) for which \(q_t(\theta_{j-1}) \leq R_{t,T} \leq q_t(\theta_j)\). SPOCQ is the market’s willingness to pay to receive a dollar in the event that the future return falls between the \(\theta_{j-1}\) and \(\theta_j\) quantiles. It equals the probability of this event multiplied by the average of the projected pricing kernel conditional on this event. The time variation in SPOCQ is driven entirely by the willingness to pay for insurance against this event because we hold the probability of a return in this interval fixed. Under risk neutrality, this state price would never change, and it would always equal the probability of the event occurring, \(\theta_j - \theta_{j-1}\).

If we divide SPOCQ by the event probability \(\theta_j - \theta_{j-1}\), then we obtain the mean of the projected pricing kernel conditional on the relevant quantiles of the return distribution. Later in the paper, we refer to \(E_t[M^*_{t,T}|\mathcal{R}_{t,T}^{j-1,j}]\) as the quantile pricing kernel (QPK) because it equals the mean of the pricing kernel conditional on a range of the return distribution dictated by the quantiles \(\theta_j\) and \(\theta_{j-1}\).

### 2.2 Interpreting Time Variation in SPOCQ

When projecting the pricing kernel onto returns on a broad index like the S&P 500, as we do in this paper, it is tempting to assume that the projection entails only a small loss in information. In the language of CAPM, it is tempting to assume that the S&P 500 is a good proxy for the market portfolio. In this section, we show how the time variation in SPOCQ
can reveal the extent to which this assumption holds. We develop our argument using a simple model in which the stochastic discount factor is linear in a single state variable. A prominent example of such a model is the conditional CAPM in Jagannathan and Wang (1996). Although this is a simple model, its intuition extends directly to more general models, such as the model of long run risk in Bansal and Yaron (2004).

Suppose the stochastic discount factor is given by

\[ M_{t,T} = \frac{1}{R_{f,t,T}} - W_T, \]  

(9)

where \( E_t[W_T] = 0 \) and the risk-free rate \( R_{f,t,T} \) may vary over time. This formulation imposes the equilibrium condition \( E_t[M_{t,T}] = 1/R_{f,t,T} \). S&P 500 returns are generated by

\[ R_{t,T} = \alpha_t + \beta_t W_T + \epsilon_{t,T}, \]  

(10)

where \( \beta_t \equiv \text{cov}_t[R_{t,T}, W_T] / \sigma_{W,t}^2 \). The idiosyncratic errors are such that \( E_t[W_T \epsilon_{t,T}] = 0 \) and \( \sigma_{W,t}^2 \equiv \text{var}_t[W_T] \). In equilibrium, the S&P 500 risk premium is

\[ E_t[R_{t,T}] - R_{f,t,T} = -R_{f,t,T} \text{cov}_t[R_{t,T}, M_{t,T}] = R_{f,t,T} \beta_t \sigma_{W,t}^2. \]  

(11)

Using (5), we obtain \( M^*_t,T \) by projecting the pricing kernel onto S&P 500 returns. Because equations (9) and (10) are linear, the projection formula mimics ordinary least squares regression. Hence, we have

\[ M_{t,T} = \alpha^M_t + \beta^M_t R_{t,T} + \epsilon^M_{t,T}, \]  

(12)

\[ \alpha^M_t = \frac{1}{R_{t,t}} - \beta^M_t E_t[R_{t,T}], \]  

(13)

\[ \beta^M_t = \frac{\text{cov}_t[R_{t,T}, M_{t,T}]}{\sigma_{R,t}^2} = -\frac{\text{cov}_t[R_{t,T}, W_T]}{\sigma_{R,t}^2}, \]  

(14)

Note also that \( E_t[R_{t,T} \epsilon^M_{t,T}] = 0 \) and \( \sigma_{R,t}^2 \equiv \text{var}_t(R_{t,T}) = \beta_t^2 \sigma_{W,t}^2 + \sigma_{\epsilon,t}^2 \). In addition, recognizing that \( M^*_t,T \equiv E \left[ R_{f,t,T} M_{t,T} | R_{t,T} \right] \), we show in Appendix A.1 that

\[ M^*_t,T = 1 - \gamma_t (R_{t,T} - E_t[R_{t,T}]) \]  

(15)

\[ \gamma_t \equiv R_{f,t,T} \frac{\beta_t \sigma_{W,t}^2}{\beta_t^2 \sigma_{W,t}^2 + \sigma_{\epsilon,t}^2}. \]  

(16)
We focus on two implications of equations (11) and (16). First, an increase in the volatility of the state variable ($\sigma_{W,t}$) steepens the slope of the projected pricing kernel and increases expected returns. Second, an increase in idiosyncratic volatility ($\sigma_{\varepsilon,t}$) decreases the slope of the projected pricing kernel and has no effect on expected returns. When idiosyncratic volatility is large, variation in returns provides less information about the pricing kernel than when idiosyncratic volatility is low, which implies a smaller $\gamma_t$ coefficient in the equation for the projected pricing kernel.

To see how the two sources of volatility affect SPOCQ, we assume $R_{t,T}$ is conditionally normally distributed. From (8), we have

$$SPOCQ_{t,T}(0,\theta) = \theta E_t[M_{t,T}^{*}\big| R_{t,T} < q_t(\theta)]$$

(17)

Using (15) along with the formula for the mean of a truncated normal distribution and some algebra provided in Appendix A.1, we obtain

$$SPOCQ_{t,T}(0,\theta) = \theta + k\gamma_t \sigma_{R,t},$$

(18)

where $k \equiv \phi(\Phi^{-1}(\theta))$ is a constant equal to the standard normal density function evaluated at the $\theta$ quantile. Consider the effect of time-varying volatility on this statistic. Assuming that the S&P 500 is positively correlated with the state variable ($\beta_t > 0$), the derivative of $SPOCQ_{t,T}(0,\theta)$ with respect to $\sigma_{W,t}$ is positive, while the derivative with respect to $\sigma_{\varepsilon,t}$ is negative. Thus, the left-tail SPOCQ increases with systematic volatility and decreases with idiosyncratic volatility—see Appendix A.2 for details. This result can also be seen by noting that equations (11) and (16) imply that $\gamma_t \sigma_{R,t}$ equals the Sharpe ratio. Systematic volatility increases the Sharpe ratio and therefore increases SPOCQ, whereas idiosyncratic volatility decreases the Sharpe ratio and decreases SPOCQ.

In contrast to SPOCQ, evaluating the risk-neutral distribution at a fixed point $R$ yields a more complicated expression that obfuscates changes in the pricing kernel

$$F_t^*(R) = \Phi\left(\frac{R - E_t[R_{t,T}]}{\sigma_{R,t}}\right) + \phi\left(\frac{R - E_t[R_{t,T}]}{\sigma_{R,t}}\right) \gamma_t \sigma_{R,t}$$

(19)

Overall, the relationship between $F_t^*(R)$ and the two types of volatility is ambiguous (see Appendix A.2). However, for values of $R$ in the left tail, an increase in volatility tends to increase $F_t^*(R)$, whether or not the additional volatility comes from an increase in systematic ($\sigma_{W,t}$) or idiosyncratic ($\sigma_{\varepsilon,t}$) volatility. We illustrate this phenomenon in Figure 3, which plots...
$SPOCQ_{i,T}(0,25)$ and $F^*_t(R)$, where $R$ corresponds to the 0.25 quantile when $\sigma_W = 0.2$ and $\sigma_\epsilon = 0.1$, i.e., the point at which the two curves cross. The figure shows that both types of volatility have very similar effects on $F^*_t(R)$ because increasing volatility increases the probability of observing a return less than $R$, and this effect dominates any effect of changing volatility on expected returns. In contrast, SPOCQ increases with systematic volatility and decreases with idiosyncratic volatility.

The world, however, is more complex than the model presented in this section; S&P 500 returns are not conditionally Gaussian and the pricing kernel is nonlinear. In fact, numerous papers have shown that the projected pricing kernel is non-monotonic, a result known as the pricing kernel puzzle. Nonetheless, the insights from this section extend readily to more complex settings. If systematic volatility increases, then a greater proportion of return variation is associated with the state variables, which implies that returns convey more information about the pricing kernel. It follows that the discount applied to large negative returns increases and the discount applied to large positive returns decreases. Therefore, the left-tail SPOCQ increases and the right-tail SPOCQ decreases. However, increases in idiosyncratic volatility reduce the information that returns convey about the pricing kernel and cause the projected pricing kernel to shrink towards its mean of one. In the conditional CAPM model above, this effect causes the left-tail (right-tail) SPOCQ to decrease (increase) with idiosyncratic volatility.

### 3 Estimating the Components of SPOCQ

To obtain SPOCQ, we need estimates of the risk-neutral distribution function $F^*_t(\cdot)$ and the conditional quantile $q_t(\theta)$ at which to evaluate this function. In Section 3.1, we describe how we estimate $F^*_t(\cdot)$ from the cross-section of options prices. We present our approach for estimating $q_t(\theta)$ in Section 3.2.

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3 For brevity here and in the rest of the paper, we adopt the notational convention of referring to quantiles as integers, i.e., $SPOCQ_{i,T}(0,25)$ rather than $SPOCQ_{i,T}(0,0.25)$.

4 Jackwerth (2000) is one of the earliest examples of the pricing-kernel puzzle literature; see Hens and Reichlin (2012) for a more recent discussion. Beare and Schmidt (2012), Härdle, Okhrin, and Wang (2010), and Golubev, Härdle, and Timofeev (2008), have formally tested and rejected the null of a monotonically decreasing pricing kernel.

5 Left-tail SPOCQ is $SPOCQ_{i,T}(0,\theta)$ for small $\theta$, and right-tail SPOCQ is $SPOCQ_{i,T}(\theta,100)$ for large $\theta$. 
3.1 Risk-neutral Distribution Implied by Options Prices

A large literature, with seminal contributions by Banz and Miller (1978) and Breeden and Litzenberger (1978), has developed methods for estimating the risk-neutral distribution directly from options prices. Our approach relies heavily on this literature. Using $X$ and $S_T$ to denote the strike and underlying price at the expiration date $T$, the price of a European put option may be written as

$$P_t(X, T) = E_t[M_{t,T} \times \max (X - S_T, 0)] = E_t[M_{t,T}] \int_{-\infty}^{X} (X - S_T)dF^*_t(S_T). \quad (20)$$

We define the adjusted put option price

$$\tilde{P}_t(X, T) \equiv \frac{1}{E_t[M_{t,T}]}P_t(X, T) \quad (21)$$

and take the derivative with respect to the strike price to get

$$\frac{\partial \tilde{P}_t(X, T)}{\partial X} = \int_{-\infty}^{X} dF^*_t(S_T) = F^*_t(X). \quad (22)$$

Put-call parity produces a parallel expression in terms of call prices

$$\tilde{P}_t(X, T) = \frac{C_t(X, T) - S_t + E_t[M_{t,T}]}{E_t[M_{t,T}]}X, \quad (23)$$

which leads us to define the adjusted call price as

$$\tilde{C}_t(X, T) \equiv \frac{C_t(X, T) - S_t}{E_t[M_{t,T}]} + X, \quad (24)$$

such that

$$\frac{\partial \tilde{C}_t(X, T)}{\partial X} = F^*_t(X). \quad (25)$$

As a result, we obtain the risk-neutral distribution $F^*_t(S_T)$ by estimating the first derivative of the adjusted call and put option price curves, $\tilde{C}_t(X, T)$ and $\tilde{P}_t(X, T)$, with respect to $X$.

We use a mixture of logistic distributions to approximate the risk-neutral distribution of the adjusted option prices in (21) and (24). We opt for mixture distributions because they offer flexible approximations to unknown distributions (Marron and Wand (1992)). By using a parametric distribution, we avoid the problems that other curve-fitting methods (e.g., splines) have in estimating the tails, and getting the estimated distribution to integrate to
one—see Figlewski (2010). The logistic distribution is appealing because its integral exists in closed form, which enables us to work with the observed options pricing curve directly.

Instead of fitting a distribution to the derivatives of the adjusted option prices, we fit the integral of a distribution to the adjusted options prices themselves. Fitting the curve before differentiating the option pricing curve is important because it avoids arbitrary assignment of the point at which the derivative applies. Thus, by using a mixture of logistic distributions, we are able to fit a flexible function to the adjusted option prices, and simultaneously impose the restriction that the derivative is a distribution.

Specifically, using $X_T$ to indicate the exercise price of an option that expires at $T$, we fit the following function

$$F^*_t (X_T) = \sum_{j=1}^{J} \omega_{jt} \Lambda_{jt} (X_T; \mu_{jt}, \sigma_{jt}),$$  \hspace{1cm} (26)

where $\Lambda_{jt} (\cdot)$ is the logistic distribution. We do so by fitting the options pricing curve to the integral of $F^*_t (X_T)$, i.e., we specify the model

$$o_{it} = \sum_{j=1}^{J} \omega_{jt} \tilde{\Lambda}_{jt} (X_{iT}; \mu_{jt}, \sigma_{jt}) + \varepsilon_{it},$$  \hspace{1cm} (27)

where $o_{it}$ denotes the $i^{th}$ adjusted options price that we observe, $X_{iT}$ is the strike price of that option, and

$$\tilde{\Lambda}_{jt} (X_T; \mu_{jt}, \sigma_{jt}) = \int \Lambda_{jt} dX_T = \sigma_{jt} \ln \left( \exp \left( \frac{X_T}{\sigma_{jt}} \right) + \exp \left( \frac{\mu_{jt}}{\sigma_{jt}} \right) \right).$$  \hspace{1cm} (28)

We impose $\sum_{j=1}^{J} \omega_{jt} = 1$, with $\omega_{jt} \geq 0$, $j = 1, \ldots, J$.

We fit the model in (27) separately for each trading date $t$. In our application, we set $J = 2$, so we fit the distribution using five parameters. This number of parameters provides considerable flexibility when compared to parametric distributions, such as the normal (two parameters), skew-normal and student-t (three parameters), as well as skew-t (four parameters; see Azzalini and Capitanio (2003)). Appendix A.3 provides the details of our method to obtain estimates of the parameters for the logistic distribution in Equation (28) using constrained non-linear least squares.
3.2 Conditional Quantiles

At each trading date \( t \), we require quantiles of the return a trader could earn by holding the asset until \( T \). Existing models for estimating conditional quantiles include quantile regression (Koenker and Bassett (1978), Koenker (2005)) univariate CAViaR models (Engle and Manganelli (2004)), multi-quantile CAViaR models (White, Kim, and Manganelli (2010)), and fully parametric GARCH models (e.g., Christoffersen, Heston, and Jacobs (2013)). Among these candidate models, the best may depend on the details of a particular application, especially on the sequence of dates \( t \) and \( T \).

The S&P 500 options have one expiration date per month; hence, we use a horizon of 28 calendar days in our empirical analysis, i.e., \( T - t = 28 \). We therefore have a sequence of monthly observations on a 28-day return, which implies a time series of non-overlapping returns. In other applications, researchers may hold \( T \) constant and vary \( t \) to follow a single option contract through its life, or look at longer horizons so that the time horizons are overlapping (e.g., monthly data on options with 3 months to expiration).

Other details are also relevant. S&P 500 option payouts are determined by value of the index at the market open on the third Friday of the month. We are interested specifically in returns from the close of business four weeks prior to expiration (a Friday) until the open four weeks ahead. Thus, we model the conditional quantiles of the log return series for time \( t = 1, \ldots, T \) defined by

\[
    r_{t,T} \equiv 100 \times \ln(S_{T}^{open}/S_{t}^{close}),
\]

where \( S_{t} \) denotes the value of the S&P 500 index on date \( t \), and \( T \) occurs 28 calendar days after \( t \).

For our baseline analysis, we use a quantile-regression model to estimate conditional quantiles. We tried several CAViaR models, but found that they did not pass diagnostic tests so we discarded them. Christoffersen, Heston, and Jacobs (2013) find that a GARCH model of daily returns works well for pricing S&P 500 options. We also estimated their model and used it to project at each date \( t \) the distribution of the return over the ensuing 28 days. However, we found that the quantiles generated by this method were much too smooth; they failed both conditional and unconditional diagnostic tests. Specifically, this GARCH model severely underestimated the probability of tail events, especially when volatility was high.

We use volatility as the main explanatory variable in our quantile regressions following Christoffersen, Hahn, and Inoue (2001), Engle and Patton (2001), and Adrian and Brunnermeier (2011), among others. Specifically, we use the square root of realized continuous
variation of daily returns (volatility) for the S&P index over the last 20 trading days.\footnote{Our measure of continuous variation follows Bollerslev and Todorov (2011) and is described in detail in Appendix A.7. We use realized continuous variation rather than realized total variation because we found that removing jumps significantly improved the fit of the model.} The quantile regression model is then given by

\[ r_{t,T} = x_t' \beta_\theta + \varepsilon_{t,T}^\theta, \]  

(30)

and the conditional quantiles are \( q_t(\theta) = x_t' \beta_\theta \). We also estimated quantile regression models that included other explanatory variables computed over the past 20 trading days, such as empirical quantiles, estimated jump tails as in Bollerslev and Todorov (2011), and lagged index returns. These variables often failed to be statistically significant and including them did not change our results.

Following Buchinsky (1998), we estimate our quantile regressions via GMM using the following moment condition

\[ E[x_t (h_t(\theta) - \theta)] = 0 \]  

(31)

where \( h_t(\theta) \) is the hit function, which equals one if \( r_{t,T} < x_t' \beta_\theta \) and zero otherwise.\footnote{Buchinsky (1998) writes this condition in the equivalent form \( E[x_t (\theta - 1/2 + 1/2 sgn(y_t - x_t' \beta_\theta))] = 0 \). See Equation (4) in the paper.} This moment condition defines the parameter vector \( \beta_\theta \) that makes hits unpredictable given a linear function of \( x_t \).

We assess the fit of our conditional-quantile models using a series of backtests that have been employed extensively in the evaluation of Value-at-Risk (VaR) model performance.\footnote{Campbell (2007) and Christoffersen (2010) offer compact recent reviews of backtesting.} If a conditional-quantile model is correctly specified, then hits \( h_t(\theta) \) occur with conditional probability \( \theta \), and thus no variable should be able to predict hits. Thus, the backtests are designed to test the null hypothesis that hits are not predictable. We provide details on the backtests in Appendix A.4.

The SPOCQ itself also provides a way to evaluate our quantile estimates. If the conditional-quantile model is correctly specified, then SPOCQ represents the value of a dollar in the event that \( r_{t,T} \) lies below the \( q_t(\theta) \) quantile of the distribution—an event with probability \( \theta \). If, on the other hand, the quantile model is misspecified, then the estimated SPOCQ represents an event with a different probability than \( \theta \). For example, if the probability that \( r_{t,T} \leq q_t(\theta) \) exceeds \( \theta \), then our estimated SPOCQ will be too large. This large SPOCQ value reflects a high probability rather than a high value for the event under consideration. Therefore, if the quantile model is well specified, then SPOCQ should not be able to predict
Thus, we augment the moment condition in (31) with the following moment condition implied by the lack of such predictability

\[ E \left[ F_t^* (x_t' \beta_0) (h_t(\theta) - \theta) \right] = 0. \] (32)

If the quantile regression model is correctly specified, then we should not be able to reject the null hypothesis that (32) is a valid over-identifying moment condition using a standard J-test (Hansen (1982)).

### 3.3 Data

The data for the S&P 500 are from the Commodity Research Bureau and span January 1990 to April 2012. Figure A.1 is a time series plot of the daily closing values for the index and Table A.1 provides summary statistics for the return series of interest in our GMM estimation of the quantile regressions, \( r_{t,T} \equiv 100 \times ln(SP_{T}^{open}/SP_{t}^{close}) \), where \( SP_{T}^{open} \) and \( SP_{t}^{close} \), are the open and close index values that are 28 calendar days apart for the 268 option trading dates in our sample.

The mean monthly return is 0.6% and not statistically significant. The median (1.2%) exceeds the mean and is statistically significant, which provides one indication of the negative skewness in the data. The first-order autocorrelation coefficient, AR(1), equals -0.02. The AR(1) coefficient for the squared returns, on the other hand, is more than an order of magnitude larger and with opposite sign (0.25), which is consistent with volatility clustering, a feature of many financial asset return series (e.g., see Engle (2004)).

The data for the S&P 500 options are from the Chicago Board Options Exchange (CBOE). The market for S&P 500 options operates between 8:30am and 3:15pm central time. At any point in time, three near-term expiration months are trading along with three additional months from the March quarterly cycle (March, June, September and December). Currently, the strike price intervals are set at 5 points (25-point intervals for distant expiration months).

S&P 500 options expire on the Saturday following the third Friday of the month. The exercise-settlement value equals the opening value of the index on the last business day (usually a Friday) before the expiration date, and trading typically ceases one day earlier (usually a Thursday). The options may be exercised only on the last business day before expiration (European style) with exercise resulting in delivery of cash on the business day.
following expiration.\footnote{Additional information about the market is available at \url{http://www.cboe.com/}.} The data in hand contain information regarding trading volume, open interest, as well as closing bid and ask quotes, for calls and puts between January 1990 and April 2012.

We measure the option price using the closing mid quote, defined as the average of the bid and ask quotes, on the Friday four weeks prior to expiration. We drop observations with mid quotes below 0.5 and those with no trading volume on day $t$ following Figlewski (2010).\footnote{We exclude “mini” contracts identified by the following codes “SXZ”, “SPB”, “LSW”, “LSX”, “LSY”, “LSZ”, “XSC”, “XSE”, “XSK”, “XSL”, “XSO”, “XSP”.}

Price information by position across the 268 trading dates in our sample is available in Table 1. The average call (put) strikes are 948 (1,083). Call mid prices are on average 167 with their put counterparts just above 70. On average, we observe 50 (45) strikes per trading date for calls (puts). Overall, the available number of strikes for both calls and puts on a given trading date increases over time from around 30 in the early 1990s to about 200 in the last two years of our sample, which is consistent with the increasing liquidity of the S&P 500 derivatives market over time. We calculate the adjusted options prices in (21) and (24), using $E_t[M_{t,T}]^{-1} = (1 + b_{t,T})$, where $b_{t,T}$ denotes the LIBOR rate that covers the interval from $t$ to $T$.

\section{Estimation results}

\subsection{Risk-Neutral Distribution: Logistic Mixture}

We fit the model in (27) for each of the 268 trading dates in our full sample calculating the adjusted options prices in (21) and (24) using the LIBOR rate that covers the interval from $t$ to $T$ as our measure of $R_{t,T}^f$.\footnote{We use the one-month London Interbank Offered Rate (LIBOR), based on U.S. Dollar series from FRED. See \url{http://research.stlouisfed.org/fred2/series/USD1MTD156N}.} We provide an example of the adjusted put and call option pricing curves in Figure 4. We constructed these pricing curves using the expressions in (21) and (24) for contracts trading on Friday, 10/24/2008, and expiring on Saturday, 11/22/2008.

In Figure 4, we have 169 strikes with calls for 142 of them and puts for all but one of them. The strikes range from 300 to 1,700 with no calls for those between 1,175 and 1,500. The average mid-quote price is 8.7 for calls and 11.4 for puts. We estimate a weight of 0.52

\footnote{To check the robustness of this measure, we repeated our analysis treating $R_{t,T}^f$ as a parameter to be estimated when we fit the risk-neutral distribution to option prices. This alternative approach does not change our results, so we proceed with LIBOR.}
on the most probable component of the logistic mixture. This component has location and scale parameters of 969 and 51, respectively. The second mixture component has a location parameter of 761 and a scale parameter of about 100. The estimated call dummy, which we include to allow for asynchronous closing times between the options and futures markets, equals -10.92, and the $R^2$ is 0.99.

We repeated the curve-fitting exercise in (27) and (28) for each of the remaining 267 trading dates in our sample. As the bottom panel of Table 1 indicates, the minimum number of strikes across both positions for a given trading date is 12, while the maximum is 338. The median number of strikes for calls (puts) is 37 (35), which provides a sufficient number of degrees of freedom for our logistic fit. Table 2 provides summary statistics for the $R^2$ when either a single logistic distribution or a mixture of two logistic distributions is used to infer the risk-neutral distribution. In the case of the mixture, we also report the weight associated with the most probable component of the mixture, which has a median value of 0.82. Fitting either a single logistic distribution or a mix of two logistic distributions leads to a median $R^2$ value greater than 0.99.

### 4.2 Conditional Quantiles: Quantile Regression

Table 3 presents GMM estimates based on (31) along with optimization diagnostics and backtests for each of the five conditional-quantile models we considered. We estimate each of the five quantiles using 268 monthly observations between January 1990 and April 2012. Each of the dates in these monthly series correspond to the trading dates of our options data.

We report results using volatility as the only predictor following the approach in Section 3.2. Volatility enters with a negative sign in the equations for the first two quantiles and with a positive sign for the remaining quantiles. The coefficient on volatility is significant at the 5% level for the 10%, 25% and 90% quantiles. We also report an LM test for the null hypothesis of constant conditional quantiles. The LM test rejects the null for the 75% quantile, which combined with the large standard error on the volatility coefficient, suggests that the 75% quantile is correlated with volatility, but that the coefficient is not precisely estimated. Hansen’s J-statistic rejects the null of correct over-identifying restrictions for the right end of the return distribution—75% and 90% quantiles—suggesting that these quantiles may not be well specified. The DQ test reinforces these findings, but most of the other backtests do not reject the model (see Appendix A.4 for details on the tests). An exception is the Bartlett and Portmanteau tests, which reject the model for the 50% quantile.
To see whether our estimated conditional quantiles deviate from the target for sustained periods of time, we plot 36-month trailing moving averages of hits in Figure 5. The dashed lines represent 80% uniform confidence bands; under a correctly specified model, we expect the entire moving average series to lie inside the bands in at least 80% of the repeated samples. The 50% and 75% quantiles generate the most sustained deviations from the target hit rate, which is consistent with the backtest results for these two quantiles. However, none of the paths are inconsistent with a correctly specified model, i.e., we observe no 36-month periods during which our model produces hits rates outside those one would expect from a correctly specified model. We find similar results for 60-month moving averages.

Our objective in estimating conditional quantiles is to approximate the beliefs of a rational trader. If our quantile model is well specified, then we should not be able to predict whether the return is likely to exceed the quantile. The failures of some of our diagnostic tests suggest predictability in the hits; at least, this seems to be the case for the 75% quantile. To the extent that our conditional quantiles are incorrect, the resulting SPOCQ series cannot be interpreted as the state price of a set of events with known and constant probability. In Appendix A.6, we show that our findings are robust to potential misspecification of the quantiles.

5 Time Variation in SPOCQ

In this section, we discuss the most salient SPOCQ features during the period 1990–2012. We estimate SPOCQ each month by evaluating the risk-neutral distribution function computed as in Section 3.1 at the estimated conditional quantiles computed as in Section 3.2. The top panel of Figure 6 shows the five conditional quantiles we obtained using the estimates in Table 3, and the bottom panel of the same figure shows the corresponding SPOCQ series.

The quantiles show significant negative skewness, which is greatest when the market is most volatile. On average, the difference between the median and the lowest quartile is around 80 percent larger than the difference between the median and the upper quartile. Similarly, the difference $q_t(50) - q_t(10)$ exceeds the difference $q_t(90) - q_t(50)$ by about 60 percent, on average. Consistent with the excess-kurtosis behavior of many financial series, the tails of

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13 We generate these bands using a Monte Carlo experiment. First, we generate an independent binary sequence of 268 observations with the relevant success probability. Second, we construct from this sequence a series of 36-observation moving averages. Third, we record the maximum and minimum values of the moving average series. Fourth, we repeat the first three steps 10,000 times. Finally, we extract the lower 10% quantile of the minimum and the upper 90% quantile of the maximum, to obtain uniform confidence bands with at least 80% coverage.

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the conditional return distribution are fatter than those implied by a normal distribution.

Under risk neutrality, the five SPOCQ series would coincide with the straight lines at 0.10, 0.25, 0.50, 0.75, and 0.90, respectively. For the quantiles lying below the median, the SPOCQ exceeds its risk-neutral counterpart in almost all months. Thus, investors are willing to pay a premium for a dollar in the event that the S&P 500 produces a negative return. This premium averages 6 cents for the bottom decile, 7 cents for the bottom quartile, and 10 cents for below-median returns.

In contrast, the SPOCQ series for above-median returns are generally inconsistent with standard notions of risk aversion. They show evidence of an upward-sloping part in the pricing kernel consistent with the pricing kernel puzzle. Given that SPOCQ(0,θ) > θ for the left-tail series, a finding of SPOCQ(0,θ) < θ in the right tail is sufficient for the pricing kernel to be upward sloping. The SPOCQ(0,90) series is less than 0.9 for about 70 percent of the sample, especially during the highly volatile period between 1996 and 2004. The SPOCQ(0,75) series exhibits more variation than SPOCQ(0,90), and it falls below 0.75 for a quarter of the sample.

To highlight changes in discount rates over time, Figure 7 shows the annual average SPOCQ for the slices of the return distribution implied by our quantiles. We standardize by the probability that returns fall in the relevant range so that each element in the figure is measured on the same scale, i.e., we plot the quantile pricing kernel (QPK) as defined in Equation (8). By showing annual averages, we smooth the noise in the monthly series. Under risk neutrality, the QPK equals one for all quantiles; values greater than one indicate a larger than average discount rate for that quantile range.

Each curve in Figure 7 is U-shaped, which illustrates the pricing kernel puzzle. The QPK values in the left tail of the distribution exceed those in the right tail, so large negative returns are discounted more than large positive returns, which is consistent with standard theory.

For the right tail of the distribution, Figure 7 shows QPK values exceeding 1.08 each year between 1998 and 2003, and from 2009 to 2012. In other words, the pricing kernel puzzle was most prominent in the years that exhibited some of the highest volatility in the sample. The first period contains the end of the dot-com frenzy, while the second covers the financial crisis and the ensuing macroeconomic stagnation. Figure 7 also shows that returns in the second quartile (25%–50%) often have higher state prices than returns just below (10%–25%), or just above (50%–75%). This anomaly is most pronounced in the low-volatility periods of 1992–1995 and 2004–2006.
The bottom-decile QPK exceeds 1.7 in each year between 1992 and 1997, and also in 2006, 2010, and 2011. As shown in the bottom panel of Figure 6, these years exhibited the lowest volatility during our sample. Across the entire period, the bottom-decile QPK exhibits a correlation of \(-0.28\) with volatility. Based on our arguments in Section 2.2, this finding implies that most variation in volatility comes from changes in idiosyncratic volatility—risk for which investors do not require compensation.\(^{14}\)

Such patterns in the bottom decile of the return distribution could arise if we underestimate the hits when volatility is low. However, this possibility is ruled out by the moment condition in (31), which imposes zero correlation between hits and volatility. For example, over the nine years that the bottom-decile QPK exceeds 1.7, returns fell below our estimated conditional bottom decile in 9 out of 108 months, which is actually slightly less than 10%. If we focus on the bottom quartile rather than the bottom decile, we obtain the same result. As shown in Figure 1, we find a strong negative correlation between volatility and the lowest quartile SPOCQ. In the 12 years that the bottom-quartile QPK exceeds 1.3, returns fell below our estimated conditional bottom quartile in 33 out of 144 months (23%).

We are unaware of any papers in the literature that report this negative correlation. Moreover, this result contradicts recent models that specify the pricing kernel as a linear function of squared returns, such as in Christoffersen, Heston, and Jacobs (2013). However, Bakshi, Madan, and Panayotov (2010) present in their Figure 1 results consistent with our findings.\(^{15}\) They use a nonparametric method to estimate the average projected pricing kernel in two separate periods. Their plots show a flatter pricing kernel in the high-volatility 1997–2007 period than in the low-volatility 1988–96 period. Because their focus is on non-monotonicity, they do not discuss this result in the text of their paper.

The recent financial crisis is a notable exception to this left-tail pattern. During the months that exhibited the highest volatility (2008/09–2009/04), the bottom decile and quartile SPOCQs were about their average values for the entire sample. Thus, volatility had a higher systematic component during the financial crisis than in other high volatility periods. Given the prolonged slump that followed the crisis, it is not surprising that returns in this period had a larger systematic component than other high-volatility periods. If anything, it is surprising that the idiosyncratic component was large enough to keep the left-tail SPOCQs near their sample averages.

\(^{14}\)We also computed state prices at unconditional quantiles as in (19), i.e., we evaluated the risk-neutral distribution at the same point each month. Consistent with the predictions of Section 2.2, the resulting state prices were positively correlated with volatility in both tails.

\(^{15}\)We reproduce this figure in Appendix Figure A.10.
The SPOCQ series in Figure 6 exhibit substantial month-to-month variation. The series for the four lower quantiles each have standard deviations between 0.03 and 0.06 and exhibit month-to-month changes that frequently exceed 0.05. We hesitate to ascribe meaning to this high-frequency variation. Conditional on the estimated risk-neutral distribution, there is a one-to-one mapping between SPOCQ and the conditional quantiles. Therefore, less variation in the estimated quantile series implies more variation in SPOCQ series and vice versa. If the true conditional quantiles contain a component that is not predictable based on lagged observables, our estimated quantiles series will be too smooth and the implied SPOCQ series will be too noisy.

Estimation error in the risk-neutral distribution would also generate estimation error in SPOCQ. Such error could arise from the fact that we use mid quotes to proxy for options prices. If the true options pricing curve is closer to the ask in some months and closer to the bid in other months, then using mid quotes could induce noise in SPOCQ. To address this possibility, we re-estimated our SPOCQ series using only bid quotes and using only ask quotes. Compared to those from mid quotes, the resulting SPOCQ series were very similar for the bids and slightly more volatile for the ask. This finding suggests that estimation error in the risk-neutral distribution is not the source of the high-frequency fluctuations in Figure 6.

6 Connecting SPOCQ to Risk Factors

The preceding sections illustrate substantial time variation in SPOCQ. In particular, high stock-return volatility is associated with lower state prices for negative returns and higher state prices for returns in the top quartile of the distribution. Based on our earlier discussion, we interpret the changes in volatility for our sample as predominantly due to changes in idiosyncratic rather than systematic volatility. If our interpretation is correct, then this result should not be explained by other variables that are related to the pricing kernel, i.e., variables that reflect the underlying state of the macroeconomy and financial markets. Thus, we regress our monthly SPOCQ series on a set of variables that capture (i) volatility, (ii) sentiment, (iii) liquidity, and (iv) macroeconomic/financial risk.
6.1 Preliminaries

A key insight offered by intertemporal equilibrium models, such as the ICAPM of Merton (1973), is that the state variables of the return generating process are state variables of the pricing kernel. Variables that generate extra returns are the ones that are correlated with the stochastic discount factor (pricing kernel). Therefore, we base our set of regressors on those variables that have been shown to exhibit significant predictive power for expected excess stock returns. Expected returns increase when the projected pricing kernel rotates clockwise, so we expect variables that increase expected returns to have positive effects on the left-tail SPOCQ and negative effects on the right-tail SPOCQ.

In the following paragraphs, we provide a brief overview and some further motivation for the variables we consider. Additional details, including data sources, are available in Appendix A.7.

(i) Volatility

Basic portfolio theory posits that investors seek compensation bearing risk in the form of high volatility. Our results in this paper suggest that much of the variation in volatility is idiosyncratic and therefore not priced. Although many studies find volatility to be a weak predictor of returns, numerous papers do find support for the pricing of volatility; see, for example, Ang, Hodrick, Xing, and Zhang (2006) and Adrian and Rosenberg (2008). In addition, several studies have shown that tail jumps characterize the behavior of asset returns (e.g., Andersen, Bollerslev, and Diebold (2007)) and that jump risk is priced (Bollerslev and Todorov (2011)). As Constantinides, Jackwerth, and Savov (2011) point out, variables from this strand of literature may proxy for a factor that is a non-linear function of the market return. Therefore, in addition to the continuous-variation volatility variable we use in estimating SPOCQ, we also include a tail jump variable. As before, we measure continuous variation using the logarithm of the square root of the realized continuous variation in the S&P 500 index daily log returns. We construct a measure of tail jumps using the square root of the difference between the realized standard deviation and the realized continuous variation of the S&P 500 daily log returns over the previous 20 trading days.

(ii) Sentiment

In classical asset pricing models with complete markets, prices are generated by a rational-
expectations equilibrium. Investor sentiment, which is a possibly erroneous belief about the future, has no effect on asset prices because arbitrage eliminates from prices any sentiment signals unrelated to fundamentals. Behavioral theories, on the other hand, incorporate investor sentiment in equilibrium asset pricing; see, for example, Delong, Shleifer, Summers, and Waldman (1990) and Baker and Wurgler (2006). Following Lemmon and Portniaguina (2006), we use the Michigan Consumer Confidence Index as our first sentiment measure. We also use the S&P 500 return over the past 20 trading days as a proxy for sentiment, based on the notion that changes in the pricing kernel in response to recent stock returns may reflect mood more than fundamentals.\footnote{Other possibilities for sentiment include the closed-end fund discount (CEFD), the bull-bear spread based on the survey of American Association of Individual Investors, the Consumer Board Consumer Confidence Index, the Yale/Shiller crash confidence index, the Duke/CFO survey, and the investor sentiment index in Baker and Wurgler. Preliminary regressions with these variables did not change qualitatively our results.}

(iii) Liquidity

Evidence that less liquid assets have higher expected returns has been presented by numerous authors (e.g., Korajcyk and Sadka (2008) and Pástor and Stambaugh (2003)). We employ two measures of liquidity from our S&P 500 options data, one based on the logarithm of the dollar trading volume and one based on the logarithm of open interest on the relevant trading date. Our hypothesis is that, when the market is less liquid for options with strikes in a particular part of the distribution, then writers of those options require a greater premium. For example, a less liquid market for out-of-the-money call options may raise the price of those options thereby raising the SPOCQ in the right tail of the return distribution. This is one potential explanation for the pricing kernel puzzle, albeit an explanation that would be incomplete without a theory for liquidity fluctuations.

(iv) Macroeconomic/Financial Risk

We use five variables to capture macroeconomic or financial risk factors. First, we use the log of the dividend yield. The price-dividend ratio is among the most commonly used predictors of stock returns (e.g., Fama and French (1988) and Campbell and Shiller (1988)). Using the Campbell-Shiller approximation for the log price-dividend ratio, and assuming that prices and dividends are cointegrated, Lettau and Ludvigson (2010) offer a concise statistical explanation for why the ratio works. They point out that the dividend yield is the cointegrating residual, so by the Granger representation theorem it should be able to predict changes in one or both of the cointegrated variables. As with the liquidity variables, an association between the dividend yield and the SPOCQ provides an incomplete explanation without a theory for what causes changes in dividend yield. Our remaining four variables
connect more directly to macroeconomic risk.

Following Lettau and Ludvigson (2001a) and Lettau and Ludvigson (2001b), we use the variable known as CAY, which is the residual of the cointegrating relation for log consumption (C), log asset wealth (A), and log labor income (Y). Similarly to the dividend yield, assuming that consumption, asset wealth, and labor income are cointegrated, the cointegrating residual has to forecast future consumption growth, future returns to asset wealth, or future return to labor income growth. Lettau and Ludvigson find no evidence of CAY being able to forecast consumption growth or labor income growth, they do find evidence of CAY being able to forecast excess returns on the aggregate stock market (see Section 2.6.2 in Lettau and Ludvigson (2010)).

Uncertainty about the future state of the economy may increase risk aversion, which is a prediction of the model in Pástor and Veronesi (2012). To account for this possibility, we include the daily news-based Economic Policy Uncertainty index from the Policy Uncertainty Project at Stanford University. More specifically, we use the logarithm of the Uncertainty index summed over the last 28 calendar days as one of our explanatory variables.

Finally, we use two interest-rate variables to capture macroeconomic conditions. The first variable is the default spread and captures the risk premium associated with corporate bond defaults. We measure default spread using the difference between BAA and AAA corporate bond yields; numerous studies have found default spreads to exhibit predictive power for stock returns (e.g., Chen (1991) and Ferson and Harvey (1991)). The second variable is the term spread, the difference between the 10-year and the 3-month Treasury yields, a commonly used predictor of macroeconomic growth (e.g., Stock and Watson (1989)). The default and term spreads serve as proxies for expected business conditions (Campbell and Diebold (2009)) and have been linked to the pricing kernel (Engle and Rosenberg (2002)).

We use the QPK rather than SPOCQ as the dependent variable in our regressions so as to standardize the units and thereby make the coefficients comparable across regressions. We estimate

\[ QPK_{t,T}(\theta_{low}, \theta_{high}) = \frac{SPOCQ_t(\theta_{low}, \theta_{high})}{\theta_{high} - \theta_{low}} = \alpha + \mathbf{x}_t' \beta + \epsilon_t, \]  

(33)

We use Newey-West standard errors (12 lags) to correct for autocorrelation, and we standardize all explanatory variables such that their mean is zero and their standard deviation is one to ease comparison of their coefficients.

To keep the discussion tractable and to avoid overfitting the data, we report results for three specifications. The specifications differ in the number of elements of \( \mathbf{x}_t \). Specifica-
tion 1 (most parsimonious) reproduces our volatility correlations by including volatility as the only regressor. In specification 2, we examine whether our liquidity, sentiment, and macroeconomic/financial variables explain the QPK time variation as well as volatility does. In specification 3, (least parsimonious), we check whether the regressors of specification 2 maintain their explanatory power once we add the two volatility variables. Specifications 2 and 3 involving CAY are based on a smaller number of observations because we don’t have CAY data for the four months in 2012.  

For the remainder of our discussion, we refer to the three specifications, as SPEC1, SPEC2, and SPEC3, respectively. For the left part of the distribution, we focus on the bottom-decile (0–10%) and bottom-quartile (0–25%) QPKs. For the right part of the distribution, we focus on the top quartile (75%–100%) and the top decile (90%–100%) QPKs. The mean of the QPK across the whole distribution equals one, so the results for the two tails of the distribution imply the patterns for the middle. For example, if a variable raises the QPK in both the top and bottom quartiles, then it must lower the QPK on average in the middle of the distribution. Therefore, we omit results for the middle of the distribution.

### 6.2 Left-Tail Results

Table 4 presents results for the left part of the distribution. SPEC1 shows negative and statistically significant coefficients on volatility in both cases, which restates the discussion in Section 5 and the baseline correlations reported in Table A.3. SPEC2 shows that the sentiment, liquidity and macro/finance variables have substantial explanatory power for the QPK. The adjusted $R^2$ is larger for SPEC2 than for SPEC1 in both cases.

Comparing SPEC1 and SPEC3, the coefficient on volatility is substantially more negative for the bottom decile and somewhat more negative for the bottom quartile. One reason the bottom-decile coefficient becomes more negative is that tail jumps enter with a significant positive coefficient. In other words, tail jumps are associated with an increase in downside risk aversion. SPEC1 confounds the positive effects of tail jumps with the negative effects of idiosyncratic continuous volatility because it only contains a single volatility variable. Hence, when we control for tail jumps, we get a stronger suggestion that variation in continuous

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18 We also checked whether persistence in preference matters by including the lag of the pricing kernel among the explanatory variables in all three specifications. The lag of the pricing kernel enters with a significant coefficient (0.165) in SPEC1 for the bottom quartile, in all three specifications for the upper quartile (values between 0.129 and 0.249), as well as in SPEC1 (0.201) and SPEC2 (0.155) for the upper decile. In all other cases, the coefficient is not significant at the 5% level. Including the lag of the pricing kernel does not compromise the explanatory power of volatility in any case.
volatility is predominantly idiosyncratic. Tail jumps are insignificant in the bottom-quartile regressions. Together, these results are consistent with our interpretation that a negative volatility coefficient implies that much of the variation in volatility is idiosyncratic.

Consumer confidence exhibits significant positive relationship with the pricing kernel for the bottom decile that disappears when we control for volatility. The lagged return, which we also classify as a sentiment indicator, has a significant negative coefficient in the bottom decile. This result suggests that a high return over the past month reduces aversion to events in the tails of the return distribution. However, the sentiment variables have small and insignificant coefficients in the bottom-quartile regressions, so their relevance appears to be restricted to the far left tail.

Our liquidity variables measure characteristics of trading in options with strikes in the relevant quantile, i.e., they differ across quantiles. These variables enter the model with opposite signs and are statistically significant in the bottom decile and quartile for SPEC3. The opposite signs and similar magnitudes of these coefficients, along with the fact that these variables enter in logs, suggest that volume relative to open interest is what affects the SPOCQ. In particular, high volume relative to open interest is associated with greater aversion to downside risk. We do not interpret this as a liquidity effect because it is of the opposite sign to our hypothesized liquidity effect. Rather, these results suggest that heavy trading activity in the tails is symptomatic of higher risk aversion.

SPEC2 produces results for the dividend yield that appear consistent with the return predictability literature (e.g., Fama and French (1988) and Campbell and Shiller (1988)). A high dividend yield indicates a high value of left-tail states and therefore high expected returns. However, for both the lowest decile and quartile, this coefficient becomes insignificant once we control for volatility in SPEC3.

The CAY and policy uncertainty variables produce similar results to each other. They have a significant positive coefficient of about 0.08 for the bottom decile. The coefficient is almost twice as large for the bottom decile as for the bottom quartile. Thus, these variables enter the pricing kernel through their association with far-left-tail events.

The term spread is positively associated with the left-tail pricing kernel (both SPEC2 and SPEC3).

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19 As we discuss in detail in Appendix A.7, for each trading date, we calculated the moneyness for each put and call strike for options with one month to expiration. We then assigned the open interest for the call and put options in the 6 moneyness bins implied by the estimates of the conditional quantiles in Section 4.2.

20 The CAY variable is measured during the month whereas the QPK is measured on a particular day mid month. This difference could explain some of the weakness in explanatory power. A similar point could be made for the consumer sentiment index.
SPEC3), which implies that steepening of the term-structure slope coincides with high risk aversion. SPEC3 shows that a high default spread implies high state prices in both the bottom decile and quartile, which is consistent with the findings of Engle and Rosenberg (2002) for countercyclicality of risk aversion. However, this result is dependent on volatility being in the model. When we remove volatility from the model (SPEC2), the coefficients on the default spread switch sign suggesting that the explanatory power of the default spread is easily confounded with that of volatility.

### 6.3 Right-Tail Results and the Pricing Kernel Puzzle

Table 5 contains the right-tail counterparts of the left-tail regressions in Table 4. For SPEC1, we see the positive correlation between volatility and the right-tail SPOCQ. Because the coefficient for the top quartile (0.14) exceeds in absolute value the coefficient in the bottom quartile (-0.10), the implied effect in the middle of the distribution should also be negative. Thus, when volatility increases, the value of a dollar tends to increase in upper-quartile states of the world and decrease in all other states.

Table 5 shows that the volatility coefficient becomes much smaller when we control for the tail jump, sentiment, liquidity and macro/finance variables—in the case of the upper decile, the coefficient becomes negative (albeit, not significant). Unlike the left tail, the right-tail quantile estimates in Table 3 failed some diagnostic tests, which means that some variation in the right-tail SPOCQ could stem changes in the probability of right-tail events rather than changes in the projected pricing kernel. However, the positive correlation between volatility and right-tail SPOCQ is robust to alternative quantile models. For example, this result holds when we use the moment condition in (32) to estimate conditional quantiles while imposing zero correlation between SPOCQ and hits (see Appendix A.6).

Similar to the results for the left tail, the tail-jump variable is positively related to the QPK. This finding suggests that tail jumps are associated with an increase in the pricing kernel in both tails of the distribution. Christoffersen, Heston, and Jacobs (2013) find that volatility raises both the left and the right end of the pricing kernel. Our result suggests that it is only the volatility that stems from tail jumps that has this feature.

Based on the results for SPEC2, the sentiment, liquidity and macro/finance variables have substantial explanatory power for the right-tail QPK. The adjusted $R^2$ is much larger for SPEC2 than for SPEC1 in both cases.

Consumer confidence exhibits a weak and statistically insignificant relationship with the
The coefficient on the lagged return is negative and statistically significant, which, when combined with the left-tail results, suggests that a high return over the past month reduces aversion to tail events. Of the liquidity variables, volume enters with a significant positive coefficient, but open interest has a small and insignificant coefficient. Thus, high volume is associated with greater aversion to tail risk. As with the left tail, these results suggest that heavy trading activity in the tails is symptomatic of higher state prices.

Expected returns increase when the projected pricing kernel rotates clockwise, placing a larger discount on negative returns and a smaller discount on positive returns. Consistent with the return predictability literature, increases in the dividend yield or term spread are associated with a clockwise rotation in the pricing kernel thereby increasing expected returns. These variables enter with negative coefficients in the right tail and positive coefficients in the left tail. Policy uncertainty and the default spread have insignificant coefficients in SPEC3 for both cases.

The CAY variable has a significant positive coefficient for both the top quartile and the top decile. The top-decile coefficient is especially large at 0.20 in SPEC3. Typically, we would expect a variable that is positively associated with expected returns to rotate the pricing kernel clockwise, i.e., to have positive coefficients in the left tail and negative coefficients in the right tail. Given the failures of our baseline quantile-regression model to pass all diagnostic tests in the right tail, one could argue that the positive right-tail coefficients here reflect a higher probability of the event rather than a shift in the pricing kernel. However, when we added CAY as a predictor to our quantile regression models, we estimated insignificant coefficients for the 75% and 90% conditional quantiles. Thus, our results do not suggest a clear association between CAY and expected returns.

The regression results in Table 5 suggest factors that affect the average of the pricing kernel on the right side of the return distribution, but do not address the pricing kernel puzzle directly. To better capture time variation in the pricing kernel puzzle, we repeated the regressions in Table 5 using the difference between the right-tail QPK and third-quartile QPK as the dependent variable. The results, which we report in Appendix Table A.2 are similar to those in Table 5. Volume, CAY, and tail jumps have strong positive coefficients, whereas lagged returns and dividend yield have strong negative coefficients. After controlling for these variables, volatility is insignificant. This finding suggests that the magnitude of the pricing kernel puzzle is more closely related to systematic risk factors than idiosyncratic.

---

21 For example, consider an increase in $\beta_t$ in equations (11) and (16).

22 Specifically, we estimate models with $QPK_{i,T}(75,100) - QPK_{i,T}(50,75)$ and $QPK_{i,T}(90,100) - QPK_{i,T}(50,75)$ as the dependent variable. For open interest and dollar volume, we use measures for the right tail, i.e., the same variables as in Table 5.
volatility.

7 Conclusion

We track the option-implied stochastic discount factor for S&P 500 returns between January 1990 and April 2012. We do so by developing a set of statistics, the State Prices of Conditional Quantiles (SPOCQ), which reflect the market’s willingness to pay for insurance against outcomes across various quantiles of the return distribution. We construct the SPOCQ series by evaluating the risk-neutral return distribution at the conditional quantiles of the physical return distribution. There is a direct and straightforward link between the pricing kernel and SPOCQ. The projected pricing kernel over a range of the return distribution is equal to the SPOCQ divided by the probability mass associated with such a range. We label this ratio the Quantile Pricing Kernel (QPK).

To our knowledge, we offer the first attempt in the literature to examine the time variation in the discount factor using quantiles. Consistent with the robust evidence presented in other studies, we find that the projected pricing kernel is non-monotonic. Moreover, our focus on quantiles allows us to discover that volatility relates to projected pricing kernel in a surprising way. High volatility is associated with a relatively low value of left-tail returns. This finding implies that much of the variation in volatility for our sample was idiosyncratic to the S&P 500 and not associated with the pricing kernel.

References


Koenker, R., 2005, Quantile Regression (Cambridge University Press).


———, 2001b, Resurrecting the (c)capm: A cross-sectional test when risk premia are time-varying, Journal of Political Economy 109, 1238–1287.


Tables and Figures

Table 1. Summary statistics for the S&P 500 options

The columns under the heading “Strike Prices” show the minimum, mean, and maximum observed prices across 268 trading dates, which are four weeks prior to option expiration for each month between Jan-1990 and Apr-2012. The columns under “Average Prices” show the mean bid and ask prices, and the mean of the midpoint between the bid and ask prices.

<table>
<thead>
<tr>
<th>Position</th>
<th>Number of trading dates</th>
<th>Strike Prices</th>
<th>Average Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>min</td>
<td>mean</td>
</tr>
<tr>
<td>Call</td>
<td>268</td>
<td>50</td>
<td>948</td>
</tr>
<tr>
<td>Put</td>
<td>268</td>
<td>250</td>
<td>1,083</td>
</tr>
<tr>
<td>Total</td>
<td>268</td>
<td>50</td>
<td>999</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Position</th>
<th>Number of strikes per trading date</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean</td>
</tr>
<tr>
<td>Call</td>
<td>50</td>
</tr>
<tr>
<td>Put</td>
<td>45</td>
</tr>
<tr>
<td>Total</td>
<td>59</td>
</tr>
</tbody>
</table>

Table 2. Summary statistics for the the fit of the adjusted call and put option prices

The table summarizes the results for the curve-fitting exercises we performed to estimate the risk-neutral distribution on each of the 268 trading dates, which are four weeks prior to option expiration for each month between Jan-1990 and Apr-2012. We show the minimum, maximum, and median estimates of the weight parameter when we use a mixture of two logistic distributions. The two rightmost columns summarize the fit of the single and mix models using the standard $R^2$.

<table>
<thead>
<tr>
<th>Weight</th>
<th>R-squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixture</td>
<td>Single</td>
</tr>
<tr>
<td>Min</td>
<td>0.500</td>
</tr>
<tr>
<td>Max</td>
<td>0.998</td>
</tr>
<tr>
<td>Median</td>
<td>0.823</td>
</tr>
</tbody>
</table>
Table 3. GMM quantile regression: estimates and model diagnostics

We report GMM estimates and model diagnostics for the quantile-regression model using the moment conditions (31) and (32). The standard errors are in parentheses and are calculated using the GMM sandwich expression with numerical derivatives of the moments. We report p-values in squared brackets. The asterisks indicate statistical significance as follows: 1% (***) , 5% (**), 10% (*). The $R^2$-type goodness-of-fit measure in Panel B follows Koenker and Machado (1999). The details of the tests in Panel C are available in Appendix A.4. The details for the optimization design underlying the GMM estimation, including the diagnostics in Panel D, are available in Appendix A.5.

<table>
<thead>
<tr>
<th>Variable</th>
<th>10%</th>
<th>25%</th>
<th>50%</th>
<th>75%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility</td>
<td>-5.328 ***</td>
<td>-2.224 ***</td>
<td>0.456</td>
<td>1.596</td>
<td>3.557 ***</td>
</tr>
<tr>
<td>Constant</td>
<td>0.003</td>
<td>0.005</td>
<td>0.009 **</td>
<td>0.017 **</td>
<td>0.022 ***</td>
</tr>
</tbody>
</table>

Panel A: Estimates

<table>
<thead>
<tr>
<th>Measure</th>
<th>10%</th>
<th>25%</th>
<th>50%</th>
<th>75%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>J-statistic</td>
<td>0.102</td>
<td>0.459</td>
<td>0.001</td>
<td>4.824 **</td>
<td>7.504 ***</td>
</tr>
<tr>
<td>LM test</td>
<td>8.598 ***</td>
<td>3.193 *</td>
<td>0.598</td>
<td>5.143 **</td>
<td>9.333 ***</td>
</tr>
<tr>
<td>Hits</td>
<td>10.4%</td>
<td>25.7%</td>
<td>49.3%</td>
<td>76.1%</td>
<td>90.3%</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.068</td>
<td>0.015</td>
<td>0.002</td>
<td>0.018</td>
<td>0.101</td>
</tr>
</tbody>
</table>

Panel B: Measures of Fit

<table>
<thead>
<tr>
<th>Test</th>
<th>10%</th>
<th>25%</th>
<th>50%</th>
<th>75%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>DQ test</td>
<td>0.139</td>
<td>0.691</td>
<td>0.104</td>
<td>5.484 **</td>
<td>9.897 ***</td>
</tr>
<tr>
<td>VQR test</td>
<td>6.975 **</td>
<td>0.557</td>
<td>1.160</td>
<td>4.023</td>
<td>3.272</td>
</tr>
<tr>
<td>LR IND test</td>
<td>4.562</td>
<td>2.773</td>
<td>3.784</td>
<td>1.855</td>
<td>1.617</td>
</tr>
<tr>
<td>LR CC test</td>
<td>4.621</td>
<td>2.852</td>
<td>3.844</td>
<td>2.036</td>
<td>1.644</td>
</tr>
<tr>
<td>Bartlett's test</td>
<td>0.628</td>
<td>0.832</td>
<td>1.355 *</td>
<td>0.862</td>
<td>0.732</td>
</tr>
</tbody>
</table>

Panel C: Diagnostic Tests for Model Fit

<table>
<thead>
<tr>
<th>Test</th>
<th>10%</th>
<th>25%</th>
<th>50%</th>
<th>75%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|g|_\infty$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$g^Hg^{-1}$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$\kappa(H)$</td>
<td>2.85E+04</td>
<td>4.61E+04</td>
<td>3.99E+04</td>
<td>3.64E+05</td>
<td>1.12E+04</td>
</tr>
<tr>
<td>obs.</td>
<td>268</td>
<td>268</td>
<td>268</td>
<td>268</td>
<td>268</td>
</tr>
</tbody>
</table>
Table 4. Quantile pricing kernel regressions: lower part of the return distribution

We report OLS estimates and model diagnostics for the quantile pricing kernel regressions in Section 6 that pertain to the lower part of the return distribution. The standard errors are in parentheses and are adjusted for autocorrelation using the Newey-West formula (12 lags). The p-values are in squared brackets. The asterisks indicate statistical significance as follows: 1% (***(1)), 5% (**), 10% (*). We standardize all explanatory variables such that their mean is zero and their standard deviation is one to ease comparison of their coefficients.

<table>
<thead>
<tr>
<th>Variable</th>
<th>0–10%</th>
<th>0–25%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SPEC1</td>
<td>SPEC2</td>
</tr>
<tr>
<td>Volatility</td>
<td>-0.148***</td>
<td>-0.265***</td>
</tr>
<tr>
<td></td>
<td>(0.048)</td>
<td>(0.040)</td>
</tr>
<tr>
<td>Tail jumps</td>
<td>0.075***</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td>(0.020)</td>
<td>(0.012)</td>
</tr>
<tr>
<td>Returns</td>
<td>-0.130***</td>
<td>-0.106***</td>
</tr>
<tr>
<td></td>
<td>(0.026)</td>
<td>(0.015)</td>
</tr>
<tr>
<td>Consumer sentiment</td>
<td>0.124**</td>
<td>0.043</td>
</tr>
<tr>
<td></td>
<td>(0.055)</td>
<td>(0.030)</td>
</tr>
<tr>
<td>Log open interest</td>
<td>-0.023</td>
<td>-0.117***</td>
</tr>
<tr>
<td></td>
<td>(0.043)</td>
<td>(0.022)</td>
</tr>
<tr>
<td>Log dollar volume</td>
<td>0.219***</td>
<td>0.211***</td>
</tr>
<tr>
<td></td>
<td>(0.047)</td>
<td>(0.047)</td>
</tr>
<tr>
<td>Log dividend yield</td>
<td>0.187***</td>
<td>-0.007</td>
</tr>
<tr>
<td></td>
<td>(0.044)</td>
<td>(0.061)</td>
</tr>
<tr>
<td>CAY</td>
<td>0.066</td>
<td>0.086**</td>
</tr>
<tr>
<td></td>
<td>(0.040)</td>
<td>(0.040)</td>
</tr>
<tr>
<td>Log uncertainty index sum</td>
<td>0.052</td>
<td>0.083***</td>
</tr>
<tr>
<td></td>
<td>(0.038)</td>
<td>(0.039)</td>
</tr>
<tr>
<td>Default spread</td>
<td>-0.088**</td>
<td>0.083**</td>
</tr>
<tr>
<td></td>
<td>(0.035)</td>
<td>(0.042)</td>
</tr>
<tr>
<td>Term spread</td>
<td>0.083***</td>
<td>0.059**</td>
</tr>
<tr>
<td></td>
<td>(0.027)</td>
<td>(0.029)</td>
</tr>
<tr>
<td>Constant</td>
<td>1.682***</td>
<td>1.683***</td>
</tr>
<tr>
<td></td>
<td>(0.038)</td>
<td>(0.025)</td>
</tr>
<tr>
<td>obs.</td>
<td>268</td>
<td>264</td>
</tr>
<tr>
<td>R² adj.</td>
<td>0.113</td>
<td>0.307</td>
</tr>
<tr>
<td></td>
<td>[0.002]</td>
<td>[0.000]</td>
</tr>
</tbody>
</table>
Table 5. Quantile pricing kernel regressions: upper part of the return distribution

We report OLS estimates and model diagnostics for the quantile pricing kernel regressions in Section 6 that pertain to the upper part of the return distribution. The standard errors are in parentheses and are adjusted for autocorrelation using the Newey-West formula (12 lags). The p-values are in squared brackets. The asterisks indicate statistical significance as follows: 1% (***) 5% (**), 10% (*). We standardize all explanatory variables such that their mean is zero and their standard deviation is one to ease comparison of their coefficients.

<table>
<thead>
<tr>
<th>Variable</th>
<th>75% – 100%</th>
<th>90% – 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SPEC1</td>
<td>SPEC2</td>
</tr>
<tr>
<td>Volatility</td>
<td>0.138***</td>
<td>0.054**</td>
</tr>
<tr>
<td></td>
<td>(0.022)</td>
<td>(0.023)</td>
</tr>
<tr>
<td>Tail jumps</td>
<td>0.072***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.023)</td>
</tr>
<tr>
<td>Returns</td>
<td>-0.105***</td>
<td>-0.079***</td>
</tr>
<tr>
<td></td>
<td>(0.013)</td>
<td>(0.009)</td>
</tr>
<tr>
<td>Consumer sentiment</td>
<td>-0.010</td>
<td>-0.016</td>
</tr>
<tr>
<td></td>
<td>(0.037)</td>
<td>(0.032)</td>
</tr>
<tr>
<td>Log open interest</td>
<td>-0.042</td>
<td>-0.031</td>
</tr>
<tr>
<td></td>
<td>(0.029)</td>
<td>(0.026)</td>
</tr>
<tr>
<td>Log dollar volume</td>
<td>0.150***</td>
<td>0.130***</td>
</tr>
<tr>
<td></td>
<td>(0.034)</td>
<td>(0.033)</td>
</tr>
<tr>
<td>Log dividend yield</td>
<td>-0.110***</td>
<td>-0.085***</td>
</tr>
<tr>
<td></td>
<td>(0.032)</td>
<td>(0.028)</td>
</tr>
<tr>
<td>CAY</td>
<td>0.089***</td>
<td>0.078***</td>
</tr>
<tr>
<td></td>
<td>(0.026)</td>
<td>(0.021)</td>
</tr>
<tr>
<td>Log uncertainty index sum</td>
<td>0.071**</td>
<td>0.033</td>
</tr>
<tr>
<td></td>
<td>(0.029)</td>
<td>(0.023)</td>
</tr>
<tr>
<td>Default spread</td>
<td>0.028*</td>
<td>-0.010</td>
</tr>
<tr>
<td></td>
<td>(0.016)</td>
<td>(0.014)</td>
</tr>
<tr>
<td>Term spread</td>
<td>-0.066***</td>
<td>-0.044***</td>
</tr>
<tr>
<td></td>
<td>(0.020)</td>
<td>(0.018)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.927***</td>
<td>0.930***</td>
</tr>
<tr>
<td></td>
<td>(0.025)</td>
<td>(0.020)</td>
</tr>
<tr>
<td>obs.</td>
<td>268</td>
<td>264</td>
</tr>
<tr>
<td>(R^2) adj.</td>
<td>0.225</td>
<td>0.546</td>
</tr>
<tr>
<td>F-test</td>
<td>40.920</td>
<td>35.190</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.000]</td>
</tr>
</tbody>
</table>
Figure 1. Scatter plot of the SPOCQ against the logarithm of volatility

Note: We measure volatility using the square root of realized continuous variation of daily returns over the last 20 trading days. We compute this statistic using the method in Bollerslev and Todorov (2011), which we describe in Appendix A.7.

Figure 2. State Price of Conditional Quantiles (SPOCQ) interpretation

Note: We obtain SPOCQ at a particular quantile ($\theta_j$) by first inverting the physical return distribution to find the quantile $q_t(\theta_j)$ and then evaluating the risk-neutral distribution at $q_t(\theta_j)$. 
Figure 3. Interpreting Variation in SPOCQ

Note: Using Equations (9)–(16), we plot SPOCQ against the standard deviation of returns ($\sigma_{R,t}$). We set $R_{t,T}^f = 1.03$, $\beta_t = 1$, and $W \sim N(0, \sigma_W^2)$. In the top panel, we set $\sigma_W = 0.2$ and $\sigma_\epsilon \in (0, 0.3]$. In the bottom panel, we set $\sigma_W \in [0.15, 0.3]$ and $\sigma_\epsilon = 0.1$. For the dashed lines, we evaluate the risk-neutral distribution at $-0.15$, which corresponds to the 25% quantile when $\sigma_W = 0.2$ and $\sigma_\epsilon = 0.1$. 

(a) Constant Systematic Volatility and Changing Idiosyncratic Volatility

(b) Changing Systematic Volatility and Constant Idiosyncratic Volatility
Figure 4. Constrained-NLS fit of adjusted put and call options prices

Note: The top panel of the figure shows the put option prices on the contracts traded on October 24, 2008 for expiration on November 22, 2008. Each put price is adjusted (discounted) by the risk-free rate, for which we use LIBOR. The bottom panel shows the parity-adjusted call prices. We use a dark line to indicate the fit achieved by the logistic mixture distribution.
Figure 5. Rolling 36-month hit rates across the five quantiles

Note: Each point is the hit rate over the past 36 months. Dashed lines indicate 80% uniform confidence bands, i.e., under correct specification, the probability that any value lies inside the bands is at least 0.8. We generate these bands using a Monte Carlo simulation.
Figure 6. GMM conditional quantiles and SPOCQ

Note: The top panel of the figure shows the monthly time series of the conditional quantiles based on GMM estimation in Section 4.2. The bottom panel of the figure shows the monthly time series of the implied SPOCQ. Each monthly observation corresponds to a trading date in our options data. For brevity, we write quantiles as integers.
Figure 7. Quantile pricing kernels over time

Note: Each plot in the figure shows the average quantile pricing kernel by year. We calculate the annual average quantile pricing kernel across the ranges of the return distribution indicated by the horizontal axis using $QPK_{t,T}(\theta_{low}, \theta_{high}) = SPOCQ_{t,T}(\theta_{low}, \theta_{high})/(\theta_{high} - \theta_{low})$, where $t$ denotes a month. Each monthly observation corresponds to a trading date in our options data. For brevity, we write quantiles as integers.
A  Online Appendix-Not for Publication
A.1 Time Variation in SPOCQ: Derivations

i. Projected Pricing Kernel

\[ M^*_t,T = R^f_t \left( \frac{1}{R^f_{t,T}} + \frac{\text{cov}_t[R_{t,T}, W_T]}{\sigma_{R,t}^2} E_t[R_{t,T}] \right) - R^f_t \frac{\text{cov}_t[R_{t,T}, W_T]}{\sigma_{R,t}^2} R_{t,T} \]

\[ = 1 + R^f_t \frac{\text{cov}_t[R_{t,T}, W_T]}{\sigma_{R,t}^2} E_t[R_{t,T}] - R^f_t \frac{\text{cov}_t[R_{t,T}, W_T]}{\sigma_{R,t}^2} R_{t,T} \]

\[ = 1 - R^f_t \frac{\text{cov}_t[R_{t,T}, W_T]}{\sigma_{R,t}^2} (R_{t,T} - E_t[R_{t,T}]) \]

\[ \equiv 1 - \gamma_t (R_{t,T} - E_t[R_{t,T}]) \tag{34} \]

where

\[ \gamma_t \equiv R^f_t \frac{\beta_t \sigma_{W,t}^2}{\beta_t^2 \sigma_{W,t}^2 + \sigma_{R,t}^2} \tag{35} \]

ii. SPOCQ

Using \( E^*_t = E[| R_{t,T} < q_t(\theta) ] \), we have

\[ E_t[M^*_t,T | R_{t,T} < q_t(\theta)] = E_t[M^*_t,T | R_{t,T} < q_t(\theta)] \]

\[ = 1 - E^*_t[\gamma_t(R_{t,T} - E_t[R_{t,T}])] \]

\[ = 1 - \gamma_t E^*_t[(R_{t,T} - E_t[R_{t,T}])] \]

\[ = 1 - \gamma_t(E^*_t[R_{t,T}] - E^*_t[E_t[R_{t,T}]])) \]

\[ = 1 - \gamma_t(E^*_t[R_{t,T}] - E_t[R_{t,T}]) \tag{36} \]

The formula for the mean of a truncated normal implies

\[ E_t^*[R_{t,T}] = E_t[R_{t,T}] - \sigma_{R,t} \frac{\phi(\Phi^{-1}(\theta))}{\theta} \tag{37} \]

Therefore, we now have

\[ E_t[M^*_t,T | R_{t,T} < q_t(\theta)] = 1 - \gamma_t(E_t^*[R_{t,T}] - E_t[R_{t,T}]) \]

\[ = 1 - \gamma_t(E_t[R_{t,T}] - \sigma_{R,t} \frac{\phi(\Phi^{-1}(\theta))}{\theta} - E_t[R_{t,T}]) \]

\[ = 1 - \gamma_t \left( -\sigma_{R,t} \frac{\phi(\Phi^{-1}(\theta))}{\theta} \right) \tag{38} \]
where $k \equiv \phi(\Phi^{-1}(\theta))$.

### A.2 The Effects of Volatility in the Conditional CAPM

#### i. SPOCQ

We start with Equation (18) in the main text

$$SPOCQ_t(0, \theta) = \theta + k\gamma_t \sigma_{R,t}. \quad (40)$$

Substituting for $\gamma_t$ and $\sigma_{R,t}$, we obtain

$$SPOCQ_t(0, \theta) = \theta + k \left( \frac{R^*_t \beta_t \sigma^2_{W,t}}{\beta^2_t \sigma^2_{W,t} + \sigma^2_{\epsilon,t}} \right) \sigma_{R,t}$$

$$= \theta + k \left( \frac{R^*_t \beta_t \sigma^2_{W,t}}{\beta^2_t \sigma^2_{W,t} + \sigma^2_{\epsilon,t}} \right) \sqrt{\beta^2_t \sigma^2_{W,t} + \sigma^2_{\epsilon,t}}$$

$$= \theta + \frac{kR^*_t \beta_t \sigma^2_{W,t}}{\sqrt{\beta^2_t \sigma^2_{W,t} + \sigma^2_{\epsilon,t}}} \sigma_{R,t}. \quad (41)$$

Assuming that the S&P 500 is positively correlated with the state variable ($\beta_t > 0$), we have

$$\frac{\partial SPOCQ_t(0, \theta)}{\partial \sigma_{W,t}} = \frac{kR^*_t \beta_t (2\beta_t \sigma^2_{\epsilon,t} \sigma_{W,t} + \beta^3_t \sigma^3_{W,t})}{\left(\beta^2_t \sigma^2_{W,t} + \sigma^2_{\epsilon,t}\right)^{3/2}} > 0 \quad (42)$$

$$\frac{\partial SPOCQ_t(0, \theta)}{\partial \sigma_{\epsilon,t}} = -\frac{\beta_t kR^*_t \sigma_{\epsilon,t} \sigma_{W,t}}{\left(\beta^2_t \sigma^2_{W,t} + \sigma^2_{\epsilon,t}\right)^{3/2}} < 0. \quad (43)$$

#### ii. Risk Neutral Distribution at a Fixed Point

We start with Equation (19) in the main text

$$F^*_t(R) = \Phi \left( \frac{R - E_t[R_{t,T}]}{\sigma_{R,t}} \right) + \phi \left( \frac{R - E_t[R_{t,T}]}{\sigma_{R,t}} \right) \gamma_t \sigma_{R,t} \quad (44)$$
For the purpose of the calculations below, we define

$$g(\sigma_{R,t}, \sigma_{\varepsilon,t}) = \frac{R - E_t[R_{t,T}]}{\sigma_{R,t}} = R - \frac{R_{t,T}^f - \beta_t \sigma_{W,t}^2}{\sqrt{\beta_t^2 \sigma_{W,t}^2 + \sigma_{\varepsilon,t}^2}},$$

(45)

where we use $E_t[R_{t,T} - R_{t,T}^f] = \beta_t \sigma_{W,t}^2$. We now have the following

$$\frac{\partial F_t^*(R)}{\partial \sigma_{W,t}} = \phi(\cdot) \frac{\partial g(\cdot)}{\partial \sigma_{W,t}} + \phi'(\cdot) \frac{\partial g(\cdot)}{\partial \sigma_{W,t}} \gamma_t \sigma_{R,t} + \phi(\cdot) \frac{\partial (\gamma_t \sigma_{R,t})}{\partial \sigma_{W,t}}$$

(46)

$$\frac{\partial F_t^*(R)}{\partial \sigma_{\varepsilon,t}} = \phi(\cdot) \frac{\partial g(\cdot)}{\partial \sigma_{\varepsilon,t}} + \phi'(\cdot) \frac{\partial g(\cdot)}{\partial \sigma_{\varepsilon,t}} \gamma_t \sigma_{R,t} + \phi(\cdot) \frac{\partial (\gamma_t \sigma_{R,t})}{\partial \sigma_{\varepsilon,t}}$$

(47)

Assuming $\beta_t > 0$, we have

$$\frac{\partial g(\cdot)}{\partial \sigma_{W,t}} = -\beta_t \sigma_{W,t} (-R + \beta_t R_{t,T}^f \sigma_{\varepsilon,t}^2 + \beta_t^2 R_{t,T}^f \sigma_{W,t}^2) < 0$$

(48)

$$\frac{\partial g(\cdot)}{\partial \sigma_{\varepsilon,t}} = \frac{\sigma_{\varepsilon,t} (-R + \beta_t R_{t,T}^f \sigma_{W,t}^2)}{(\beta_t^2 \sigma_{W,t}^2 + \sigma_{\varepsilon,t}^2)^{3/2}}$$

(49)

$$\frac{\partial (\gamma_t \sigma_{R,t})}{\partial \sigma_{W,t}} = \frac{R_{t,T}^f (2\beta_t \sigma_{\varepsilon,t}^2 \sigma_{W,t} + \beta_t^3 + \sigma_{W,t}^3)}{(\beta_t^2 \sigma_{W,t}^2 + \sigma_{\varepsilon,t}^2)^{3/2}} > 0$$

(50)

$$\frac{\partial (\gamma_t \sigma_{R,t})}{\partial \sigma_{\varepsilon,t}} = \frac{-\beta_t R_{t,T}^f \sigma_{\varepsilon,t}^2}{(\beta_t^2 \sigma_{W,t}^2 + \sigma_{\varepsilon,t}^2)^{3/2}} < 0$$

(51)

Overall, the sign of the derivatives in (46) and (47) is ambiguous. Figure 3 shows a positive relationship between $F_t^*(R)$ and $\sigma_{\varepsilon,t}$, as well as between $F_t^*(R)$ and $\sigma_{W,t}$ for a subset of the parameter space when $R = -0.15$.

### A.3 Details for the Logistic Fit of the Risk-Neutral Distribution

For each date $t$, we have adjusted options prices $\tilde{C}_t(X,T)$ and $\tilde{P}_t(X,T)$ for a set of strike prices derived following the steps in Section 3.1. Collecting these prices in the $N_t \times 1$ vector of option prices $o_t$, we specify the model

$$o_t = \sum_{j=1}^{J} \omega_{jt} \tilde{X}_j t(X_{it}; \mu_{jt}, \sigma_{jt}) + \varepsilon_{it},$$

(52)
where the subscript \(i\) denotes the \(i^{th}\) element of \(o_t\). We then solve the constrained non-linear least squares (CNLS) problem

\[
\min_{\omega_t, \mu_t, \sigma_t} \frac{1}{N_t} \sum_{i=1}^{N_t} \left( o_{it} - \sum_{j=1}^{J} \omega_{jt} \sigma_{jt} \ln \left( \exp \left( \frac{X_{it} R}{\sigma_{jt}} \right) + \exp \left( \frac{\mu_{jt}}{\sigma_{jt}} \right) \right) \right)^2,
\]

\(53\)

\[
s.t. \sum_{j=1}^{J} \omega_{jt} = 1, \quad \omega_{jt} \geq 0, j = 1, \ldots, J.
\]

We estimate \(F^*_t (X, T)\) by taking the parameters estimated from (53) and plugging them into (26). We solve this CNLS problem separately for each date \(t\) that occurs one month ahead of an option expiration.\(^{23}\)

Two practical difficulties arise in the setup described above. First, we don’t observe the exact closing options prices. Instead, we observe the end-of-day bid and ask quotes. We use the mid point between these quotes as our options price. This choice implies that we will have less than a perfect fit in our least squares problem, but we do not expect a systematic bias. To improve the accuracy of our option price data, we omit option prices implying mid quotes of less than 0.5 and those with no trading volume on day \(t\) following Figlewski (2010). Our second difficulty stems from the fact that option trading closes 15 minutes after the stock market at 3:15pm central time. This institutional fact creates an asynchronicity when we calculate the adjusted call price in (24) due to the presence of \(S_t\). We address the asynchronicity issue by assuming that information in the last 15 minutes of options trading does not change the shape of the pricing curve, but may shift it up or down by an unknown constant. We can estimate this constant because we observe multiple call option prices shifted by the same constant. Thus, we adjust our CNLS problem to the following

\[
\min_{a_t, \omega_t, \mu_t, \sigma_t} \frac{1}{n} \sum_{i=1}^{n} \left( o_{it} - a_t d_{it} - \sum_{j=1}^{J} \omega_{jt} \sigma_{jt} \ln \left( \exp \left( \frac{X_{it} R}{\sigma_{jt}} \right) + \exp \left( \frac{\mu_{jt}}{\sigma_{jt}} \right) \right) \right)^2,
\]

\(54\)

\[
s.t. \sum_{j=1}^{J} \omega_{jt} = 1, \quad \omega_{jt} \geq 0, j = 1, \ldots, J
\]

where \(d_{it}\) equals one for call prices and zero otherwise, while \(a_t\) is an unknown shift of the pricing curves that we estimate.\(^{24}\) The error term in (52) captures noise in the data that

\(^{23}\)Throughout the paper, we use the term month to mean four weeks.

\(^{24}\)The adjusted call price that addresses the timing discrepancy may be written as \(\tilde{C}_{it} (X, T) = (1 + r_{t,T}) (C_{it} (X, T) - (S_t + \delta^{timing}))\). The term \(a_t d_{it}\) in our estimating equation plays the role of \(\delta^{timing}\).
may result from stale quotes, our use of the bid-ask mid point rather than the unobserved fundamental price, or from approximation error. We assess the fit of the logistic mixtures using the following R-squared measure

$$R^2_t = 1 - \frac{\sum_{i=1}^{N_t}(o_{it} - \hat{o}_{it})^2}{\sum_{i=1}^{N_t}(o_{it} - \bar{o}_t)^2}. \quad (55)$$

In terms of notation, $N_t$ denotes the number of observations for the date under consideration and $\bar{o}_t = (1/N_t)\sum_{i=1}^{N_t}o_{it}$.

### A.4 Performance of the Conditional-Quantile Models: Backtests

In this Appendix, we discuss the backtests we employ to evaluate the performance of the conditional-quantile models discussed in Section 3.2. We start with Engle and Manganelli (2004) who propose a dynamic-quantile (DQ) test, where instruments are incorporated to increase the power of the backtest. The instruments differ depending on the model we estimate. To allow for different instruments, we write the “full” instrument matrix as follows $X \equiv (Z,W)$. Matrix $Z$ includes the explanatory variables in the GMM model used to estimate the quantiles (when applicable) and $W$ includes the test variables that may predict hits but are not included in the model. The elements of the matrices $Z$ and $W$ for the five models we estimated are as follows

- **Baseline.** $Z$: constant and volatility. $W$: SPOCQ.
- **SPOCQ in Moments.** $Z$: constant and volatility. $W$: SPOCQ.
- **Kitchen Sink.** $Z$: constant, volatility, monthly lagged return, tail jumps, default spread, and term spread. $W$: SPOCQ.
- **GARCH.** $Z$: no elements. $W$: constant and SPOCQ.
- **Unconditional Quantile.** $Z$: constant. $W$: SPOCQ.

The Engle and Manganelli DQ statistic for each of the five quantile predictions implied by the models above is then given by

$$DQ_i = \frac{hit_i(\theta)'X_i[X_i'X_i]^{-1}X_i'hit(\theta)}{\theta(1-\theta)} \sim \chi^2(W_{iq}), \quad i = 1, \ldots , 5 \quad (56)$$
where $hit_i(\theta)$ is the hit sequence and $W_{iq}$ is the column rank of matrix $W$ for the $i$th model. Notice that the row dimension of the hit vector and the instrument matrix in Equation (56) equals the full sample (268).

We also implement the refinements of the backtest in Kupiec (1995) due to Christoffersen (1998). Kupiec tests whether the average number of hits is equal to the quantile under consideration. The hits follow a Bernoulli distribution and the null can be tested using the following likelihood-ratio (LR), also known as the unconditional coverage, statistic

$$LR_{uc} = -2\ln \left[ L(\theta)/L(\hat{\pi}) \right] \sim \chi^2(1)$$

$$L(\theta) = (1 - \theta)^{T_0}\theta^{T_1}$$

$$L(\hat{\pi}) = (1 - T_1/T)^{T_0}(T_1/T)^{T_1}$$

where $\hat{\pi} = (1/T)\sum_{t=1}^{T} I(r_{t,T} < \hat{q}(\theta)) = T_1/T$, and $T_0 = T - T_1$. The test rejects the null of a correctly specified model if the average number of hits is statistically different from $\theta$.

The test proposed by Kupiec fails to account for “hit clustering” over time, a property that would invalidate the conditional-quantile model. Christoffersen (1998) extended the unconditional coverage test to account for serial independence of the hits. Under the maintained assumption that hits follow a first-order Markov process with transition probability matrix given by

$$\Pi_1 = \begin{bmatrix} 1 - \pi_{01} & \pi_{01} \\ 1 - \pi_{11} & \pi_{11} \end{bmatrix},$$

the null hypothesis of independence can be tested using the following LR statistic

$$LR_{ind} = -2\ln \left[ L(\hat{\pi})/L(\hat{\Pi}_1) \right] \sim \chi^2(1)$$

$$L(\Pi_1) = (1 - \pi_{01})^{T_{00}}(\pi_{01})^{T_{01}}(1 - \pi_{11})^{T_{10}}(\pi_{11})^{T_{11}},$$

where $L(\hat{\pi})$ is the likelihood under the alternative from the $LR_{uc}$ test, and $L(\Pi_1)$ is the likelihood function of the first-order Markov process. Furthermore, $T_{ij}$, $i,j = 0,1$, is the number of observations with a $j$ following an $i$ (e.g., hit, non-hit). Differentiating with respect to $\pi_{01}$ and $\pi_{11}$, we obtain the maximum likelihood estimates, $\hat{\pi}_{01} = T_{01}/(T_{00} + T_{01})$, and $\hat{\pi}_{11} = T_{11}/(T_{10} + T_{11})$, in order to evaluate $L(\hat{\Pi}_1)$. 

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Christoffersen also offered a test for the joint null of correct unconditional coverage and independence using a conditional coverage LR statistic

\[ LR_{cc} = LR_{uc} + LR_{ind} \sim \chi^2(2), \tag{59} \]

Rejecting the null is suggestive of the need to improve the conditional-quantile model to address the hit clustering. The test is somewhat restrictive since it takes into account only first-order serial correlation in the hits, as opposed to more general dynamics, which is the case in the DQ test in Engle and Manganelli.

Gaglianone, Lima, Linton, and Smith (2011) propose a Wald test for the null of correct specification, which they label the VQR test. The authors document superior finite-sample performance of the VQR test relative to its LM-type counterparts. The VQR test is implemented via a quantile regression of the outcome variable—\( r_{t,T} \) in our case—on the predicted quantile and testing whether the coefficient on the predicted quantile equals one and the intercept equals zero. The VQR test applies to out-of-sample evaluation of conditional quantiles estimated from quantile regression, or to in-sample evaluation of conditional quantiles estimated using alternative models.

We implement the out-of-sample version of the VQR test on the last 68 observations of our sample using a quantile regression of the return \( r_{t,T} \) on a constant and the conditional-quantile prediction that we obtain from our model. The VQR test collapses to a Wald test for the joint null hypothesis that the coefficient of the constant is zero and the coefficient of the conditional-quantile prediction is one in each of the five quantile regressions. We obtain an estimate of the variance-covariance matrix that enters the expression for the Wald test using a Gaussian kernel with the bandwidth suggested by Chamberlain (1994). The Wald statistic for the VQR test is given by the following expression

\[ VQR = \hat{\beta}(\theta)' \hat{V}_\beta(\theta)^{-1} \hat{\beta}(\theta) \sim \chi^2(2), \tag{60} \]

where \( \hat{\beta}(\theta) \) is a 2 \times 1 vector that contains the coefficients of the quantile regression and \( \hat{V}_\beta(\theta) \) is the associated variance-covariance matrix.

Finally, we implement two well-known tests of the white-noise null hypothesis, namely the tests by Bartlett (1955) and Ljung and Box (1978), which can be implemented “right off the shelf” for the hit sequence (see Christoffersen (2010)).
A.5 GMM Estimation of Quantile Regressions

In this Appendix, we provide a brief description of the optimization design underlying the GMM estimates of the conditional-quantile regression models discussed in Section 3.2. We estimate all our models in MATLAB with two-stage GMM using the optimal weighting matrix (variance-covariance of the moments) in the second stage to construct statistics of interest, such as the J-statistic of Hansen (1982). We report standard errors using the sandwich formula (e.g., see Table 6.3 in Cameron and Trivedi (2005)) based on numerical derivatives of the moments calculated using Richardson extrapolation.

The hit function is not differentiable, which complicates estimation and inference. For computational ease, and to employ standard GMM inference, we follow Amemiya (1982) and remove the cusps in the indicator function using a smooth-function approximation (see also Horowitz (1998)). Similar to Amemiya, we use the logistic function $h^f_t(\theta)$ in place of $h_t(\theta)$ in our moment condition, where

$$h^f_t(\theta) = \frac{\exp(c(r_{t,T} - \mathbf{x}_t'\beta))}{1 + \exp(c(r_{t,T} - \mathbf{x}_t'\beta))}$$

for a large constant $c$. We ensured that our non-linear searches converged to a local minimum by employing the following approach. For the just-identified models, we first obtained parameter estimates using qreg in STATA to estimate a quantile regressions of the return series ($r_{t,T}$) using a constant and volatility as explanatory variables. Subsequently, we used the STATA parameter estimates as starting values for the derivative-free Simplex (fminsearch) algorithm in MATLAB. The Simplex parameter estimates in MATLAB served as starting values for the Quasi-Newton (fminunc) algorithm in MATLAB. The Quasi-Newton estimates from MATLAB are the estimates we report in the paper.

The optimization diagnostics provided in Panel D of Table 3 confirm that our non-linear search algorithms converged to a local minimum. These diagnostics are the $inf$ norm of the gradient, $\|g\|_\infty$, a scale invariant weighted gradient stopping-criterion, $g'H^{-1}g$, with $H$ being the Hessian, and the Hessian condition number $\kappa(H) = \lambda_{\max}(H)/\lambda_{\min}(H)$, which is

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25 Horowitz employs the so-called smoothed least absolute deviation (SLAD) estimator in order to apply bootstrap theory on the LAD estimator. The LAD estimator is the same as quantile regression for $\theta = 0.5$. The SLAD smooths the cusps in the indicator function by replacing the indicator function with a smooth function, which is analogous to the integral of a kernel function used in nonparametric estimator.

26 We set $c$ equal to 10,000 times the standard deviation of returns. With this value, $h^f_t < 0.0001$ for any returns within 0.001 standard deviations of $\mathbf{x}_t'\beta$. 

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the ratio of its largest to its smallest eigenvalue.\footnote{All the Hessian eigenvalues should be positive at a local minimum. At the same time, as long as the condition number does not exceed $1/\sqrt{\epsilon}$, where $\epsilon$ is the machine numerical precision, there are no concerns regarding numerical issues surrounding the estimates. We used MATLAB to estimate our models. For MATLAB, $\epsilon = 2.2E-16$, such that $1/\sqrt{\epsilon} = 6.7E + 0.7$.}

The optimization diagnostics described above are all based on numerical gradients and Hessians that are available as byproducts in the MATLAB \texttt{fminunc} routine. Our MATLAB optimization settings are as follows: \texttt{TolX=}1E-6, \texttt{TolFun=}1E-6, \texttt{MaxFunEvals=}50,000, and \texttt{MaxIter=}10,000. We use the BFGS Hessian update with \texttt{fminunc}. We checked that our GMM parameter estimates obtained smoothing the cusps of the indicator function are the same with those obtained using \texttt{qreg}, which is based on the linear-programming formulation of the quantile regressions, within numerical precision. All MATLAB and STATA codes with accompanying log files for estimation results are available from the authors upon request.

### A.6 Robustness of Main Findings

Figures 6 and 7 show that the discount rate applied to large negative returns tends to be smaller when volatility is high and the upward-sloping portion of the pricing kernel tends to be steeper when volatility is high. In this section, we investigate the robustness of these findings to alternative quantile models. Our first and second models are variants of quantile regression that use more predictors (the kitchen sink model) and different moment conditions (the SPOCQ-in-moments model). The other two models are rather different. We report results from the following four models.

1. **Kitchen Sink.** Quantile regressions based on (31), except with a richer set of predictors. In addition to volatility, we use the monthly lagged return, tail jumps in the past month, corporate bond default spread, and the Treasury term spread as predictors. See Appendix A.7 for details on these variables.

2. **SPOCQ in Moments.** Estimates of the conditional quantiles using GMM with volatility as the only predictor and the moment conditions $E[h_t(\theta)] = \theta$ and (32). The latter imposes zero correlation between SPOCQ and hits.

3. **GARCH.** The Heston and Nandi (2000) GARCH model, the discrete analog of the
Heston (1993) GARCH

\[ r_t = r + \left( \mu - \frac{1}{2} \right) \sigma_t + \sqrt{\sigma_t} z_t \]

\[ \sigma_t = \omega + \beta \sigma_{t-1} + \alpha (z_{t-1} - \gamma \sqrt{\sigma_{t-1}})^2 \]  \( (62) \)

where \( r_t \) denotes the daily log return and \( z_t \sim iidN(0,1) \).

4. **Unconditional Quantiles** We evaluate the risk-neutral distribution at unconditional quantiles, i.e., at the same point each month. This is a special case of a quantile regression with the constant as the only explanatory variable.

Christoffersen, Heston, and Jacobs (2013) argue that the Heston and Nandi (2000) GARCH model prices S&P 500 options well. This model allows skewness in the return distribution through the parameter \( \gamma \); a positive \( \gamma \) causes negative returns to have a larger effect on variance than positive returns. The model also has a volatility risk premium (\( \mu \)) and the feature that it approaches Black-Scholes in the limit as volatility becomes constant. Following Christoffersen, Heston, and Jacobs (2013), we fit this model to daily returns for 1990/01–2012/04 and estimate conditional quantiles by simulating the 28-day-ahead return distribution implied by the model.

The Unconditional Quantiles model produces results that match the \( F^*(R) \) curve in Figure 3. Rather than estimating conditional quantiles, we evaluate the risk-neutral density at the same points each month. To make the results comparable to the rest of our analysis, we use unconditional quantiles as our evaluation points. This exercise allows us to illustrate the insights gained by evaluating the risk-neutral density at conditional quantiles.

Table A.3 presents results from these alternative models along with the baseline results reported in the previous section.\(^{28}\) The SPOCQ-in-Moments model produces similar hit rates and volatility-SPOCQ correlations to the baseline. It passes all but one of the diagnostic tests, which indicates that it fits the conditional quantiles well. Unlike the baseline, it passes the diagnostic tests for the 75% quantile. Given that the SPOCQ-in-Moments and baseline models generate similar volatility-SPOCQ correlations, this result suggests that any misspecification of the 75% quantile in our baseline model has a small effect on our results.

The Kitchen Sink model fails some of the diagnostic tests for the 25%, 50%, and 75% quantiles, which suggests that it fits the conditional quantiles less well than the baseline.

\(^{28}\)Figures A.2-A.9 are the analogs to Figures 1 and 6 for each of the four alternative models.
Nonetheless, it also produces negative volatility-SPOCQ correlations in the left tail and positive correlations in the right tail.

The GARCH model fails the diagnostic tests. Panel A shows that it severely underestimates hits in the tails. It produces conditional quantiles for the bottom quartile and decile that hit in just 5.6% and 17.5% of months. The hit rates are similarly low for the right tail quantiles. The DQ and Portmanteau tests also reject the GARCH model for most quantiles. The SPOCQ series produced by the GARCH model produce different volatility-SPOCQ correlations to the quantile regression models. In particular, the correlation is positive in the left tail.

Examining the GARCH quantiles further, we find that they are too smooth, especially in the left tail. When volatility is above its median, the hit rate on the 10% quantile is 0.07, and it drops to 0.04 when volatility is below its median. Similarly, for the 25% quantile, hits occur in 22% of above-median-volatility months, but in only 13% of below-median-volatility months. Thus, in high volatility months, the GARCH-implied SPOCQ is high because the probability of the event is high and not because the pricing kernel is high. In essence, the GARCH model produces quantiles that have shifted from correctly-specified conditional quantiles towards the unconditional quantiles. Consistent with the predictions of Section 2.2, the unconditional-quantile model produces positive correlations between SPOCQ and volatility in both tails, as does the GARCH model.

A.7 Data in the Quantile Pricing Kernel Regressions

This Appendix provides the details behind the data used in the quantile pricing kernel (QPK) regressions of Section 6.

1. Volatility: We constructed a measure of continuous variation for each trading date in our options data using information from the Commodity Research Bureau (CRB) for the closing values of the S&P 500 as follows.\(^\text{29}\) We first calculated the daily log return. We then constructed a “non-tail” daily log return. The non-tail daily log return equals the daily log return if the absolute value of the daily log return falls below 1.8% and zero, otherwise. The threshold follows the parameterization in Bollerslev and Todorov (2011). The continuous variation is the square root of the sum of the squared non-tail daily log returns over the last 20 tradings days divided by 20. The volatility variable in our QPK regressions is the log of the continuous variation.

\(^\text{29}\)http://www.crbtrader.com/.
2. **Tail Jumps:** Our measure of tail jumps equals the square root of the difference between the realized standard deviation and the realized continuous variation of the S&P 500 daily log returns over the last 20 trading days using data from CRB for the closing values of the S&P 500. The realized standard deviation is the square root of the sum of squared daily log returns over the last 20 trading days divided by 20.

3. **Most recent returns:** Our measure of most recent returns equals the cumulative daily log returns over the last 20 trading days using data from the CRB on the closing values of the S&P 500.

4. **Consumer Sentiment:** We use monthly data for the University of Michigan Consumer Sentiment available from the FRED website.\[^{30}\]

6. **Open Interest:** We constructed our measure of open interest using data for S&P 500 call and put options for 268 trading dates between January 1990 and April 2012 from the Chicago Board Options Exchange (CBOE).\[^{31}\] We limited our attention to contracts with one-month expiration. For each trading date, we calculated the moneyness for each put and call option strike. We then assigned the open interest for the call and put options in the 6 moneyness bins (MBins) implied by the estimates of the conditional quantiles in Section 4.2.\[^{32}\] For example, the 0–10% MBin contains all options with moneyness less than or equal to the moneyness implied by the 10% conditional quantile. We used put options for the 0–10% and 0–25% MBins. We used call options for the 75%-100% and 90%-100% MBins. The variable that appears in the QPK regressions is the logarithm of the open interest for the corresponding MBin. We added one to the open interest before taking its logarithm to address the presence of zeros.

7. **Dollar Volume:** We constructed our measure of dollar volume using data for the S&P 500 call and put options for 268 trading dates between January 1990 and April 2012 from CBOE. The dollar volume is the product of the option price by the volume for each option contract. We aggregated the dollar volume across options for each of the 6 MBins constructed as described in (5). Similar to the open interest, we used put options for the 0–10% and 0–25% MBins. We used call options for the 75%-100% and 90%-100% MBins. The variable that appears in the QPK regressions is the logarithm of the dollar volume for the corresponding MBin. We added one to the dollar volume before taking its logarithm to address the presence of zeros.

\[^{30}\]http://research.stlouisfed.org/fred2/series/UMCSENT/.
\[^{31}\]http://www.cboe.com/.
\[^{32}\]The moneyness implied by each conditional quantile is its exponential.
8. **Dividend Yield:** We calculated the dividend yield for the S&P 500 using the index closing value on the last trading date of the previous quarter and the sum of the cash dividends per share for the 4 last quarters. For example, we calculated the dividend yield for January, February, and March 1990 using the index closing value on 12/31/1989, and the cash dividends per share on 12/31/1989, 09/30/1989, 06/30/1989, and 03/31/1989, as reported by Standard and Poor’s.33

9. **CAY:** Quarterly data for the CAY are available from Martin Lettau’s website.34 We use the CAY value of the previous quarter for the three months of the current quarter. For example, we use the 1989Q4 CAY value for January, February, and March 1990.

10. **Uncertainty Index:** Our uncertainty measure utilizes the daily news-based index from the Economic Policy Uncertainty project at Stanford University.35 For each of the trading dates in our option data, we calculated the logarithm of the index sum for the last 28 calendar days.

11. **Default Spread:** We calculated the default spread as the difference between Moody’s BAA and AAA corporate bond yields for each of the trading dates in our sample using information from the FRED website.36

12. **Term Spread:** The term spread is the difference between the 10-year and the 3-month yield curve rates for each of the trading dates in our options data using yields from the Treasury.37

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34 [http://faculty.haas.berkeley.edu/lettau/data/cay_q_11Q3.txt](http://faculty.haas.berkeley.edu/lettau/data/cay_q_11Q3.txt).
A.8 Appendix Tables and Figures

Table A.1. Summary statistics for the S&P 500 returns

The table characterizes the S&P 500 returns data that we use to construct the quantiles underlying our SPOCQ statistics. Returns are defined as 100 times the log change in the S&P 500 index from the close on one Friday to the open on Friday four weeks hence. We report statistics for the monthly return series $r_{t,T}$ based on 268 option trading dates between January 19, 1990 and April 20, 2012. The asterisk (*) denotes significance at 5% based on a block bootstrap with 2-year block size.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Value</th>
<th>Statistic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obs.</td>
<td>268</td>
<td>10% quantile</td>
<td>-4.4155 *</td>
</tr>
<tr>
<td>Min</td>
<td>-28.6642</td>
<td>25% quantile</td>
<td>-1.1954 *</td>
</tr>
<tr>
<td>Mean</td>
<td>0.6078</td>
<td>50% quantile</td>
<td>1.2075 *</td>
</tr>
<tr>
<td>Max</td>
<td>11.8445</td>
<td>75% quantile</td>
<td>2.9270  *</td>
</tr>
<tr>
<td>Std. Deviation</td>
<td>4.5851 *</td>
<td>90% quantile</td>
<td>5.0591  *</td>
</tr>
<tr>
<td>AR(1) returns</td>
<td>-0.0210</td>
<td>Ann. mean returns</td>
<td>7.9014</td>
</tr>
<tr>
<td>AR(1) sq returns</td>
<td>0.2475</td>
<td>Ann. return volat.</td>
<td>16.5318</td>
</tr>
</tbody>
</table>
Table A.2. Quantile pricing kernel regressions: explaining the slope

We report OLS estimates and model diagnostics for the quantile pricing kernel regressions in Section 6 that pertain to the slope of the pricing kernel in the upper part of the return distribution. The columns labeled 75%-100% use \( QPK_{t,T}(75, 100) - QPK_{t,T}(50, 75) \) as the dependent variable and the columns labeled 90%-100% use \( QPK_{t,T}(90, 100) - QPK_{t,T}(50, 75) \) as the dependent variable. The standard errors are in parentheses and are adjusted for autocorrelation using the Newey-West formula (12 lags). The p-values are in squared brackets. The asterisks indicate statistical significance as follows: 1% (***) , 5% (**) , 10% (*). We standardize all explanatory variables such that their mean is zero and their standard deviation is one to ease comparison of their coefficients.

<table>
<thead>
<tr>
<th>Variable</th>
<th>75% – 100%</th>
<th>90% – 100%</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SPEC1</td>
<td>SPEC2</td>
<td>SPEC3</td>
<td>SPEC1</td>
</tr>
<tr>
<td>Volatility</td>
<td>0.166***</td>
<td>0.045*</td>
<td>0.166***</td>
<td>0.166***</td>
</tr>
<tr>
<td></td>
<td>(0.024)</td>
<td>(0.027)</td>
<td>(0.039)</td>
<td>(0.041)</td>
</tr>
<tr>
<td>Tail jumps</td>
<td>0.088***</td>
<td>0.128***</td>
<td>0.089***</td>
<td>0.109***</td>
</tr>
<tr>
<td></td>
<td>(0.017)</td>
<td>(0.017)</td>
<td>(0.017)</td>
<td>(0.017)</td>
</tr>
<tr>
<td>Returns</td>
<td>-0.140***</td>
<td>-0.109***</td>
<td>-0.246***</td>
<td>-0.205***</td>
</tr>
<tr>
<td></td>
<td>(0.017)</td>
<td>(0.017)</td>
<td>(0.026)</td>
<td>(0.023)</td>
</tr>
<tr>
<td>Consumer sentiment</td>
<td>-0.005</td>
<td>-0.015</td>
<td>-0.066</td>
<td>-0.089</td>
</tr>
<tr>
<td></td>
<td>(0.046)</td>
<td>(0.046)</td>
<td>(0.073)</td>
<td>(0.073)</td>
</tr>
<tr>
<td>Log open interest</td>
<td>-0.024</td>
<td>-0.013</td>
<td>-0.114*</td>
<td>-0.088</td>
</tr>
<tr>
<td></td>
<td>(0.048)</td>
<td>(0.048)</td>
<td>(0.058)</td>
<td>(0.057)</td>
</tr>
<tr>
<td>Log dollar volume</td>
<td>0.161***</td>
<td>0.142***</td>
<td>0.376***</td>
<td>0.338***</td>
</tr>
<tr>
<td></td>
<td>(0.052)</td>
<td>(0.051)</td>
<td>(0.071)</td>
<td>(0.067)</td>
</tr>
<tr>
<td>Log dividend yield</td>
<td>-0.130***</td>
<td>-0.111***</td>
<td>-0.149***</td>
<td>-0.175***</td>
</tr>
<tr>
<td></td>
<td>(0.040)</td>
<td>(0.037)</td>
<td>(0.056)</td>
<td>(0.061)</td>
</tr>
<tr>
<td>CAY</td>
<td>0.123***</td>
<td>0.116***</td>
<td>0.232***</td>
<td>0.246***</td>
</tr>
<tr>
<td></td>
<td>(0.031)</td>
<td>(0.026)</td>
<td>(0.045)</td>
<td>(0.041)</td>
</tr>
<tr>
<td>Log uncertainty index sum</td>
<td>0.094**</td>
<td>0.053*</td>
<td>0.096</td>
<td>0.063</td>
</tr>
<tr>
<td></td>
<td>(0.037)</td>
<td>(0.030)</td>
<td>(0.063)</td>
<td>(0.055)</td>
</tr>
<tr>
<td>Default spread</td>
<td>0.045**</td>
<td>0.011</td>
<td>0.003</td>
<td>0.014</td>
</tr>
<tr>
<td></td>
<td>(0.021)</td>
<td>(0.017)</td>
<td>(0.038)</td>
<td>(0.031)</td>
</tr>
<tr>
<td>Term spread</td>
<td>-0.051**</td>
<td>-0.028</td>
<td>-0.081**</td>
<td>-0.060</td>
</tr>
<tr>
<td></td>
<td>(0.026)</td>
<td>(0.024)</td>
<td>(0.035)</td>
<td>(0.038)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.197***</td>
<td>0.201***</td>
<td>0.265***</td>
<td>0.277***</td>
</tr>
<tr>
<td></td>
<td>(0.029)</td>
<td>(0.025)</td>
<td>(0.048)</td>
<td>(0.034)</td>
</tr>
<tr>
<td>obs.</td>
<td>268</td>
<td>264</td>
<td>264</td>
<td>268</td>
</tr>
<tr>
<td>R² adj.</td>
<td>0.200</td>
<td>0.518</td>
<td>0.565</td>
<td>0.073</td>
</tr>
<tr>
<td>F-test</td>
<td>48.330</td>
<td>37.520</td>
<td>34.450</td>
<td>18.560</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.000]</td>
<td>[0.000]</td>
<td>[0.000]</td>
</tr>
</tbody>
</table>
We report various diagnostics for the models discussed in Section A.6 based on the full sample. Panel A provides the hit rates. Panel B provides the p-values for the DQ statistic of *Engle and Manganelli (2004)* (top) and the Portmanteau (Q) statistic (bottom). The DQ statistic is based on a constant and the implied SPOCQ series. The Portmanteau statistic is based on 12 lags of the hit sequence. The asterisks indicate statistical significance at 5% level. Panel C provides the correlation between volatility and each of the SPOCQ series implied by the model under consideration.

**Table A.3. Model comparisons**

Panel A: Hit Rates

<table>
<thead>
<tr>
<th>Model</th>
<th>10%</th>
<th>25%</th>
<th>50%</th>
<th>75%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>10.4%</td>
<td>25.0%</td>
<td>50.0%</td>
<td>75.0%</td>
<td>89.9%</td>
</tr>
<tr>
<td>SPOCQ in Moments</td>
<td>10.8%</td>
<td>25.4%</td>
<td>49.3%</td>
<td>75.0%</td>
<td>90.3%</td>
</tr>
<tr>
<td>Kitchen Sink</td>
<td>9.3%</td>
<td>24.6%</td>
<td>49.6%</td>
<td>74.3%</td>
<td>88.4%</td>
</tr>
<tr>
<td>GARCH</td>
<td>5.6%</td>
<td>17.5%</td>
<td>50.0%</td>
<td>82.8%</td>
<td>95.5%</td>
</tr>
<tr>
<td>Unc. Quantiles</td>
<td>9.7%</td>
<td>25.0%</td>
<td>50.0%</td>
<td>75.0%</td>
<td>90.3%</td>
</tr>
</tbody>
</table>

Panel B: Diagnostic Tests of Quantile Fit

<table>
<thead>
<tr>
<th>Model</th>
<th>(0,10)</th>
<th>(0,25)</th>
<th>(0,50)</th>
<th>(75,100)</th>
<th>(90-100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>-0.275</td>
<td>-0.387</td>
<td>-0.341</td>
<td>0.406</td>
<td>0.226</td>
</tr>
<tr>
<td>SPOCQ in Moments</td>
<td>-0.251</td>
<td>-0.449</td>
<td>-0.341</td>
<td>0.371</td>
<td>0.270</td>
</tr>
<tr>
<td>Kitchen Sink</td>
<td>-0.313</td>
<td>-0.153</td>
<td>-0.326</td>
<td>0.212</td>
<td>0.110</td>
</tr>
<tr>
<td>GARCH</td>
<td>0.558</td>
<td>0.313</td>
<td>-0.113</td>
<td>0.274</td>
<td>0.261</td>
</tr>
<tr>
<td>Unc. Quantiles</td>
<td>0.839</td>
<td>0.624</td>
<td>-0.514</td>
<td>0.729</td>
<td>0.793</td>
</tr>
</tbody>
</table>
Figure A.1. Daily S&P 500 closing values
Figure A.2. Conditional quantiles and SPOCQ: SPOCQ in Moments
Figure A.3. Conditional quantiles and SPOCQ: Kitchen Sink
Figure A.4. Conditional quantiles and SPOCQ: GARCH
Figure A.5. Unconditional Quantiles and SPOCQ
Figure A.6. SPOCQ in Moments

Figure A.7. Kitchen Sink
Figure A.8. GARCH

Figure A.9. Unconditional Quantiles
Figure A.10. Mimicking portfolio for the pricing kernel from Bakshi, Madan, and Panayotov (2010)

Note: We reproduce here Figure 1 in Bakshi, Madan, and Panayotov (2010). Consistent with our findings, the pricing kernel associated with negative returns is flatter in the high-volatility (1996–2007) period than in the low-volatility (1988–1996) period; compare and contract Panel B and Panel D.